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Article *in* Differential Equations · July 2007

DOI: 10.1134/S0012266107070038

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ISSN 0012-2661, Differential Equations, 2007, Vol. 43, No. 7, pp. 905–915. © Pleiades Publishing, Ltd., 2007.
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ORDINARY DIFFERENTIAL EQUATIONS

On the Basis Property of the System of Eigenfunctions of a Spectral Problem with Spectral Parameter in the Boundary Condition

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Received February 6, 2006

DOI: 10.1134/S0012266107070038

1. INTRODUCTION

Consider the spectral problem

$$y^{(4)}(x) - (q(x)y'(x))' = \lambda y(x), \qquad 0 < x < l, \tag{1.1}$$

$$y'(0)\cos\alpha - y''(0)\sin\alpha = 0, \tag{1.2a}$$

$$y(0)\cos\beta + Ty(0)\sin\beta = 0, \tag{1.2b}$$

$$y'(l)\cos\gamma + y''(l)\sin\gamma = 0, \tag{1.2c}$$

$$(a\lambda + b)y(l) - (c\lambda + d)Ty(l) = 0, \tag{1.2d}$$

where λ is a spectral parameter;

$$Ty \equiv y''' - qy';$$

$q(x)$ is a positive absolutely continuous function on the interval $[0, l]$; α, β, γ, a, b, c, and d are real constants; and $0 \le \alpha, \beta, \gamma \le \pi/2$.

Throughout the following, we assume that

$$\sigma = bc - ad > 0. \tag{1.3}$$

In the present paper, we study basis properties of the system of eigenfunctions of the boundary value problem (1.1), (1.2) in the spaces $L_p(0, l)$ $(1 < p < \infty)$.

Boundary value problems for second- and fourth-order ordinary differential operators with a spectral parameter in boundary conditions were studied in a series of papers (e.g., see [1–17]). A number of problems in mathematical physics can be reduced to such problems (e.g., see [2–10]).

Basis properties of the system of eigenfunctions of the Sturm–Liouville problem with a spectral parameter in the boundary condition were studied in [10–16] in various function spaces, and the existence of eigenvalues, estimates of eigenvalues and eigenfunctions, and expansion theorems were considered in [1, 6, 8, 9] for fourth-order ordinary differential operators with a spectral parameter in a boundary condition.

To study the basis properties of the system of eigenfunctions of the boundary value problem (1.1), (1.2) in the space $L_p(0, 1)$ $(1 < p < \infty)$, one should use the oscillation properties of eigenfunctions and asymptotic formulas for eigenvalues and eigenfunctions of this problem. Note that the oscillation properties of solutions were used in [11] to analyze the basis properties of the system of eigenfunctions of a second-order differential operator with a potential and with spectral parameter in the boundary condition in the space L_2. An adaptation of the classical result on the oscillation properties of eigenfunctions to problems with a spectral parameter in the boundary condition was represented in [11]. A detailed exposition of the classical result on the oscillation properties of the eigenfunctions of a second-order differential operator with a potential can be found in the monograph [18].

2. OSCILLATION PROPERTIES OF THE EIGENFUNCTIONS OF THE BOUNDARY VALUE PROBLEM (1.1), (1.2)

Lemma 2.1 [17]. *For each $\lambda \in \mathbb{C}$, there exists a unique (up to a constant factor) nontrivial solution $y(x, \lambda)$ of problem* (1.1), (1.2a)–(1.2c).

Remark 2.1 [17]. Without loss of generality, one can assume that the solution $y(x, \lambda)$ of problem (1.1), (1.2a)–(1.2c) is an entire function of λ for each $x \in [0, l]$.

Lemma 2.2. *The eigenvalues of the boundary value problem* (1.1), (1.2) *are real and simple and form an at most countable set without finite limit points.*

Proof. Let $y(x, \lambda)$ be a nontrivial solution of problem (1.1), (1.2a)–(1.2c). The eigenvalues of problem (1.1), (1.2) are the roots of the equation

$$(a\lambda + b)y(l, \lambda) - (c\lambda + d)Ty(l, \lambda) = 0. \tag{2.1}$$

Let λ^* be a nonreal eigenvalue of problem (1.1), (1.2). Then $\bar{\lambda}^*$ is also an eigenvalue of this problem, since the coefficients $q(x)$, α, β, γ, a, b, c, and d are real; moreover, $y\left(x, \bar{\lambda}^*\right) = \overline{y\left(x, \lambda^*\right)}$. By virtue of (1.1), we have

$$(Ty(x, \mu))' y(x, \lambda) - (Ty(x, \lambda))' y(x, \mu) = (\mu - \lambda) y(x, \mu) y(x, \lambda).$$

By integrating this relation from 0 to l, by using the formula for the integration by parts, and by taking into account conditions (1.2a)–(1.2c), we obtain

$$y(l, \lambda)Ty(l, \mu) - y(l, \mu)Ty(l, \lambda) = (\mu - \lambda) \int_0^l y(x, \mu) y(x, \lambda) dx. \tag{2.2}$$

By setting $\mu = \bar{\lambda}^*$ and $\lambda = \lambda^*$ in (2.2), we obtain

$$y\left(l, \lambda^*\right) \overline{Ty\left(l, \lambda^*\right)} - \overline{y\left(l, \lambda^*\right)} Ty\left(l, \lambda^*\right) = \left(\bar{\lambda}^* - \lambda^*\right) \int_0^l \left|y\left(x, \lambda^*\right)\right|^2 dx. \tag{2.3}$$

Since λ^* is a root of Eq. (2.1), we have the relation

$$Ty\left(l, \lambda^*\right) = \left((a\lambda^* + b)/(c\lambda^* + d)\right) y\left(l, \lambda^*\right).$$

