

Article

Parity–Time Symmetry in Bidirectionally Coupled Semiconductor Lasers

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Abstract: We report on the numerical analysis of intensity dynamics of a pair of mutually coupled, single-mode semiconductor lasers that are operated in a configuration that leads to features reminiscent of parity–time symmetry. Starting from the rate equations for the intracavity electric fields of the two lasers and the rate equations for carrier inversions, we show how these equations reduce to a simple 2×2 effective Hamiltonian that is identical to that of a typical parity–time (PT)-symmetric dimer. After establishing that a pair of coupled semiconductor lasers could be PT-symmetric, we solve the full set of rate equations and show that despite complicating factors like gain saturation and nonlinearities, the rate equation model predicts intensity dynamics that are akin to those in a PT-symmetric system. The article describes some of the advantages of using semiconductor lasers to realize a PT-symmetric system and concludes with some possible directions for future work on this system.

Keywords: parity–time symmetry; semiconductor laser; intensity dynamics

1. Introduction

Semiconductor lasers (SCLs) with optical injection and feedback, as well as coupled SCLs, have been basic paradigms for investigating nonlinear dynamics for the last several years [1]. The dynamical response of these SCL systems has been shown to include low frequency fluctuations (LFFs), periodic doubling routes to chaos, and the occurrence of unstable attractors, and the dynamics have been exploited for chaotic encryption, random number generation, linewidth reduction, and optical waveform production [2]. Independent of these studies on SCLs, there has been enormous interest in systems that are described by non-hermitian Hamiltonians that arise in open systems, i.e., systems that are coupled to the environment [3–11]. Typically, the Hamiltonian in quantum mechanics is hermitian because one deals with closed systems, and the hermiticity leads to real eigenvalues, orthogonal eigenfunctions, unitary evolution, and conservation of probability. As soon as one deals with realistic systems, by including, say, dissipation, one has to work with non-hermitian Hamiltonians, and the varying dynamics that result in systems that are described by such Hamiltonians have attracted much attention in recent years. Part of this interest is driven by the fundamental physics inherent in such systems and part of it by their predicted applications. The optics community has been particularly interested in one type of non-hermitian Hamiltonians called the parity–time (PT) symmetric Hamiltonians, which are a class of Hamiltonians that are symmetric under combined operations of parity (P) and time-reversal (T). The pioneering work of Bender and co-workers, and others [6–11], demonstrating that a non-hermitian Hamiltonian may have a real energy spectrum provided it is parity (P) and time-reversal (T) symmetric, has led to tremendous interest in experimental

realizations of PT-symmetric laboratory systems [12–19]. Many experimental realizations have been in the optical domain, largely because PT symmetry requires systems with balanced gain and loss, which are ubiquitous in optics. Thus, much effort has been put into developing integrated structures with appropriate gain and loss properties. The typical PT-symmetric dimer [12] consists of two coupled oscillators wherein the gain in one oscillator is exactly equal to the loss in the other. The resulting 2×2 Hamiltonian matrix that describes this system then has complex diagonal elements, which are complex conjugates of each other and represent gain and loss in each oscillator, and the off-diagonal elements are real and equal and represent the coupling between the oscillators.

In a typical PT-symmetric system, say a pair of evanescently coupled waveguides in which one waveguide has gain and the other an equal amount of loss [12], one finds that as the gain/loss parameter is varied, there is a critical value, called the PT threshold, at which the eigenvalues of the Hamiltonian transition from being real to complex. In the regime where the eigenvalues are real, the norm of the wavefunction is bounded, and once the eigenvalues are complex, the norm grows abruptly. In our work, we use this abrupt transition as a metric for the PT threshold.

One outcome of the studies on PT symmetry is that many of the features of these systems are a result of the exceptional point (EP) behaviors of the underlying Hamiltonian [20–23]. Coupled lasers are especially attractive for the experimental realization of PT-symmetric models and exceptional point (EP) behaviors, and a few recent experiments have fabricated synthetic microcavity lasers on an integrated chip and reported the PT-symmetric properties of the system [24]. The laser configuration is typically designed to exploit the balance between the gain and loss of the lasers in order to extract unexpected behaviors that arise when the system undergoes an abrupt PT phase transition or, more generally, approaches an EP. It is anticipated that the outcomes of our work will be important for systems described by non-hermitian rate equations, local and nonlocal, and their laboratory implementations.

