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On the basicity of unitary system of exponents in the variable exponent Lebesgue spaces

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Abstract. *In the present work it is considered the system of functions* $a(t)\,e^{int} - b(t)\,e^{-int}$, $n \in N$, *with complex-valued coefficients* $a(\cdot)\,;b(\cdot) : [0,\pi] \to C$. *Sufficient conditions on the coefficients of the system are found in order for the system to form a basis in Lebesgue spaces with variable exponent.*

Keywords. exponential systems · basisness, variable exponents · generalized Lebesgue spaces

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1 Introduction

When solving the PDEs of mixed type by Fourier method there frequently appear systems of sines and cosines of the following form

$$\{\cos(n+\alpha)\,t\}_{n\in Z_+}\,, \tag{1.1}$$

$$\{\sin(n+\alpha)\,t\}_{n\in N}\,, \tag{1.2}$$

where α is a real number(here, thereafter$N$ is the set of all natural numbers, $Z_+ = \{0\} \bigcup N$). Justification of the Fourier method requires to study the basicity properties of such systems in some function spaces. Some examples of such equations and concrete systems of trigonometric-type functions that appear after applying Fourier method can be found, for example, in [22–24,28]. The basicity properties of the systems (1.1) and (1.2) are well studied in Lebesgue and Sobolev spaces, as well as, in their weighted settings [3–9,20,21,25, 26,29,30].

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During the last two decades, non-standard function spaces became an extremely popular subject because of their appearance in modern problems of analysis and qualitative theory of PDEs. Introduction of Lebesgue spaces with variable exponents at the end of last century and variety of extraordinary results obtained therein were the main motivation and the inception of this new tendency in analysis. We only mention the monograph [15] and the comprehensive bibliography therein, where thoroughly treatment of these issues can be found.

In the present work it is considered the system of functions

$$a\,(t)\,e^{int} - b\,(t)\,e^{-int}\,, n \in N, \tag{1.3}$$

with omplex-valued coefficients $a\,(\cdot)\,; b\,(\cdot)\;:\;[0,\pi]\;\to\;C$. The system of perturbed exponential functions, which is the sytem of eigenfunctions of some second order ordinary differential operator is a special case of the sytem (1.3). Sufficient conditions on the coefficients of the system (1.3) are found in order for the system to form a basis in Lebesgue spaces with variable exponent.

Notice that, similar problems for the double system of exponents with complex-valued coefficients in Lebesgue spaces with variable exponent were earlier studied in [11–14, 27]. The basicity properties of the systems (1.1) and (1.2) in classical Lebesgue spaces were studied in [20,24].

2 Preliminaries

We use the following standard denotations: $Z-$the set of all integers; $R-$the set of all real numbers; $C-$complex plane; $(\bar{\cdot})-$complex conjugate of $(\,.\,)$; $\delta_{nk}-$Kronecker delta; $\chi_A\,(\cdot)\,-$the indicator function of the set A . $\omega \equiv \{z \in C : |z| < 1\}-$ the unit disc; $\partial\omega \equiv \{z \in C : \; |z| = 1\}-$ the unit circle.

Let $p : [-\pi, \pi] \to [1, +\infty)\,-$be a Lebesgue measurable function. We denote by $\mathscr{L}_0$ the set of all Lebesgue measurable functions on $[-\pi, \pi]$. Set

$$I_p\,(f) \overset{def}{\equiv} \int_{-\pi}^{\pi} |f\,(t)|^{p(t)}\,dt$$

and

$$\mathscr{L} \equiv \{f \in \mathscr{L}_0 : I_p\,(f) < +\infty\}\,.$$

If $p^+ = \sup\underset{[-\pi,\pi]}{vrai}\,p\,(t) < +\infty$, then $\mathscr{L}$ is a linear space with respect to pointwise linear operations. $\mathscr{L}$ is a Banach space with respect to the norm

$$\|f\|_{p(\cdot)} \overset{def}{\equiv} \inf\left\{\lambda > 0 : I_p\left(\frac{f}{\lambda}\right) \le 1\right\},$$

and we denote it by $L_{p(\cdot)}$. Set

$$WL \overset{def}{\equiv} \{p : p(-\pi) = p(\pi); \exists C > 0, \quad \forall t_1, t_2 \in [-\pi, \pi] : |t_1 - t_2| \le \tfrac{1}{2} \Rightarrow$$
$$\Rightarrow |p\,(t_1) - p\,(t_2)| \le \tfrac{C}{-\ln|t_1 - t_2|}\right\}.$$

Throughout the paper $q\,(\cdot)$ denotes the conjugate function of $p\,(\cdot)$, that is, $\frac{1}{p(t)} + \frac{1}{q(t)} \equiv 1$. Let $p^- = \inf\underset{[-\pi,\pi]}{vrai}\,p\,(t)$. The following generalized Hölder's inequality holds

$$\int_{-\pi}^{\pi} |f(t)\, g(t)|\, dt \le c\left(p^-; p^+\right) \|f\|_{p(\cdot)} \|g\|_{q(\cdot)},$$

where $c\left(p^-; p^+\right) = 1 + \frac{1}{p^-} - \frac{1}{p^+}$.

The following fact plays a key role in obtaining main results of paper.

Proposition 2.1 **([15])** *If $p(\cdot) : 1 < p^- \le p^+ < +\infty$, then the set $C_0^\infty(-\pi, \pi)$ (the set of all infinitely differentiable functions with compact support) is dense in $L_{p(\cdot)}$.*

Let

$$\rho(t) = \prod_{k=1}^{m} |t - t_k|^{\alpha_k},$$

be a weight function, where $\{t_k\}_1^m \subset [-\pi, \pi]$ is a set of disjoint points.