In view of this relation, from (2.3), we obtain

$$\frac{\left(\bar{\lambda}^* - \lambda^*\right)(ad - bc)}{\left|c\lambda^* + d\right|^2} \left|y\left(l, \lambda^*\right)\right|^2 = \left(\bar{\lambda}^* - \lambda^*\right) \int_0^l \left|y\left(x, \lambda^*\right)\right|^2 dx.$$

Since $\bar{\lambda}^* \neq \lambda^*$, we have the relation

$$\frac{-\sigma \left|y\left(l, \lambda^*\right)\right|^2}{\left|c\lambda^* + d\right|^2} = \int_0^l \left|y\left(x, \lambda^*\right)\right|^2 dx,$$

which contradicts condition (1.3). Therefore, $\lambda^* \in \mathbb{R}$.

The entire function occurring on the left-hand side in Eq. (2.1) does not vanish for nonreal λ. Consequently, it does not vanish identically. Therefore, its zeros form an at most countable set without finite limit points.

Let us show that Eq. (2.1) has only simple roots. Indeed, if λ^* is a multiple root of Eq. (2.1), then

$$(a\lambda^* + b)\, y\,(l, \lambda^*) - (c\lambda^* + d)\, Ty\,(l, \lambda^*) = 0, \tag{2.4}$$

$$ay\,(l, \lambda^*) + (a\lambda^* + b)\, \frac{\partial}{\partial \lambda} y\,(l, \lambda^*) - cTy\,(l, \lambda^*) - (c\lambda^* + d)\, \frac{\partial}{\partial \lambda} Ty\,(l, \lambda^*) = 0. \tag{2.5}$$

By dividing both sides of relation (2.2) by $(\mu - \lambda)$ $(\mu \neq \lambda)$ and by passing to the limit as $\mu \to \lambda$, we obtain

$$y(l, \lambda) \frac{\partial}{\partial \lambda} Ty(l, \lambda) - Ty(l, \lambda) \frac{\partial}{\partial \lambda} y(l, \lambda) = \int_0^l y^2(x, \lambda) dx. \tag{2.6}$$

Since $\sigma \neq 0$ [see condition (1.3)], we have $(a\lambda^* + b)^2 + (c\lambda^* + d)^2 \neq 0$. Suppose that $a\lambda^* + b \neq 0$. Then, by expressing $y\,(l, \lambda^*)$ and $\frac{\partial}{\partial \lambda} y\,(l, \lambda^*)$ from (2.4) and (2.5), respectively, and by substituting them into relation (2.6) for $\lambda = \lambda^*$, we obtain

$$-\frac{\sigma}{(a\lambda^* + b)^2} \left(Ty\,(l, \lambda^*)\right)^2 = \int_0^l y^2\,(x, \lambda^*)\, dx,$$

which is impossible in view of condition (1.3).

The case in which $c\lambda^* + d \neq 0$ can be considered in a similar way. The proof of Lemma 2.2 is complete.

We introduce the boundary condition

$$y(l) \cos \delta - Ty(l) \sin \delta = 0, \qquad \delta \in [0, \pi/2]. \tag{1.2d$'$}$$

Along with problem (1.1), (1.2), consider the boundary value problem (1.1), (1.2a)–(1.2c), (1.2d′).

Theorem 2.1 [19] (see also [17]). *The eigenvalues of the boundary value problem* (1.1), (1.2a)–(1.2c), (1.2d′) *are simple and form an infinitely increasing sequence* $\{\mu_n(\delta)\}_{n=1}^{\infty}$; *moreover,* $\mu_n(\delta) \geq 0$, $n \in \mathbb{N}$. *The eigenfunction* $v_n^{(\delta)}(x)$ *corresponding to the eigenvalue* $\mu_n(\delta)$ *has* $n-1$ *simple zeros in the interval* $(0, l)$.

Remark 2.2 [17]. One has $\mu_1(\delta) > 0$ for $\delta \in [0, \pi/2)$ and $\delta = \pi/2$, $\beta \in [0, \pi/2)$; $\mu_1(\pi/2) = 0$ for $\beta = \pi/2$.

Let $y(x, \lambda)$ be a nontrivial solution of problem (1.1), (1.2a)–(1.2c). Obviously, the eigenvalues $\mu_n(\delta)$, $n \in \mathbb{N}$, of problem (1.1), (1.2a)–(1.2c), (1.2d′) are zeros of the entire function

$$y(l, \lambda) \cos \delta - Ty(l, \lambda) \sin \delta.$$

We set $\mu_n = \mu_n(0)$ and $\nu_n = \mu_n(\pi/2)$, $n \in \mathbb{N}$. Note that the function $f(\lambda) = Ty(l, \lambda)/y(l, \lambda)$ is defined for

$$\lambda \in D = (\mathbb{C} \backslash \mathbb{R}) \cup \left(\bigcup_{n=1}^{\infty} (\mu_{n-1}, \mu_n) \right),$$

where $\mu_0 = -\infty$.

It was shown in [17] that the function $f(\lambda)$ is continuous and strictly increasing on each interval (μ_{n-1}, μ_n), $n \in \mathbb{N}$, and that $\lim_{\lambda \to -\infty} f(\lambda) = -\infty$.

By $m(\lambda)$, we denote the number of zeros of $y(x, \lambda)$ in the interval $(0, l)$. Lemma 7 in [17] and Theorems 3.3, 4.2, and 5.4 in [19] readily imply the following assertion.

Lemma 2.3. *If* $\lambda \in (\mu_{n-1}, \mu_n]$, $n \in \mathbb{N}$, *then* $m(\lambda) = n - 1$.