In this paper, we report a realization of a time-delayed, non-hermitian system in a bulk optical system that is comprised of two optically coupled semiconductor lasers (SCLs), and a numerical investigation of the properties of this system. In particular, we show that the rate equation model that is typically used to describe these coupled lasers [25] can, under certain conditions, lead to an effective non-hermitian Hamiltonian that is strongly reminiscent of the Hamiltonians that arise in the study of conventional PT-symmetric systems. Our work demonstrates that the coupled SCL system possesses many of the features that PT-symmetric systems do. We note that our system is completely classical, and yet it has features of PT symmetry because many aspects of PT symmetry are a result of the characteristics of exceptional points in the governing Hamiltonian. The predictions of our numerical work can be implemented in commercially available, off-the-shelf SCLs, since it does not require any specially fabricated components with tailored properties. Furthermore, as we will show, the important PT parameters can be easily controlled in the laboratory, making coupled SCLs very useful for studying PT symmetry.

Among the key features of our system are the fact that unlike other PT-symmetric systems, which rely on coupling a system with gain to an identical one with loss, our configuration couples two lasers in which the frequency detuning between the two lasers and the coupling strength between them, respectively, are the relevant parameters. The advantage is that in contrast to other systems where a precise balance between gain and loss has to be engineered, our system always has the frequency detuning of one laser exactly equal and opposite in sign to the frequency detuning of the second laser, thereby guaranteeing that the diagonal elements of the effective PT Hamiltonian are equal and opposite in sign. PT-symmetric systems are of interest for making materials with unidirectional optical propagation [26], single mode lasing action [27], and the spontaneous generation of photons in a PT-symmetric medium by a vacuum field [28]. Due to the miniature size of SCLs and well established fabrication methods for incorporating several lasers and associated components on chips, our work may lead to PT-symmetric photonics on a chip.

2. Numerical Model

Our system is described by a rate equation model that is based on the Lang–Kobayashi model [25] wherein we assume that the two lasers are nearly identical in all of their characteristics, single-mode, and operate at slightly different frequencies, ω_1 and ω_2 . We write the rate equations in a frame that is rotating at the average frequency θ of the two lasers, i.e., $\theta = (\omega_1 + \omega_2)/2$ [29]. The rate equations describing the normalized complex electric fields, $E_{1,2}(t)$, and the normalized excess carrier densities, $N_{1,2}(t)$, may be written as follows [29]:

$$\frac{dE_1}{dt} = (1 + i\alpha)N_1(t)E_1(t) + i\Delta\omega E_1(t) + \kappa \exp(-i\theta\tau)E_2(t - \tau), \quad (1)$$

$$\frac{dE_2}{dt} = (1 + i\alpha)N_2(t)E_2(t) - i\Delta\omega E_2(t) + \kappa \exp(-i\theta\tau)E_1(t - \tau), \quad (2)$$

$$T \frac{dN_1}{dt} = J_1 - N_1(t) - (1 + 2N_1(t))|E_1(t)|^2, \quad (3)$$

$$T \frac{dN_2}{dt} = J_2 - N_2(t) - (1 + 2N_2(t))|E_2(t)|^2, \quad (4)$$

where α is the linewidth enhancement factor [28], τ is the time delay in coupling due to physical separation between the lasers, $J_{1,2}$ is the injection current above threshold, and T is the ratio of the carrier lifetime to the photon lifetime. The model used in Equations (1)–(4) is a phenomenological model [30,31] that has been quite accurate in modeling the dynamical response of semiconductor lasers subject to optical injection and in reproducing the intensity response of mutually coupled SCLs. A detailed and rigorous model has been described in Ref. [29] for bidirectionally coupled SCLs, where the authors start from Maxwell’s equations, apply appropriate boundary conditions, and obtain the time evolution of electric field amplitudes in each laser cavity. Equations for the time evolution of the carrier inversion in each laser are also obtained. Ref. [29] has shown that under the assumptions of (i) weak coupling between the lasers, (ii) both lasers operating at nearly identical optical frequencies, (iii) both lasers having equal gain coefficients despite a slight detuning between them, and (iv) neglecting multiple feedbacks, the rigorous model reduces to the phenomenological model.