Lemma 2.1 *Let $p \in C[-\pi, \pi]$ and $p(t) > 0,\ \forall t \in [-\pi, \pi]$. Then the function $\xi(t) = |t - c|^\alpha$ belongs to $L_{p(\cdot),\rho}$, if $\alpha > -\frac{1}{p(c)}$, and $c \ne \tau_k,\ \forall\, k = \overline{1, m}$, as well as if $\alpha + \alpha_{k_0} > -\frac{1}{p(c)}$, and $c = \tau_{k_0}$.*

We will need some facts from the theory of Hardy spaces. By $H_{p_0}^+$ we denote the usual Hardy space, where $p_0 \in [1, +\infty)$. Let $\rho : \partial\omega \to R_+ = (0, +\infty)$ be a weight function. If the weight function $\rho(\cdot)$ is defined on $(-\pi, \pi)$, then $\rho\left(e^{it}\right)$ stresses $\rho\left(e^{it}\right) \equiv \rho(t)$, $t \in (-\pi, \pi)$. Set $H_{p(\cdot),\rho}^{\pm} \equiv \left\{f \in H_1^+ : f^+ \in L_{p(\cdot),\rho}(\partial\omega)\right\}$, where f^+- is the nontangential boundary values of f on $\partial\omega$. In [10] the following was proved:

Theorem 2.1 *Let $p \in WL, p^- > 1$, and*

$$-\frac{1}{p(t_k)} < \alpha_k < \frac{1}{q(t_k)},\ k = \overline{1, m}, \tag{2.1}$$

where $\rho(t) = \prod_{k=1}^{m} |t - t_k|^{\alpha_k}$. If $F \in H_{p(\cdot),\rho}^+$, then $\exists f \in L_{p(\cdot),\rho}$:

$$F(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} K_z(t) f(t) dt, \tag{2.2}$$

here $K_z(t) \equiv \frac{1}{1 - z e^{-it}}$ is the Cauchy kernel. Conversely, if $f \in L_{p(\cdot),\rho}$, then the function F, defined by (2.2)*, belongs to $H_{p(\cdot),\rho}^+$.*

Now, define the set ${}_mH_{p(\cdot)}^-$, where $m \in Z$. Let $\Phi(z)$ be a function, analytic outside ω, which does not have an essential singularity at infinity. In other words, let

$$\Phi(z) = \Phi_0(z) + \Phi_1(z) = \sum_{n=-\infty}^{k} a_n z^n \ ,\quad k < +\infty,$$

for sufficiently large $|z|$, here $\Phi_0(z)$ and $\Phi_1(z)$ are principle and analytic parts of above series, respectively. If $k \le m$ and $\overline{\Phi_0\left(\frac{1}{\bar z}\right)}$ belongs to $H_{p(\cdot)}^+$, then we will say that $\Phi(\cdot)$ belongs to the class ${}_mH_{p(\cdot)}^-$ outside ω.

The following theorems can be proved by the same way as in the classical case.

Theorem 2.2 *Let $p \in WL, p^- > 1$,and* (2.1) *fulfils. If $f \in H^+_{p(\cdot),\rho}$, then*

$$\left\| f(re^{it}) - f^+(e^{it}) \right\|_{p(\cdot),\rho} \to 0, \ r \to 1-0,$$

where $f^+$is the interior nontangential boundary values of f on $\partial\omega$.

Theorem 2.3 *Let $p \in WL, p^- > 1$, and* (2.1) *fulfils. If $f \in {}_mH^-_{p(\cdot),\rho}$, then*

$$\left\| f(re^{it}) - f^-(e^{it}) \right\|_{p(\cdot),\rho} \to 0, \ r \to 1+0,$$

where f^- is the exterior nontangential boundary values of f on $\partial\omega$.

Subject the coefficients $a(\cdot)$ and $b(\cdot)$ of the system (1.3) to the following conditions:
i) $a^{\pm 1}(\cdot)\ ;\ b^{\pm 1}(\cdot) \in L_\infty(0,\pi)$;
ii) $\alpha(\cdot)\ ;\ \beta(\cdot)$ are piecewise continuous functions on $(0,\pi)$ with jump points $\{t_k\}_{k\in N}$ and $\{\tau_k\}_{k\in N}$, respectively; additionally it is assumed that the set $\{\tilde{s}_k\} \equiv \{t_k\} \bigcup \{\tau_k\}$ may have $\tilde{s}_0 \in (0,\pi)$ as its only limit point and the function $\tilde{\theta}(t) \equiv \beta(t) - \alpha(t)$ has a finite right and left limits at $\tilde{s}_0$;
iii) $\sum_{k=1}^{\infty} |h(\tilde{s}_k)| < +\infty$, where $h(\tilde{s}_k) = \tilde{\theta}(\tilde{s}_k - 0) - \tilde{\theta}(\tilde{s}_k + 0)$ is the jump of $\tilde{\theta}(\cdot)$ at $\tilde{s}_k$;
iv) The jumps $\left\{\tilde{h}_i\right\}$ hold $\left(\frac{\tilde{h}(\tilde{s}_i)}{2\pi} + \frac{1}{p(\tilde{s}_i)}\right) \notin Z, \forall i \in N$.

3 Main results

Consider the system

$$v_n(t) \equiv a(t)e^{int} - b(t)e^{-int}, \ \ n \in N,$$

where $a(t) = |a(t)|\, e^{i\alpha(t)}$, $b(t) = |b(t)|\, e^{i\beta(t)}$ are some complex-valued functions on $[0,\pi]$. Suppose that the coefficients $a(t)$ and $b(t)$ satisfy the conditions i)-iv). From iii) it follows that there is $r \in N$, such that

$$-\frac{2\pi}{p(\tilde{s}_k)} < \tilde{h}(\tilde{s}_k) < \frac{2\pi}{q(\tilde{s}_k)}, k = \overline{r,\infty}.$$

Enumerate the set$\{\tilde{s}_i\}_1^r$ from least to greatest and denote it by$\{s_i\}_1^r$: $0 < s_1 < ... < s_r < \pi$. Denote the corresponding jump sequence (finite) by $\{h(s_i)\}_1^r$:

$$h(s_i) = \beta(s_i - 0) - \beta(s_i + 0) + \alpha(s_i + 0) - \alpha(s_i - 0), i = \overline{1,r}.$$

Suppose that for some n_0

$$\frac{1}{p(0)} + 2(n_0 - 1) < \frac{\beta(0) - \alpha(0)}{\pi} < \frac{1}{p(0)} + 2n_0, \tag{3.1}$$

holds. Taking into account the condition iv) we define the integers $n_i, i = \overline{1,r}$ by the inequalities

$$-\frac{1}{p(s_i)} < \frac{h(s_i)}{2\pi} + n_i - n_{i-1} < \frac{1}{q(s_i)}, i = \overline{1,r}. \tag{3.2}$$