To study basis properties of the eigenfunctions of the spectral problem (1.1), (1.2) in the spaces $L_p(0,l)$ $(1<p<\infty)$, one should establish:

(i) a correspondence between the eigenfunctions of problem (1.1), (1.2) and the eigenfunctions of problem (1.1), (1.2a)–(1.2c), (1.2d′);

(ii) the structural arrangement of the eigenvalues of problem (1.1), (1.2) and problem (1.1), (1.2a)–(1.2c), (1.2d′) for $\delta = 0$ and $\delta = \pi/2$.

For $c \neq 0$, we find a positive integer N from the inequality $\mu_{N-1} < -d/c \leq \mu_N$.

Theorem 2.2. *The eigenvalues of the boundary value problem* (1.1), (1.2) *form an infinitely increasing sequence* $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$; *moreover,* $\lambda_n > 0$ *for* $n \geq 3$. *The corresponding eigenfunctions* $y_1(x), y_2(x), \ldots, y_n(x), \ldots$ *have the following oscillation properties*:

(a) *if* $c = 0$, *then* $y_n(x)$ *has exactly* $n-1$ *simple zeros*;

(b) *if* $c \neq 0$, *then* $y_n(x)$ *has exactly* $n-1$ *simple zeros for* $n \leq N$ *and exactly* $n-2$ *simple zeros for* $n > N$ *in the interval* $(0,l)$.

Proof. By virtue of the properties of the function $f(\lambda)$ and the relations $y(l,\mu_n) = 0$, $n \in \mathbb{N}$, we have

$$\lim_{\lambda \to \mu_{n-1}+0} f(\lambda) = -\infty, \qquad \lim_{\lambda \to \mu_n - 0} f(\lambda) = +\infty;$$

moreover, the function $f(\lambda)$ takes each value in $(-\infty,+\infty)$ at a unique point in the interval (μ_{n-1},μ_n). For the function $g(\lambda) = (a\lambda+b)/(c\lambda+d)$, we have $g'(\lambda) = -\sigma/(c\lambda+d)^2$. Since $\sigma > 0$, it follows that the function $g(\lambda)$ with $c = 0$ is strictly decreasing on the interval $(-\infty,+\infty)$; if $c \neq 0$, then the function $g(\lambda)$ is strictly decreasing on each of the intervals $(-\infty,-d/c)$ and $(-d/c,+\infty)$; moreover,

$$\lim_{\lambda \to -d/c-0} g(\lambda) = -\infty, \qquad \lim_{\lambda \to -d/c+0} g(\lambda) = +\infty.$$

Suppose that either $c = 0$ or $c \neq 0$ and $-d/c \notin (\mu_{n-1},\mu_n]$. It follows from the preceding considerations that in the interval (μ_{n-1},μ_n), $n \in \mathbb{N}$, there exists a unique $\lambda = \lambda_n^*$ such that

$$f(\lambda) = g(\lambda), \tag{2.7}$$

i.e., condition (1.2d) is satisfied. Therefore, λ_n^* is an eigenvalue of the boundary value problem (1.1), (1.2), and $y(x,\lambda_n^*)$ is the corresponding eigenfunction. It follows from Lemma 2.3 that $m(\lambda_n^*) = n-1$. One can readily see that if either $c = 0$ or $c \neq 0$ and $n < N$, then λ_n^* is the nth eigenvalue of the boundary value problem (1.1), (1.2); i.e., $\lambda_n = \lambda_n^*$.

Let $c \neq 0$ and $-d/c \in (\mu_{N-1},\mu_N)$. In a similar way, one can show that in each of the intervals $(\mu_{N-1},-d/c)$ and $(-d/c,\mu_N)$, there exists a unique value (λ_N and λ_{N+1}, respectively) such that relation (2.7) is valid. By Lemma 2.3, we have $m(\lambda_N) = m(\lambda_{N+1}) = N-1$.

The case in which $c \neq 0$ and $-d/c = \mu_N$ can be considered in a similar way; here one uses the fact that μ_N is also an eigenvalue of the boundary value problem (1.1), (1.2). In this case, we have $\lambda_N \in (\mu_{N-1},\mu_N)$ and $\lambda_{N+1} = \mu_N$. By Lemma 2.3,

$$m(\lambda_N) = m(\lambda_{N+1}) = N-1 \qquad [m(\lambda_{N+1}) = (N+1)-2].$$

Note that if $c \neq 0$ and $n > N$, then the unique solution λ_n^* of Eq. (2.7) in the half-open interval $(\mu_{n-1},\mu_n]$ is the $(n+1)$st eigenvalue of the boundary value problem (1.1), (1.2), i.e., $\lambda_{n+1} = \lambda_n^*$. Since $m(\lambda_n^*) = n-1$, we have $m(\lambda_{n+1}) = n-1 = (n+1)-2$.

It follows from the preceding considerations that $\lambda_n > \mu_1$ for $n \geq 3$. Consequently, by Theorem 2.1, $\lambda_n > 0$ for $n \geq 3$. The proof of Theorem 2.2 is complete.