The main result of the paper is

Theorem 3.1 *Let the coefficients* $a(\cdot)$ *and* $b(\cdot)$ *of the system* $\{v_n\}_{n\in N}$ *satisfy the conditions i)-iv), the integers* $\{n_i\}_1^r$ *are defined by* (3.1) *and*(3.2)*. Suppose that,*

$$\frac{\beta(\pi)-\alpha(\pi)}{2\pi}+\frac{1}{2p(\pi)}\notin Z. \tag{3.3}$$

If

$$-\frac{1}{p(\pi)}+2n_r<\frac{\beta(\pi)-\alpha(\pi)}{\pi}<-\frac{1}{p(\pi)}+2(n_r+1), \tag{3.4}$$

then the system $\{v_n\}_{n\in N}$ *forms a basis in* $L_{p(\cdot)}(0,\pi)$*;*
if

$$\beta(\pi)-\alpha(\pi)<-\frac{\pi}{p(\pi)}+2n_r\pi,$$

then the system $\{v_n\}_{n\in N}$ *is not complete, but is minimal in* $L_{p(\cdot)}(0,\pi)$*;*
if

$$\beta(\pi)-\alpha(\pi)>-\frac{\pi}{p(\pi)}+2(n_r+1)\pi,$$

the system is complete, but not minimal in $L_{p(\cdot)}(0,\pi)$ *.*

Proof. The proof is based on solution of the following linear conjugation problem for analytic functions in Hardy class $H^+_{p(\cdot)}\times {}_{-1}H^-_{p(\cdot)}$:

$$\left.\begin{array}{l} F^+(\tau)+G(\tau)F^-(\tau)=\psi(\arg\tau),\\ |\tau|=1, F(\infty)=0,\end{array}\right\} \tag{3.5}$$

where

$$G(e^{i\theta})=\begin{cases} b(\theta)\,a^{-1}(\theta), & 0<\theta<\pi,\\ a(-\theta)\,b^{-1}(-\theta), & -\pi<\theta<0,\end{cases}$$

$$\psi(\theta)=\begin{cases} \psi(\theta)\,a^{-1}(\theta), & 0<\theta<\pi,\\ -\psi(-\theta)\,b^{-1}(-\theta), & -\pi<\theta<0.\end{cases}$$

Hereafter $a^{-1}(\theta)$ and $b^{-1}(\theta)$ denotes the functions $a^{-1}(\theta)=|a(\theta)|^{-1}e^{-i\alpha(\theta)}$ and $b^{-1}(\theta)=|b(\theta)|^{-1}e^{-i\beta(\theta)}$, respectively, $\psi(\cdot)\in L_{p(\cdot)}(0,\pi)$ is an arbitrary function.

Consider the following function:

$$\gamma(\theta)=\begin{cases} \beta(\theta)-\alpha(\theta), & \theta\in(0,\pi),\\ \alpha(-\theta)-\beta(-\theta), & \theta\in(-\pi,0).\end{cases}$$

It is clear that, the points$\{-\tilde{s}_i\}_{i\in N}$ are also jump points for the function $\gamma(\theta)$ and the jump at the point $(-\tilde{s}_i)$ is equal to the jump of $(-h(\tilde{s}_i))$ at the point $\tilde{s}_i$. Denote the union of the sets $\{-\tilde{s}_i\}_{i\in N}$ and $\{\tilde{s}_i\}_{i\in N}$ by $\{\bar{s}_i\}$ and the corresponding jumps by $\{\bar{h}_i\}$. The point $s=0$ is also a jump point of $\gamma(\theta)$ and the jump at this point is

$$\bar{h}_0=\beta(0)-\alpha(0)-(\alpha(0)-\beta(0))=2(\beta(0)-\alpha(0)).$$

Add the point $s=0$ to the set $\{\bar{s}_i\}$ and $\bar{h}_0$ to the set $\{\bar{h}_i\}$. We construct the jump function $\gamma_1(\cdot)$ of the function $\gamma(\cdot)$:

$$\gamma_1(-\pi)=0, \gamma_1(s)=[\gamma(-\pi+0)-\gamma(-\pi)]+\sum_{-\pi<\bar{s}_k<s}\bar{h}_k$$

$$+[\gamma(s)-\gamma(s-0)], -\pi<s\le\pi, \tag{3.6}$$

where $\gamma(\bar{s}_k)$ denotes the left limit of the function $\gamma(s)$ at the point $\bar{s}_k$,

$$\beta(\pi)=\beta(\pi-0), \alpha(\pi)=\alpha(\pi-0), \beta(0)=\beta(+0), \alpha(0)=\alpha(+0).$$

In [18] it was proved the following

Lemma 3.1 *If iii) holds, then* $\gamma_0(\cdot)\in C[-\pi,\pi]$, *where* $\gamma_0(s)=\gamma(s)-\gamma_1(s)$, $\forall s\in[-\pi,\pi]$.

This lemma allows us to apply the method from [16] for solving the problem (3.5) in Hardy classes $H^+_{p(\cdot)}\times {}_mH^-_{p(\cdot)}$. First, we assume that

$$-\frac{2\pi}{p(\tilde{s}_i)}<h(\tilde{s}_i)<\frac{2\pi}{q(\tilde{s}_i)}, \forall i\in N \tag{3.7}$$

holds. By the scheme of [16], compute the quantities $h_0^{(1)}$ and $h_0^{(0)}$. We have

$$h_0^{(0)}=\gamma_0(\pi)-\gamma_0(-\pi)=\gamma(\pi-0)-\gamma_1(\pi-0)-(\gamma(-\pi+0)-\gamma_1(-\pi+0)),$$
$$h_0^{(1)}=\gamma_1(-\pi+0)-\gamma_1(\pi-0).$$

Hence,

$$h_0=h_0^{(1)}-h_0^{(0)}=\gamma(-\pi+0)-\gamma(-\pi-0)=\alpha(\pi)-\beta(\pi)-(\beta(\pi)-\alpha(\pi))=2(\alpha(\pi)-\beta(\pi)).$$

The jump of the function $\gamma_1(\cdot)$ at the point $\theta=0$ is

$$\bar{h}_0=\gamma(+0)-\gamma(-0)=2\gamma(+0)=2(\beta(0)-\alpha(0)).$$

First, assume that

$$-\frac{\pi}{p(0)}<\alpha(0)-\beta(0)<\frac{\pi}{q(0)}, \tag{3.8}$$

holds.
As it was shown in [27] under the conditions i)-iv) and (3.7), (3.8), the problem (3.5) has a unique solution in $H^+_{p(\cdot)}\times {}_{-1}H^-_{p(\cdot)}$ and this solution can be represented as

$$F(z)=\frac{Z_0(z)}{2\pi}\int_{-\pi}^{\pi}\frac{\psi(\sigma)}{Z_0^+(e^{i\sigma})}\frac{d\sigma}{1-ze^{-i\sigma}}, \tag{3.9}$$

where

$$Z_0(z)=\begin{cases} X_1(z)\,Y_1(z), & |z|<1, \\ -X_2^{-1}(z)\,Y_2^{-1}(z), & |z|>1, \end{cases}$$

is a canonical solution of the problem (3.5) belonging to $H^+_{p(\cdot)}\times {}_{-1}H^-_{p(\cdot)}$. Here the functions $X_k(\cdot)$ and $Y_k(\cdot)$, $k=1,2$ are defined as

$$X_1(z) = \exp\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi}\ln|G(s)|\,\frac{e^{is}+z}{e^{is}-z}ds\right\}, |z|<1,$$
$$X_2(z) = \exp\left\{-\frac{1}{4\pi}\int_{-\pi}^{\pi}\ln|G(s)|\,\frac{e^{is}+z}{e^{is}-z}ds\right\}, |z|>1,$$
$$Y_1(z) = \exp\left\{\frac{i}{4\pi}\int_{-\pi}^{\pi}\gamma(s)\frac{e^{is}+z}{e^{is}-z}ds\right\}, |z|<1,$$
$$Y_2(z) = \exp\left\{-\frac{i}{4\pi}\int_{-\pi}^{\pi}\gamma(s)\frac{e^{is}+z}{e^{is}-z}ds\right\}, |z|>1.$$

Divide the set $\{-h(\tilde{s}_k)\}$ into two subsets: the first consists all positives of $\{-h(\tilde{s}_k)\}$, we denote it by $\{h_k^+\}$; the second consists all negatives of $\{-h(\tilde{s}_k)\}$, we denote it by $\{-h_k^-\}$. To such a separation of $\{h_k\}$, associates a separation of the set $\{\tilde{s}_k\}$: the first subset of $\{\tilde{s}_k\}$ contains all $\{s_k^+\}$ of $\tilde{s}_k$, for which $(\,-h(\tilde{s}_k)\,)>0$; second subset contains all $\{s_k^-\}$ for which $(\,-h(\tilde{s}_k)\,)<0$. Set

$$U_1^{\pm}(\sigma) = \prod_k \left|\sin\frac{\sigma - s_k^{\pm}}{2}\right|^{\frac{h_k^{\pm}}{2\pi}},$$

$$U_0(\sigma) = \left\{\sin\frac{\sigma}{2}\right\}^{-\frac{h_0^{(0)}}{2\pi}} \exp\left\{-\frac{1}{4\pi}\int_{-\pi}^{\pi}\gamma_0(s)ctg\frac{\sigma-s}{2}ds\right\}.$$

Using it, for the boundary values of the analytic function $Z_0(z)$ the following is true:

$$\left|Z_0^{\pm}(e^{i\sigma})\right| = \left|G(e^{is})\right|^{\pm\frac{1}{2}} U_0(\sigma)\left[U_1^+(\sigma)\right]^{-1} \times U_1^-(\sigma)\left\{\sin\frac{\sigma}{2}\right\}^{-\frac{h_0}{2\pi}}.$$

The Cauchy-Plemelj formula implies that

$$F^+(e^{is}) = \frac{1}{2}\psi(s) + \frac{Z_0^+(e^{is})}{2\pi}\int_{-\pi}^{\pi}\frac{\psi(\sigma)}{Z_0^+(e^{i\sigma})}\frac{d\sigma}{1-e^{i(s-\sigma)}}. \tag{3.10}$$

We have

$$\ln Z_0^+(e^{i\sigma}) = \ln G(e^{i\sigma}) + \ln X_2^{-1}(e^{i\sigma}) + \ln Y_2^{-1}(e^{i\sigma}) = \ln G(e^{i\sigma}) - \frac{1}{2}\ln\left|G(e^{i\sigma})\right|$$
$$+\frac{i}{4\pi}\int_{-\pi}^{\pi}\ln\left|G(e^{is})\right| ctg\frac{\sigma-s}{2}ds - \frac{i}{2}\gamma(\sigma) - \frac{1}{4\pi}\int_{-\pi}^{\pi}\gamma(s)ctg\frac{\sigma-s}{2}ds = \frac{1}{2}\ln G(e^{i\sigma})$$
$$+\frac{i}{4\pi}\int_{-\pi}^{\pi}\ln G(s)ctg\frac{\sigma-s}{2}ds\ ,\ \ G(s)\equiv G\left(e^{is}\right).$$

Since

$$ctg\frac{\sigma-s}{2} = i + i\frac{2e^{is}}{e^{i\sigma}-e^{is}},$$

then

$$\ln Z_0^+(e^{i\sigma}) = \frac{1}{2}\ln G(e^{i\sigma}) + \frac{i}{4\pi}\int_{-\pi}^{\pi}\ln G(s)ctg\frac{\sigma-s}{2}ds$$
$$= \frac{1}{2}\ln G(\sigma) - \frac{1}{4\pi}\int_{-\pi}^{\pi}\ln G(s)ds + \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\ln G(s))e^{is}}{e^{is}-e^{i\sigma}}ds.$$

Further, we have

$$I\left(\sigma\right)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\ln G(s)}{1-e^{i(\sigma-s)}}ds=\frac{1}{2\pi}\int_{0}^{\pi}\ln\frac{b(\theta)}{a(\theta)}\left[\frac{1}{1-e^{i(\sigma-\theta)}}-\frac{1}{1-e^{i(\sigma+\theta)}}\right]d\theta.$$

Taking into account the identity

$$\frac{1}{1-e^{i(\theta-\varphi)}}-\frac{1}{1-e^{i(\theta+\varphi)}}=-\frac{i}{2}\frac{\cos\frac{\varphi}{2}}{\cos\frac{\theta}{2}}\left[\frac{1}{\sin\frac{\varphi-\theta}{2}}+\frac{1}{\sin\frac{\varphi+\theta}{2}}\right],$$

we get

$$I(\sigma)=-\frac{i}{4\pi}\int_{0}^{\pi}\frac{\cos\frac{\theta}{2}}{\cos\frac{\sigma}{2}}\ln\frac{b(\theta)}{a(\theta)}\left[\frac{1}{\sin\frac{\theta-\sigma}{2}}+\frac{1}{\sin\frac{\theta+\sigma}{2}}\right]d\theta. \tag{3.11}$$