Theorem 2.3. *The following relations are valid for sufficiently large* $n \in \mathbb{N}$:

$$\mu_{n-1} < \lambda_n < \nu_n < \mu_n \quad \text{if} \quad c = 0, \tag{2.8}$$

$$\mu_{n-2} < \lambda_n < \nu_{n-1} < \mu_{n-1} \quad \text{if} \quad c \neq 0 \quad \text{and} \quad a/c < 0, \tag{2.9}$$

$$\mu_{n-2} < \nu_{n-1} < \lambda_n < \mu_{n-1} \quad \text{if} \quad c \neq 0 \quad \text{and} \quad a/c \geq 0. \tag{2.10}$$

Proof. The eigenvalues ν_n, $n \in \mathbb{N}$, of the boundary value problem (1.1), (1.2a)–(1.2c), (1.2d′) for $\delta = \pi/2$ are zeros of the function $f(\lambda)$. In a similar way (see the proof of Theorem 2.2), one can show that the equation $f(\lambda) = 0$ has the unique solution $\nu_n = \mu_n(\pi/2)$ in each interval (μ_{n-1}, μ_n). Consequently,

$$\mu_{n-1} < \nu_n < \mu_n, \qquad n \in \mathbb{N}. \tag{2.11}$$

By (1.3), for sufficiently large λ, we have the inequalities

$$\frac{a\lambda + b}{c\lambda + d} < 0 \quad (\text{if} \quad a/c < 0), \qquad \frac{a\lambda + b}{c\lambda + d} > 0 \quad (\text{if} \quad a/c \geq 0).$$

Moreover, Theorem 2.1, together with the properties of the function $f(\lambda)$, implies that

$$\frac{Ty(l,\lambda)}{y(l,\lambda)} < 0 \quad (\text{if} \quad \mu_{n-1} < \lambda < \nu_n), \qquad \frac{Ty(l,\lambda)}{y(l,\lambda)} > 0 \quad (\text{if} \quad \nu_n < \lambda < \mu_n).$$

The validity of relations (2.8)–(2.10) follows from Theorem 2.2 and from the last four inequalities. The proof of Theorem 2.3 is complete.

3. ASYMPTOTIC FORMULAS FOR THE EIGENVALUES AND EIGENFUNCTIONS OF THE BOUNDARY VALUE PROBLEMS (1.1), (1.2a)–(1.2c), (1.2d′) AND (1.1), (1.2)

For brevity, we introduce the notation $s(\delta_1, \delta_2) \equiv \operatorname{sgn}\delta_1 + \operatorname{sgn}\delta_2$. We define numbers τ, ν, α_n, and β_n ($n \in \mathbb{N}$) and a function $\varphi(x,t)$, $x \in [0,l]$, $t \in \mathbb{R}$, as follows:

$$\tau = \begin{cases} 3(1 + s(\beta,\delta))/4 - 1 & \text{if} \quad s(\alpha,\gamma) = 1 \\ (5 + 2\operatorname{sgn}\alpha)/4 + (-1)^{\operatorname{sgn}\alpha}\left((-1)^{\operatorname{sgn}\beta} + (-1)^{\operatorname{sgn}\delta}\right) & \\ \qquad \times(6\operatorname{sgn}\alpha - 3)/8 - 1 & \text{if} \quad s(\alpha,\gamma) \neq 1, \end{cases}$$

$$\nu = \begin{cases} 3(1 + s(\beta,|c|))/4 & \text{if} \quad s(\alpha,\gamma) = 1 \\ (5 + 2\operatorname{sgn}\alpha)/4 + (-1)^{\operatorname{sgn}\alpha}\left((-1)^{\operatorname{sgn}\beta} + (-1)^{\operatorname{sgn}|c|}\right) & \\ \qquad \times(6\operatorname{sgn}\alpha - 3)/8 & \text{if} \quad s(\alpha,\gamma) \neq 1, \end{cases}$$

$$\alpha_n = (n - \tau)\pi/l, \qquad \beta_n = (n - \nu)\pi/l,$$

$$\varphi(x,t) = \begin{cases} \sin\left(tx + \dfrac{\pi}{2}\operatorname{sgn}\beta\right) - \cos\left(tl + \dfrac{\pi}{2}s(\beta,\gamma)\right)e^{-t(l-x)} & \text{if} \quad s(\alpha,\beta) = 1 \\ \sin tx - \cos tx + (-1)^{\operatorname{sgn}\alpha}e^{-tx} & \\ \qquad +(-1)^{1-\operatorname{sgn}\gamma}\sqrt{2}\sin\left(tl + (-1)^{\operatorname{sgn}\gamma}\pi/4\right)e^{-t(l-x)} & \text{if} \quad s(\alpha,\beta) \neq 1. \end{cases}$$

Theorem 3.1. *One has the asymptotic formulas*

$$\sqrt[4]{\mu_n(\delta)} = \alpha_n + O(1/n), \tag{3.1}$$

$$v_n^{(\delta)}(x) = \varphi(x,\alpha_n) + O(1/n), \tag{3.2}$$

$$\sqrt[4]{\lambda_n} = \beta_n + O(1/n), \tag{3.3}$$

$$y_n(x) = \varphi(x,\beta_n) + O(1/n). \tag{3.4}$$

Proof. We set $\lambda = \varrho^4$ in Eq. (1.1). By Theorem 1 in [20, p. 58], Eq. (1.1) has four linearly independent solutions $z_k(x,\varrho)$ ($k = 1,\dots,4$), which are regular with respect to ϱ (for sufficiently large ϱ) and satisfy the relations

$$z_k^{(s)}(x,\varrho) = (\varrho\omega_k)^s e^{\varrho\omega_k x}[1 + O(1/\varrho)], \qquad k = 1,\dots,4, \qquad s = 0,\dots,3, \tag{3.5}$$

where ω_k ($k = 1,\dots,4$) are distinct fourth roots of unity.

We set

$$\tau_n = \sqrt[4]{\mu_n(0)} = \sqrt[4]{\mu_n}, \qquad \sigma_n = \sqrt[4]{\mu_n(\delta)} \qquad (\delta \in (0,\pi/2]), \qquad \varrho_n = \sqrt[4]{\lambda_n}.$$

The boundary conditions (1.2a)–(1.2c), (1.2d′) are strongly regular (see [20, p. 71]).