From here it immediately follows that, the function $I(\sigma)$ is even on $(-\pi,\pi)$, i.e. $I(-\sigma)=I(\sigma),\forall\sigma\in(-\pi,\pi)$. Then from (3.11) we get

$$Z_0^+(e^{i\sigma})=G^{\frac{1}{2}}(\sigma)e^{I(\sigma)},$$

since

$$\int_{-\pi}^{\pi}\ln G(s)ds=0.$$

Suppose that

$$-\frac{1}{p(\pi)}<\frac{\beta(\pi)-\alpha(\pi)}{\pi}<\frac{1}{q(\pi)}. \tag{3.12}$$

Using the expression of $F^+(e^{i\sigma})$ and Smirnov's theorem for the classes $H^{\pm}_{p(\cdot)}$ we get that under the conditions (3.7),(3.8) and (3.12) the function $F(\cdot)$ belongs to $H^{+}_{p(\cdot)}$. Furthermore, from

$$\int_{-\pi}^{\pi}\frac{\psi(\sigma)}{Z_0^+(e^{i\sigma})}d\sigma=\int_{0}^{\pi}\frac{a^{-1}(\sigma)\psi(\sigma)}{\sqrt{b(\sigma)a^{-1}(\sigma)}}\frac{d\sigma}{e^{I(\sigma)}}+\int_{-\pi}^{0}\frac{-b^{-1}(-\sigma)\psi(-\sigma)}{\sqrt{a(-\sigma)b^{-1}(\sigma)}}\frac{d\sigma}{e^{I(\sigma)}}$$
$$=\int_{0}^{\pi}\frac{\psi(\sigma)}{\sqrt{a(\sigma)b(\sigma)}}\frac{d\sigma}{e^{I(\sigma)}}-\int_{0}^{\pi}\frac{\psi(\sigma)}{\sqrt{a(\sigma)b(\sigma)}}\frac{d\sigma}{e^{I(\sigma)}}=0,$$

it follows that, $F(\cdot)$ has

$$F^+(z)=\sum_{n=1}^{\infty}a_nz^n, \tag{3.13}$$

as its Taylor expansion on $|z|<1$, where

$$a_n=\sum_{k=1}^{n}c_kb_{n-k},$$

$$Z_0^+\left(z\right)=\sum_{k=0}^{\infty}b_kz^k,$$

$$c_k = -\frac{i}{\pi}\int_0^{\pi} \frac{\sin k\sigma}{\sqrt{a(\sigma)\, b(\sigma)}} \frac{\psi(\sigma)}{e^{I(\sigma)}} d\sigma.$$

Hence,

$$a_n = (h_n, \psi) = \int_0^{\pi} \psi(\sigma)\, h_n(\sigma)\, d\sigma,$$

where

$$h_n(\sigma) = -\frac{i}{e^{I(\sigma)}\sqrt{a(\sigma)\, b(\sigma)}} \sum_{k=1}^{n} b_{n-k} \sin k\sigma. \tag{3.14}$$

Now, consider the following integral:

$$\tilde{I}(\varphi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\psi(\sigma)}{Z_0^+(e^{i\sigma})} \frac{d\sigma}{1 - e^{i(\varphi-\sigma)}}.$$

We have

$$2\pi \tilde{I}(\varphi) = \int_0^{\pi} \frac{a^{-1}(\sigma)\,\psi(\sigma)}{\sqrt{b(\sigma)\, a^{-1}(\sigma)} e^{I(\sigma)}} \frac{d\sigma}{1 - e^{i(\varphi-\sigma)}}$$

$$+\int_{-\pi}^{0} \frac{-b^{-1}(-\sigma)\,\psi(-\sigma)}{\sqrt{a(-\sigma)\, b^{-1}(-\sigma)} e^{I(\sigma)}} \frac{d\sigma}{1 - e^{i(\varphi-\sigma)}}$$

$$= \int_0^{\pi} \frac{\psi(\sigma)}{\sqrt{a(\sigma)\, b(\sigma)} e^{I(\sigma)}} \left[\frac{1}{1 - e^{i(\varphi-\sigma)}} - \frac{1}{1 - e^{i(\varphi+\sigma)}}\right] d\sigma$$

$$= -\frac{i}{2}\int_0^{\pi} \frac{\cos\frac{\sigma}{2}}{\cos\frac{\varphi}{2}} \frac{\psi(\sigma)}{\sqrt{a(\sigma)\, b(\sigma)} e^{I(\sigma)}} \left[\frac{1}{\sin\frac{\sigma-\varphi}{2}} + \frac{1}{\sin\frac{\sigma+\varphi}{2}}\right] d\sigma.$$

From here, it immediately follows that, $\tilde{I}(\cdot)$ is an even function on $(-\pi, \pi) : \tilde{I}(-\varphi) = \tilde{I}(\varphi)$, $\varphi \in (-\pi, \pi)$. Let $s \in (0, \pi)$. From (3.10) we get that

$$F^+(e^{is}) = \frac{1}{2}\psi(s) + Z_0^+(e^{is})\tilde{I}(s) = \frac{1}{2}a^{-1}(s)\psi(s) + \sqrt{b(s)a^{-1}(s)}e^{I(s)}\tilde{I}(s),$$

$$F^+(e^{-is}) = -\frac{1}{2}b^{-1}(s)\psi(s) + \sqrt{a(s)b^{-1}(s)}e^{I(s)}\tilde{I}(s).$$

Hence

$$a(s)F^+(e^{is}) - b(s)F^+(e^{-is}) = \psi(s), s \in (0, \pi). \tag{3.15}$$

Since, $F(\cdot) \in H^+_{p(\cdot)}, p^- > 1$, then from the Theorem 2.2 we have

$$\lim_{r\to 1-0} \int_{-\pi}^{\pi} \left|F^+(re^{is}) - F^+(e^{is})\right|^{p(s)} ds = 0$$

and from

$$a_n = \frac{1}{2\pi r^n}\int_{-\pi}^{\pi} F^+(re^{i\varphi})e^{-in\varphi} d\varphi, 0 < r < 1, n \in N,$$

we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}F^{+}(e^{i\varphi})e^{-in\varphi}d\varphi=\begin{cases}a_n,\ n\geq 1\,,\\ 0,\ \ n<1.\end{cases}$$