Let $\alpha \in (0,\pi/2]$, $\beta \in (0,\pi/2]$, and $\gamma \in (0,\pi/2]$ in the boundary conditions (1.2a)–(1.2c). By Theorem 2 in [20, p. 74], for sufficiently large indices k, we have

$$\tau_{k+m_0} = (k+1/4)\pi/l + O(1/k), \tag{3.6}$$
$$\sigma_{k+m_1} = (k+1/2)\pi/l + O(1/k), \tag{3.7}$$

where m_0 and m_1 are some given integers. It follows from relations (3.6), (3.7), and (2.8) and Property 1 in [19, Sec. 4] that $m_1 = m_0 + 1$.

By taking into account relations (3.5) in the boundary conditions (1.2), for sufficiently large k, we obtain

$$\begin{aligned} \varrho_{k+m_2} &= \left(k+\frac{1}{4}\right)\frac{\pi}{l} + O\left(\frac{1}{k}\right) \quad \text{if} \quad c=0, \\ \varrho_{k+m_3} &= \left(k+\frac{1}{2}\right)\frac{\pi}{l} + O\left(\frac{1}{k}\right) \quad \text{if} \quad c\neq 0, \end{aligned} \tag{3.8}$$

where m_2 and m_3 are some fixed integers. It follows from (2.8)–(2.10) and (3.6)–(3.8) that $m_2 = m_0 + 1$ and $m_3 = m_0 + 2$.

Thus, for sufficiently large k, we have the formulas

$$\sigma_{k+m_0+1} = (k+1/2)\pi/l + O(1/k), \tag{3.9}$$
$$\varrho_{k+m_0+1} = \begin{cases} (k+1/4)\pi/l + O(1/k) & \text{if} \quad c=0 \\ (k-1/2)\pi/l + O(1/k) & \text{if} \quad c\neq 0. \end{cases} \tag{3.10}$$

Further, by taking into account relations (3.5), (3.6), (3.9), and (3.10), we obtain the asymptotic formulas [20, pp. 84–87]

$$\begin{aligned} \left(v^{(0)}_{k+m_0}(x)\right)^{(s)} &= \left(\left(k+\frac{1}{4}\right)\frac{\pi}{l}\right)^s \left\{\sin\left[\left(k+\frac{1}{4}\right)\frac{\pi}{l}x + \frac{\pi}{2}s\right]\right. \\ &\quad \left. -\cos\left[\left(k+\frac{1}{4}\right)\frac{\pi}{l}x + \frac{\pi}{2}s\right] + (-1)^{s+1}e^{-(k+1/4)\pi x/l}\right\} + O(1/k), \end{aligned} \tag{3.11}$$

$$\begin{aligned} v^{(\delta)}_{k+m_0+1}(x) &= \sin\left(k+\frac{1}{2}\right)\frac{\pi}{l}x - \cos\left(k+\frac{1}{2}\right)\frac{\pi}{l}x - e^{-(k+1/2)\pi x/l} \\ &\quad + (-1)^{k+m_0+1}e^{-(k+1/2)\pi(l-x)/l} + O(1/k) \qquad (\delta\in(0,\pi/2]), \end{aligned} \tag{3.12}$$

$$y_{k+m_0+1}(x) = \begin{cases} \sin(k+1/4)\pi x/l - \cos(k+1/4)\pi x/l \\ \qquad -e^{-(k+1/4)\pi x/l} + O(1/k) & \text{if} \quad c=0 \\ \sin(k-1/2)\pi x/l - \cos(k-1/2)\pi x/l \\ \qquad -e^{-(k-1/2)\pi x/l} + (-1)^{k+m_0}e^{-(k-1/2)\pi(l-x)/l} + O(1/k) & \text{if} \quad c\neq 0. \end{cases} \tag{3.13}$$

Let us find m_0. Let $k = 2m$, where m is a sufficiently large fixed positive integer. Consider the function

$$v^{(0)}_{2m+m_0}(x) = \sin\left(2m+\frac{1}{4}\right)\frac{\pi}{l}x - \cos\left(2m+\frac{1}{4}\right)\frac{\pi}{l}x - e^{-(2m+1/4)\pi x/l} + O\left(\frac{1}{m}\right).$$

We introduce the notation $t = (2m+1/4)\pi x/l$, $x\in[0,l]$; then

$$x = tl/((2m+1/4)\pi), \qquad t\in[0,(2m+1/4)\pi].$$

We set

$$\varphi(t) = v^{(0)}_{2m+m_0}\left(\frac{tl}{(2m+1/4)\pi}\right); \tag{3.14}$$

then

$$\varphi(t) = \sin t - \cos t - e^{-t} + O(1/m). \tag{3.15}$$

By taking into account (3.11), we obtain

$$\varphi'(t) = \cos t + \sin t + e^{-t} + O(1/m). \tag{3.16}$$

Let $t \in [2r\pi, 2(r+1)\pi]$, $r = 0, 1, \ldots, m-1$. We fix a number r and set $\xi = t - 2r\pi$, whence it follows that $t = \xi + 2r\pi$, $0 \le \xi \le 2\pi$. We set

$$\psi(\xi) = \varphi(\xi + 2r\pi) = \sin\xi - \cos\xi - e^{-(\xi+2r\pi)} + O(1/m). \tag{3.17}$$

It follows from (3.17) and (3.16) that

$$\psi'(\xi) = \cos\xi + \sin\xi + e^{-(\xi+2r\pi)} + O(1/m). \tag{3.18}$$

By taking into account (3.17) and (3.18), we obtain the relations $\psi(0) < 0$, $\psi(\pi/2) > 0$, $\psi(\pi) > 0$, $\psi(3\pi/2) < 0$, $\psi(2\pi) < 0$, $\psi'(\xi) > 0$ for $\xi \in (0, \pi/2)$, $\psi(\xi) > 0$ for $\xi \in (\pi/2, \pi)$, $\psi'(\xi) < 0$ for $\xi \in (\pi, 3\pi/2)$, and $\psi(\xi) < 0$ for $\xi \in (3\pi/2, 2\pi)$. Hence it follows that the function $\psi(\xi)$ has exactly two zeros in the interval $[0, 2\pi)$; consequently, the function $\varphi(t)$ has exactly $2m$ zeros in the interval $(0, 2m\pi)$.

Let $t \in [2m\pi, 2m\pi + \pi/4)$. By virtue of (3.15) and (3.16), $\varphi(2m\pi) < 0$, $\varphi(2m\pi + \pi/4) = 0$, and $\varphi'(t) > 0$. Consequently, the function $\varphi(t)$ has no zeros in the interval $[2m\pi, 2m\pi + \pi/4)$.

By virtue of the preceding considerations, we find that the function $\varphi(t)$ has exactly $2m$ zeros in the interval $(0, (2m + 1/4)\pi)$. Consequently, by the relation (3.14), the function $v^{(0)}_{2m+m_0}(x)$ has exactly $2m$ zeros in the interval $(0, l)$. Then, by Theorem 2.2, the function $v^{(0)}_{2m+m_0}(x)$ corresponds to the eigenvalue μ_{2m+1} of the boundary value problem (1.1), (1.2a)–(1.2c), (1.2d′) for $\delta = 0$. This implies that $m_0 = 1$.

By setting $m_0 = 1$ in (3.6), (3.9)–(3.13), we obtain the assertion of Theorem 3.1 for α, β, γ belonging to $(0, \pi/2]$. The remaining cases can be considered in a similar way. The proof of Theorem 3.1 is complete.

4. ON THE MINIMALITY OF THE SYSTEM OF EIGENFUNCTIONS OF THE BOUNDARY VALUE PROBLEM (1.1), (1.2)

Theorem 4.1. *Let s be an arbitrary fixed positive integer. Then the system $\{y_n(x)\}$ ($n = 1, 2, \ldots$; $n \ne s$) is minimal in the space $L_p(0, l)$ ($1 < p < \infty$).*

Proof. It suffices to show that there exists a system $\{u_n(x)\}$ ($n = 1, 2, \ldots$; $n \ne s$) biorthogonal to the system $\{y_n(x)\}$ ($n = 1, 2, \ldots$; $n \ne s$).

By virtue of (2.2), for arbitrary positive integers n and m, we have

$$y_m(l)Ty_n(l) - y_n(l)Ty_m(l) = (\lambda_n - \lambda_m)(y_n, y_m), \tag{4.1}$$

where

$$(y_n, y_m) = \int_0^l y_n(x) y_m(x)\,dx.$$

Consider the case in which $c = 0$. It follows from (4.1) and (1.2d) that

$$(y_n, y_m) = (a/d) y_n(l) y_m(l), \qquad m \ne n. \tag{4.2}$$

We define elements of the system $\{u_n(x)\}$ ($n = 1, 2, \ldots$; $n \ne s$) by the relation

$$u_n(x) = \left[y_n(x) - \frac{y_n(l)}{y_s(l)} y_s(x) \right] \Big/ \left[\|y_n\|_2^2 - \frac{a}{d} y_n^2(l) \right], \tag{4.3}$$

where $\|\cdot\|_p$ stands for the norm in the space $L_p(0,l)$. By (4.2), we have

$$(u_n, y_m) = \delta_{n,m}, \tag{4.4}$$

where $\delta_{n,m}$ is the Kronecker delta.

Now consider the case in which $c \neq 0$. Let $\lambda_n \neq -d/c$ and $\lambda_m \neq -d/c$. Then, by (4.1) and (1.2d), we have

$$y_m(l)Ty_n(l) - y_n(l)Ty_m(l) = -\frac{\sigma(\lambda_n - \lambda_m)\, y_n(l)y_m(l)}{(c\lambda_n + d)(c\lambda_m + d)}. \tag{4.5}$$

Suppose that $\lambda_{N+1} = -d/c$. By (1.2d), we have $y_{N+1}(l) = 0$. From (4.5) with $m \neq N+1$, we obtain

$$y_m(l)Ty_{N+1}(l) - y_{N+1}(l)Ty_m(l) = y_m(l)Ty_{N+1}(l). \tag{4.6}$$

By comparing relations (4.1), (4.5), and (4.6), we find that

$$(y_n, y_m) = \begin{cases} -\dfrac{\sigma y_n(l)y_m(l)}{(c\lambda_n + d)(c\lambda_m + d)} & \text{if } \lambda_n \neq -\dfrac{d}{c},\ \lambda_m \neq -\dfrac{d}{c} \\ -\dfrac{c y_m(l)Ty_{N+1}(l)}{c\lambda_m + d} & \text{if } n = N+1,\ \lambda_{N+1} = -\dfrac{d}{c} \end{cases} \tag{4.7}$$

for $m \neq n$.

Let $\lambda_{N+1} \neq -d/c$. Obviously, $\lambda_n \neq -d/c$ for all $n = 1, 2, \ldots$ We define the elements of the system $\{u_n(x)\}$ $(n = 1, 2, \ldots;\ n \neq s)$ by the relation

$$u_n(x) = \left[y_n(x) - \frac{(c\lambda_s + d)\, y_n(l)}{(c\lambda_n + d)\, y_s(l)} y_s(x)\right] \Big/ \left[\|y\|_2^2 + \frac{\sigma y_n^2(l)}{(c\lambda_n + d)^2}\right]. \tag{4.8}$$

By using (4.7), one can readily justify relation (4.4).