It is clear that $\tilde{p}(-\pi)=\tilde{p}(\pi)$ and $\tilde{p}^{-}>1$, then, $\tilde{p}\in WL$. Then by the results of the works [11–14,31], the system $\left\{e^{in\varphi}\right\}_{n\in Z}$ forms a basis in $L_{\tilde{p}(\cdot)}(-\pi,\pi)$. Taking into account the above equalities we get that, the series

$$\sum_{n=1}^{\infty}a_n e^{in\varphi}$$

converges to the function $F^{+}\left(e^{i\varphi}\right)$ in $L_{\tilde{p}(\cdot)}\left(-\pi,\pi\right)$. Then, by (3.15) we get that, the series $\sum_{n=1}^{\infty}a_n\vartheta_n\left(s\right)$ converges to the function $\psi\left(s\right)$ in the space $L_{p(\cdot)}\left(0,\pi\right)$.

Now, as the function $\psi\left(\theta\right)$ we take $\vartheta_n\left(\theta\right)$, here $n\in N$ is arbitrary, but fixed. The (unique) solution is given by (3.9), where $\psi\left(\theta\right)=\vartheta_n\left(\theta\right)$. Furthermore, in that case, the solution of the problem (3.5) is the function

$$F\left(z\right)=\begin{cases}z^{n},\ \ |z|<1,\\ z^{-n},\ \ |z|>1.\end{cases}\tag{3.16}$$

Comparing (3.13) with (3.16) gives us

$$\left(h_n,\vartheta_m\right)=\int_0^{\pi}h_n\left(\theta\right)\ \vartheta_m\left(\theta\right)d\theta=\delta_{nm},\ \ n,m=\overline{1,\infty}.$$

It is clear that

$$\frac{h\left(s_r^{-}\right)}{2\pi}<\frac{1}{q\left(s_r^{-}\right)},\ \ \forall r.$$

Hence, from the expression for $h_n\left(\theta\right),\ \ n=\overline{1,\infty}$, and from the Lemma 2.1 we get that, $h_n\left(\theta\right)\in L_{q(\cdot)},\forall n=\overline{1,\infty}$.

Therefore, we get that the pair of systems $\left\{\vartheta_n\left(\theta\right)\right\}$ and $\left\{h_n\left(\theta\right)\right\}$ is biorthogonal and hence, $\left\{\vartheta_n\right\}_{n\in N}$ forms a basis in $L_{p(\cdot)}\left(0,\pi\right)$.

If at least one of the conditions holds

$$\frac{1}{p\left(0\right)}-2<\frac{\beta\left(0\right)-\alpha\left(0\right)}{\pi}<\frac{1}{p\left(0\right)}-1,$$

$$1-\frac{1}{p\left(\pi\right)}<\frac{\beta\left(\pi\right)-\alpha\left(\pi\right)}{\pi}<-\frac{1}{p\left(\pi\right)}+2,$$

then, rewriting $\tilde{I}\left(\varphi\right)$ in the form

$$\tilde{I}\left(t_0\right)=\frac{i}{2\pi}\int_{|t|=1}\frac{\psi\left(\arg t\right)}{Z_0^{+}\left(t\right)\left(t-t_0\right)}dt=\frac{i}{4\pi}\int_0^{\pi}\frac{\cos\frac{\theta}{2}}{\cos\frac{\varphi}{2}}e^{-I(\theta)}$$
$$\times\frac{\psi\left(\theta\right)}{\sqrt{A\left(\theta\right)B\left(\theta\right)}}\left[\frac{1}{\sin\frac{\theta-\varphi}{2}}+\frac{1}{\sin\frac{\theta+\varphi}{2}}\right]d\theta,$$

$$\tilde{I}\left(\varphi\right)=-\frac{i}{4\pi}\int_0^{\pi}\frac{\sin\frac{\sigma}{2}}{\sin\frac{\varphi}{2}}\frac{\psi\left(\sigma\right)}{\sqrt{A\left(\sigma\right)B\left(\sigma\right)}e^{I(\sigma)}}\left[\frac{1}{\sin\frac{\sigma-\varphi}{2}}-\frac{1}{\sin\frac{\sigma+\varphi}{2}}\right]d\sigma,$$

and using the Theorem 2.1 we get that, in these cases $F(z) \in H_{p(\cdot)}$. The inclusion $F(z) \in H_{p(\cdot)}$ under the conditions (3.1)-(3.4), follows from the Riesz-Thorin interpolation theorem in generalized Lebesgue spaces.

Set

$$\tilde{\beta}(t) = \beta(t) - \mu(2t - \pi), \quad \tilde{\alpha}(t) + \mu(2t - \pi), \qquad \frac{1}{2} \le \mu \le 1.$$

The functions $\tilde{\beta}(\cdot)$ and $\tilde{\alpha}(\cdot)$ obey all conditions of above considered case. Therefore, according to above we have

$$\int_0^\pi \vartheta_n\left(t, \tilde{\beta}(t), \tilde{\alpha}(t)\right) h_m\left(t, \tilde{\beta}(t), \tilde{\alpha}(t)\right) dt = \delta_{nm}, \quad n, m = \overline{1, \infty}. \tag{3.17}$$

As the left-hand side of the last equality analytically depends on μ, if $-\varepsilon < \mu \le 1$ (ε is sufficiently small positive number), this holds true for $\mu = 0$, as well. Hence, it was proved that, under the conditions (3.1), (3.7) and (3.2), where $n_0 = n_r = 0$, the system $\{\vartheta_n\}_{n \in N}$ forms a basis in the space $L_{p(\cdot)}(0, \pi)$.

Now, consider the general case. Define $g(\theta)$ to be

$$g(\theta) \equiv \begin{cases} e^{-in_0\theta}, & 0 < \theta < s_1, \\ e^{-in_1\theta}, & s_1 < \theta < s_2, \\ \dots\dots\dots\dots\dots\dots & \\ e^{-in_r\pi}, & s_r < \theta < \pi. \end{cases}$$

Consider the following system:

$$\varphi_m(\theta) \equiv g(\theta)\ \vartheta_m(\theta), \quad m = \overline{1, \infty}.$$

Rewrite $\varphi_m(\theta)$ in the form:

$$\varphi_m(\theta) \equiv \begin{cases} |a(\theta)|\, e^{i(\alpha(\theta) + n_0\pi)}\, e^{im\theta} - |b(\theta)|\, e^{i(\beta(\theta) - n_0\pi)}\, e^{-im\theta}, & \theta \in (0, s_1), \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots & \\ |a(\theta)|\, e^{i(\alpha(\theta) + n_r\pi)}\, e^{im\theta} - |b(\theta)|\, e^{i(\beta(\theta) - n_r\pi)}\, e^{-im\theta}, & \theta \in (s_r, \pi). \end{cases}$$

It is easy to check that, the functions

$$\tilde{a}(\theta) = g^{-1}(\theta)\, a(\theta), \quad \tilde{b}(\theta) = g(\theta)\ b(\theta).$$

obey all conditions of the theorem for $n_0 = n_i = 0, \quad i = \overline{1, \infty}$. Then, the system $\{\varphi_m(\theta)\}, \ m = \overline{1, \infty}$, forms a basis in $L_{p(\cdot)}(0, \pi)$. Since $|g(\theta)| \equiv 1$, we get that, the system $\{\vartheta_n(\theta)\}, \quad n = \overline{1, \infty}$, also forms a basis in $L_{p(\cdot)}(0, \pi)$ in the general case.

Now, consider the case when $\beta(\pi) - \alpha(\pi) < -\frac{\pi}{p(\pi)}$. By substituting the index of the system $\{\vartheta_n\}_{n \in N}$ by

$$n = m + \left[\frac{\beta(\pi) - \alpha(\pi)}{2\pi} + \frac{1}{2p(\pi)}\right],$$

and setting

$$\tilde{\alpha}(\theta) = \alpha(\theta) + \left[\frac{\beta(\pi) - \alpha(\pi)}{2\pi} + \frac{1}{2p(\pi)}\right] \theta,$$

$$\tilde{\beta}(\theta) = \beta(\theta) - \left[\frac{\beta(\pi) - \alpha(\pi)}{2\pi} + \frac{1}{2p(\pi)}\right]\theta,$$

(here, square brackets indicate the greatest integer function), we get a new system, so that

$$m \geq 2, \quad \frac{\tilde{\beta}(\pi) - \tilde{\alpha}(\pi)}{\pi} \in \left(-\frac{1}{p(\pi)}, 2 - \frac{1}{p(\pi)}\right).$$

Therefore, this system is not complete in $L_{p(\cdot)}(0,\pi)$, since, $\vartheta_m(\theta)$, $\quad m = \overline{1,\infty}$, forms a basis in $L_{p(\cdot)}$. Now, let

$$\beta(\pi) - \alpha(\pi) > 2\pi - \frac{\pi}{p(\pi)}.$$

If $\beta(\pi) - \alpha(\pi) > 2\pi - \frac{\pi}{p}$, then substitute

$$n = m + 1 + \left[\frac{\beta(\pi) - \alpha(\pi)}{2\pi} - 1 + \frac{1}{2p(\pi)}\right],$$

$$\tilde{\alpha}(\theta) = \alpha(\theta) + \left(\left[\frac{\beta(\pi) - \alpha(\pi)}{2\pi} - 1 + \frac{1}{2p(\pi)}\right] + 1\right)\theta,$$

$$\tilde{\beta}(\theta) = \beta(\theta) - \left(\left[\frac{\beta(\pi) - \alpha(\pi)}{2\pi} - 1 + \frac{1}{2p(\pi)}\right] + 1\right)\theta.$$

As a result we get a new system $\left\{\tilde{\vartheta}_m\right\}_{m\geq 0}$, so that

$$-\frac{\pi}{p(\pi)} < \tilde{\beta}(\pi) - \tilde{\alpha}(\pi) < 2\pi - \frac{\pi}{p(\pi)}.$$

Then by the fact, already proved it follows that the system $\left\{\tilde{\vartheta}_m\right\}_{m\in N}$ forms a basis in $L_{p(\cdot)}(0,\pi)$, hence, $\left\{\tilde{\vartheta}_m\right\}_{m\geq 0} \equiv \{\vartheta_n\}_{n\in N}$ is complete but not minimal in $L_{p(\cdot)}(0,\pi)$.

Corollary 3.1 *Let* $\gamma(t)$ *be a continuous function on* $[0,\pi]$. *Suppose*

$$\frac{\pi}{2p(0)} + (n_0 - 1)\pi < \gamma(0) < \frac{\pi}{2p(0)} + n_0\pi$$

for some integer n_0 *and* $\gamma(\pi) \neq -\frac{\pi}{2p(\pi)} + n\pi$, $\forall n \in Z$. *If*

$$-\frac{\pi}{2p(\pi)} + n\pi < \gamma(\pi) < -\frac{\pi}{2p(\pi)} + (n+1)\pi,$$

then the system of sines $\sin(nt - \gamma(t))$, $\quad n = \overline{1,\infty}$, *forms a basis in* $L_{p(\cdot)}(0,\pi)$*; if*

$$\gamma(\pi) < -\frac{\pi}{2p(\pi)} + n\pi,$$

then the system of sines $\sin(nt - \gamma(t))$, $\quad n = \overline{1,\infty}$ *is not complete in* $L_{p(\cdot)}(0,\pi)$*, but is minimal in* $L_{p(\cdot)}(0,\pi)$*; if*

$$\gamma(\pi) \geq -\frac{\pi}{2p(\pi)} + (n+1)\pi,$$

then the system of sines $\sin(nt - \gamma(t))$, $\quad n = \overline{1,\infty}$, *is complete, but not minimal in* $L_{p(\cdot)}(0,\pi)$.

The corollary easily follows from Theorem 3.1 if we set

$$a(t) = e^{-i\gamma(t)}, \quad b(t) = e^{i\gamma(t)}.$$

Note that, the Corollary 3.1 has been proved in [29] in the case when $\gamma(t)$ is of Hölder class and $p(\cdot) \equiv const$.