Let $\lambda_{N+1} = -d/c$. We define the elements of the system $\{u_n(x)\}$ $(n = 1, 2, \ldots;\ n \neq s)$ by the relation

$$u_n(x) = \left[y_n(x) - \frac{\sigma y_n(l)}{c\,(c\lambda_n + d)\, Ty_{N+1}(l)} y_{N+1}(x)\right] \Big/ \left[\|y\|_2^2 + \frac{\sigma y_n^2(l)}{(c\lambda_n + d)^2}\right] \tag{4.9}$$

for $s = N+1$, relation (4.8) for $s \neq N+1$, $n \neq N+1$, and the relation

$$u_n(x) = \left[y_{N+1}(x) - \frac{c\,(c\lambda_s + d)\, Ty_{N+1}(l)}{\sigma y_s(l)} y_s(x)\right] \Big/ \left[\|y_{N+1}\|_2^2 + \frac{c^2\,(Ty_{N+1}(l))^2}{\sigma}\right]$$

for $s \neq N+1$, $n = N+1$; in this case, one can readily see that relation (4.4) is valid. The proof of Theorem 4.1 is complete.

Lemma 4.1. *One has the asymptotic formula*

$$u_n(x) = l^{-1} y_n(x) + O(1/n). \tag{4.10}$$

Proof. By (3.4), we have the relations

$$\|y_n\|_2^2 = l + O(1/n), \qquad y_n(l) = \begin{cases} O(1/n) & \text{if } c = 0 \\ a_n + O(1/n) & \text{if } c \neq 0, \end{cases} \tag{4.11}$$

where $1 \leq |a_n| \leq 2$.

By taking into account (3.3) and (4.11), from (4.3), (4.8), and (4.9), we obtain the representation (4.10). The proof of Lemma 4.1 is complete.

5. THE BASIS PROPERTY OF THE SYSTEM OF EIGENFUNCTIONS OF THE BOUNDARY VALUE PROBLEM (1.1), (1.2) IN $L_p(0,l)$ $(1<p<\infty)$

Theorem 5.1. *Let s be an arbitrary fixed positive integer. Then the system $\{y_n(x)\}$ $(n=1,2,\ldots;\ n\neq s)$ is a basis in the space $L_p(0,l)$ $(1<p<\infty)$; moreover, if $p=2$, then this basis is unconditional.*

Proof. Recall that the boundary conditions (1.2a)–(1.2c), (1.2d′) are strongly regular. Then, by Theorem 5.1 in [21] (see also [22]), the system of eigenfunctions $\{v_n^{(\delta)}(x)\}_{n=1}^{\infty}$ of problem (1.1), (1.2a)–(1.2c), (1.2d′) is a basis in the space $L_p(0,l)$ $(1<p<\infty)$; moreover, if $p=2$, then that basis is unconditional.

Let $c=0$. We compare the system $\{y_n(x)\}$ $(n=1,2,\ldots;\ n\neq s)$ with the system $\{v_n^{(0)}(x)\}_{n=1}^{\infty}$. By Theorem 3.1, we have the inequality

$$\left\|y_{n+1}(x)-v_n^{(0)}(x)\right\|_2\leq \text{const}\times n^{-1}$$

for all sufficiently large n, which implies the convergence of the series

$$\sum_{n=1}^{s-1}\left\|y_n(x)-v_n^{(0)}(x)\right\|_2^2+\sum_{n=s}^{\infty}\left\|y_{n+1}(x)-v_n^{(0)}(x)\right\|_2^2$$

(if $s=1$, then the first sum is absent). Consequently, the system $\{y_n(x)\}$ $(n=1,2,\ldots;\ n\neq s)$ is quadratically close to the system $\{v_n^{(0)}(x)\}_{n=1}^{\infty}$. By Theorem 4.1, $\{y_n(x)\}$ $(n=1,2,\ldots;\ n\neq s)$ is a minimal system in $L_p(0,l)$ $(1<p<\infty)$. Then, by Theorem 9.9.8 in [23, p. 440], the system $\{y_n(x)\}$ $(n=1,2,\ldots;\ n\neq s)$ is an unconditional basis in $L_2(0,l)$.

The case in which $c\neq 0$ can be considered in a similar way; in this case, the system $\{y_n(x)\}$ $(n=1,2,\ldots;\ n\neq s)$ is compared with the system $\{v_n^{(\pi/2)}(x)\}_{n=1}^{\infty}$.

We set

$$\tilde v_n^{(\delta)}(x)=v_n^{(\delta)}(x)\left\|v_n^{(\delta)}(x)\right\|_2^{-1}\qquad (n=1,2,\ldots).$$

Since the boundary value problem (1.1), (1.2a)–(1.2c), (1.2d′) is self-adjoint, it follows from (3.2) that the systems $\{\tilde v_n^{(0)}(x)\}_{n=1}^{\infty}$ and $\{\tilde v_n^{(\pi/2)}(x)\}_{n=1}^{\infty}$ are uniformly bounded orthonormal bases in $L_2(0,l)$.