Corollary 3.2 *Let the conditions of Theorem 3.1 fulfil, and besides it*

$$-\frac{2\pi}{p(\tilde{s}_i)} < \tilde{h}_i < \frac{2\pi}{q(\tilde{s}_i)}, \quad i = \overline{1,\infty}$$

and for some integer n_0 (3.1) *holds, with* $\beta(\pi) - \alpha(\pi) \neq -\frac{\pi}{p(\pi)} + 2n_0\pi$. *If*

$$-\frac{1}{p(\pi)} + 2n_0 < \frac{\beta(\pi) - \alpha(\pi)}{\pi} < -\frac{1}{p(\pi)} + 2(n_0 + 1);$$

then the system $\{\vartheta_n\}_{n\in N}$ *forms a basis in* $L_{p(\cdot)}(0,\pi)$*; if*

$$\beta(\pi) - \alpha(\pi) < -\frac{\pi}{p(\pi)} + 2n_0\pi,$$

then it is not complete, but minimal in $L_{p(\cdot)}(0,\pi)$*; if*

$$\beta(\pi) - \alpha(\pi) \geq -\frac{\pi}{p(\pi)} + 2(n_0 + 1)\pi,$$

then it is complete, but not minimal in $L_{p(\cdot)}(0,\pi)$.

Remark 3.1 Theorem 3.1 can also be proved in case that, set $\{\tilde{s}_i\}$ has a finite number of limits points.

References

1. Bilalov, B.T.: *On uniform convergence of series with regard to some system of sines*, Diff. uravn. **24** (1), 175–177 (1988).
2. Bilalov, B.T.: *Basicity of some systems of functions*, Diff. uravn. **25**, 163–164 (1989).
3. Bilalov, B.T.: *Basicity of some systems of exponents, cosines and sines*, Diff. Uravn. **26** (1), 10–16 (1990).
4. Bilalov, B.T.: *Necessary and sufficient condition for completeness of some system of functions*, Diff. Uravn. **27**(1), 158–161 (1991).
5. Bilalov, B.T.: *Completeness and minimality of some trigonometric systems*, Diff. Uravn. **28**(1), 170–173 (1992).
6. Bilalov, B.T.: *Basis properties of eigen functions of some not self-adjoint differential operators*, Diff. Uravn. **30** (1), 16–21 (1994).
7. Bilalov, B.T.: *On Bases for Some Systems of Exponentials, Cosines and Sines in* L_p, *Doklady Mathematics*, **379** (2), 7–9 (2001).
8. Bilalov, B.T.: *Bases of Exponentials, Sines and Cosines*, Diff. Uravn. **39** (5), 619–622 (2003).
9. Bilalov, B.T.: *Basis properties of some systems of exponents, cosines and sines*, Sibirskiy Matem. Jurnal, **45** (2), 264–273 (2004).
10. Bilalov, B.T., Mamedov, F.I., Bandaliev, R.A.: *On classes of harmonic functions with variable exponent*, Dokl NAN Azerb. **63** (5), 16–21 (2007).
11. Bilalov, B.T., Guseynov, Z.Q.: *Bases from exponents in Lebesgue spaces of functions with variable summability exponent*, Trans. of NAS of Az. **28** (1), 43–48 (2008).

12. Bilalov, B.T., Guseynov, Z.G.: *On the basicity from exponents in Lebesgue spaces with variable exponents*, TWMS J. Pure Appl. Math. **1** (1), 14–23 (2010).
13. Bilalov, B.T., Guseynov Z.G.: *Basicity criterion for perturbed systems of exponents in Lebesgue spaces with variable summability*, Dokl. RAN, **436** (5), 586–589 (2011).
14. Bilalov, B.T., Guseynov, Z.G.: *Basicity of a system of exponents with a piece-wise linear phase in variable spaces*, Mediterr. J. Math. **9** (3), 487–498 (2012).
15. Cruz-Uribe, D.V., Fiorenza, A.: Variable Lebesgue Spaces, Foundations and Harmonic Analysis. *Springer* (2013).
16. Danilyuk, I.I.: Irregular boundary value problems in the plane, *Nauka, Moscow* (1975).
17. Devdariani, G.G.: *On basicity of a system of functions*, Diff. Uravn. **22** (1), 170–171 (1986).
18. Devdariani, G.G.: *On basicity of a trigonometric system of functions*, Diff. Uravn. **22** (1), 168–170 (1986).
19. Devdariani, G.G.: *Basicity of a system of sines*, Proc. of I.N. Venua Institute of Applied Mathematics, **19**, 21–27 (1987).
20. Moiseev, E.I.: *On basicity of systems of sines and cosines*, DAN SSSR, **275** (4), 794–798 (1984).
21. Moiseev, E.I.: *On basicity of a system of sines*, Diff. Uravn. **23** (1), 177–179 (1987).
22. Moiseev, E.I.: *On some boundary value problems for mixed type equations*, Diff Uravn. **28** (1), 123–132 (1992).
23. Moiseev, E.I.: *On solution of Frankle's problem in special domain*, Diff. Uravn. **28** (4), 682–692 (1992).
24. Moiseev, E.I.: *On existence and uniqueness of solution a classical problem*, Dokl. RAN, **336** (4), 448–450 (1994).
25. Moiseev, E.I.: *On basicity of systems of sines and cosines in a weighted space*, Diff. Uravn. **34** (1), 40–44 (1998).
26. Moiseev, E.I.: *Basicity of a system of eigenfunctions of the differential operator in the weighted space*, Diff. Uravn. **35** (2), 200–205 (1999).
27. Najafov, T.I. Nasibova N.P.: *On the Noetherness of the Riemann problem in a generalized weighted Hardy classes*, Azerb. J. of Math. **5** (2), 109–139 (2015).
28. Ponomarev, S.M.: *On the theory of boundary value problems for equations of mixed type in three-dimensional domains*. Dokl. Akad. Nauk SSSR, **246** (6), 1303–1304 (1979).
29. Pukhov, S.S. Sedletskii A.M.: *Bases of exponents, sines and cosines in weight spaces on finite interval*, Dokl. RAN, **425** (4), 452–455 (2009).
30. Sedletskii, A.M.: *Biorthogonal expansions in series of exponents on intervals of real axis*, Usp. Mat. Nauk, **37** (5), 51–95 (1982).
31. Sharapudinov, I.I.: *Some problems of approximation theory in spaces* $L^{p(x)}(E)$, Anal.Math. **33** (2), 135–153 (2007).