By using (3.1)–(3.4) and (4.10), one can readily show that

$$y_n(x)=l^{1/2}\tilde v_n(x)+O(1/n),\qquad u_n(x)=l^{-1/2}\tilde v_n(x)+O(1/n)\tag{5.1}$$

for $n\in\mathbb{N}$, $n\neq s$, where

$$\tilde v_n(x)=\begin{cases}\tilde v_n^{(0)}(x) & \text{if}\quad c=0 \text{ and } 1\leq n\leq s-1\\ \tilde v_{n-1}^{(0)}(x) & \text{if}\quad c=0 \text{ and } n\geq s+1\\ \tilde v_n^{(\pi/2)}(x) & \text{if}\quad c\neq 0 \text{ and } 1\leq n\leq s-1\\ \tilde v_n^{(\pi/2)}(x) & \text{if}\quad c\neq 0 \text{ and } n\geq s+1.\end{cases}$$

Note that the system $\{\tilde v_n(x)\}$ $(n=1,2,\ldots;\ n\neq s)$ is a uniformly bounded orthonormal basis in the space $L_2(0,l)$.

We fix $p\in(1,2)$. Since the system $\{y_n(x)\}$ $(n=1,2,\ldots;\ n\neq s)$ is a basis in $L_2(0,l)$, we find that it is complete in $L_p(0,l)$. The system $\{y_n(x)\}$ $(n=1,2,\ldots;\ n\neq s)$ is a basis in $L_p(0,l)$ if and only if

$$\left\|\sum_{n=1,\,n\neq s}^{k}(f,u_n)\,y_n\right\|_p\leq M_p\|f\|_p,\qquad k=1,2,\ldots,\tag{5.2}$$

for an arbitrary function $f(x)$ in $L_p(0,l)$, where M_p is a positive constant [24, p. 19].

By (5.1), we have

$$\left\| \sum_{n=1,\, n\neq s}^{k} (f, u_n)\, y_n \right\|_p \le \left\| \sum_{n=1,\, n\neq s}^{k} (f, \tilde{v}_n)\, \tilde{v}_n \right\|_p + \left\| \sum_{n=1,\, n\neq s}^{k} (f, \tilde{v}_n)\, O\left(\frac{1}{n}\right) \right\|_p$$
$$+ \left\| \sum_{n=1,\, n\neq s}^{k} \left(f, O\left(\frac{1}{n}\right)\right) \tilde{v}_n \right\|_p + \left\| \sum_{n=1,\, n\neq s}^{k} \left(f, O\left(\frac{1}{n}\right)\right) O\left(\frac{1}{n}\right) \right\|_p. \quad (5.3)$$

Since $\{\tilde{v}_n(x)\}$ $(n = 1, 2, \ldots;\ n \neq s)$ is an orthonormal system and is a basis in the space $L_p(0, l)$, we have the inequality

$$\left\| \sum_{n=1,\, n\neq s}^{k} (f, \tilde{v}_n)\, \tilde{v}_n \right\|_p \le \text{const} \times \|f\|_p, \quad (5.4)$$

where $f(x)$ is an arbitrary function in $L_p(0, l)$.

By Theorem 2.3 in [25, p. 154], the estimate

$$\left\| \sum_{n=1,\, n\neq s}^{k} (f, \tilde{v}_n)\, O\left(\frac{1}{n}\right) \right\|_p \le \text{const} \times \left(\sum_{n=1,\, n\neq s}^{k} |(f, \tilde{v}_n)|^q \right)^{1/q} \left(\sum_{n=1,\, n\neq s}^{k} \frac{1}{n^p} \right)^{1/p}$$
$$\le \text{const} \times \|f\|_p, \quad (5.5)$$

where $q = p(p-1)$, is valid for an arbitrary function $f(x) \in L_p(0, l)$. Next, we have

$$\left\| \sum_{n=1,\, n\neq s}^{k} \left(f, O\left(\frac{1}{n}\right)\right) \tilde{v}_n \right\|_p \le \text{const} \times \left\| \sum_{n=1,\, n\neq s}^{k} \left(f, O\left(\frac{1}{n}\right)\right) \tilde{v}_n \right\|_2$$
$$\le \text{const} \times \left(\sum_{n=1,\, n\neq s}^{k} \left| \left(f, O\left(\frac{1}{n}\right)\right) \right|^2 \right)^{1/2}$$
$$\le \text{const} \times \|f\|_1 \left(\sum_{n=1,\, n\neq s}^{k} \frac{1}{n^2} \right)^{1/2} \le \text{const} \times \|f\|_p, \quad (5.6)$$

$$\left\| \sum_{n=1,\, n\neq s}^{k} \left(f, O\left(\frac{1}{n}\right)\right) O\left(\frac{1}{n}\right) \right\|_p \le \text{const} \times \|f\|_1 \left(\sum_{n=1,\, n\neq s}^{k} \frac{1}{n^2} \right) \le \text{const} \times \|f\|_p. \quad (5.7)$$

Inequality (5.2) is a consequence of inequalities (5.3)–(5.7). This completes the proof of the basis property of the system $\{y_n(x)\}$ $(n = 1, 2, \ldots;\ n \neq s)$ in the space $L_p(0, l)$ for $1 < p < 2$.

Let $2 < p < +\infty$. Since the system $\{y_n(x)\}$ $(n = 1, 2, \ldots;\ n \neq s)$ is a basis in $L_q(0, l)$, it follows from Corollary 2 in [24, p. 20] that the system $\{u_n(x)\}$ $(n = 1, 2, \ldots;\ n \neq s)$ is a basis in the space $L_p(0, l)$. Consequently, $\{u_n(x)\}$ $(n = 1, 2, \ldots;\ n \neq s)$ is a complete system in the space $L_q(0, l)$. Further, by using the preceding considerations, one can use the basis property of the system $\{u_n(x)\}$ $(n = 1, 2, \ldots;\ n \neq s)$ in $L_q(0, l)$, which is equivalent to the basis property of the system $\{y_n(x)\}$ $(n = 1, 2, \ldots;\ n \neq s)$ in $L_p(0, l)$. The proof of Theorem 5.1 is complete.

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