The important and relevant PT parameters for our work are κ and $\Delta\omega$, which describe the coupling coefficient and the frequency detuning between the lasers, respectively. Note that in Equation (2), the coupling term accounts for the mutual coupling between the two lasers, and a phase accumulation term has been added to account for the time taken for the light to travel from one laser to the other. In our system, $\Delta\omega$ physically represents the frequency pulling that is typical of coupled lasers operating at slightly different frequencies, and κ produces amplification of light in each laser.

To motivate the connection to non-hermitian Hamiltonians in general, and PT-symmetry in particular, the rest of this paper will focus on the zero-delay case. The effects of time-delay are profound and will be discussed in a future article. When the SCLs are operating in steady state, above threshold, the inversion above transparency is zero, i.e., $N_{1,2} = 0$ [32]. Therefore, Equations (1) and (2) reduce to

$$\begin{bmatrix} \dot{E}_1 \\ \dot{E}_2 \end{bmatrix} = \begin{bmatrix} i\Delta\omega & \kappa \\ \kappa & -i\Delta\omega \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}, \quad (5)$$

where the 2×2 effective Hamiltonian is isomorphic to typical PT-symmetric Hamiltonians under a $\pi/2$ rotation about Pauli matrix σ_y , with the difference being that the diagonal elements of the matrix that normally represent gain/loss terms [12] are replaced in Equation (5) by frequency detuning between the two lasers. The SCL model is a rate equation model, in contrast to typical PT systems that are studied

by invoking the Schroedinger equation. Thus, the complex i that occurs in the Schroedinger equation is missing in our model (such systems are referred to as anti-PT systems). In our system, the diagonal elements, instead of contributing to amplification or attenuation of light, now give rise to temporal oscillations in the field. The off-diagonal elements, instead of determining the frequency of exchange between the two oscillators, now contribute to laser intensity growth.

The eigenvalues, λ , of the effective 2×2 Hamiltonian above are given by $\lambda = \pm \sqrt{\kappa^2 - \Delta\omega^2}$. For values of $|\Delta\omega| < \kappa$, the eigenvalues are real, and for $|\Delta\omega| > \kappa$, the eigenvalues are complex. Thus, the point at which $|\Delta\omega| = \kappa$ marks the PT threshold.

The reduction of the rate equations to the simplified 2×2 effective Hamiltonian answers the question of why one might expect PT-symmetric behavior in coupled SCLs. The question still remains as to whether the full rate equation model also exhibits PT-symmetric features. We will show below that despite the simplifying assumptions made to get Equation (5) and the differences in the conventional PT model and our system, the coupled SCL system does behave like a PT-symmetric system. In fact, our work to date indicates that the coupled SCL system is a very robust PT system and that the signatures of PT symmetry persist even without some of the simplifying assumptions.

3. Results

Having motivated the existence of PT-symmetric behavior in a pair of coupled SCLs, we now investigate whether the system retains any features of PT symmetry when the full set of laser rate equations is solved numerically. We restrict our discussion to the zero time-delay case, i.e., $\tau = 0$, to focus on the PT symmetry aspects of the system. The key signature we look for is whether there is an abrupt change in the intensities of the lasers at the PT threshold, i.e., when $|\Delta\omega| = \kappa$. In Figure 1a are shown the real parts of the two eigenvalues of Equation (5) vs. $\Delta\omega$ when $\tau = 0$, as well as the imaginary parts of the eigenvalues vs. $\Delta\omega$. It is seen that at $|\Delta\omega| = \kappa$, there is an abrupt change in the real eigenvalues to non-zero values. At the same time, the imaginary parts of the eigenvalues transition from non-zero to zero values at $|\Delta\omega| = \kappa$. It is clear from the behavior of the eigenvalues that as in all PT-symmetric systems, there is a threshold at which the eigenvalues transition from purely imaginary to purely real. The solutions for Equation (5) have the form $\exp(\lambda t)$, and so the real parts of the eigenvalues lead to amplification or decay of the laser intensities, depending on whether the real parts of the eigenvalues are positive or negative, respectively. Since the real parts take both positive and negative values (see Figure 1a, for example), the linear model gives physical results only if the real part of the eigenvalues is negative. For positive values of the real part of the eigenvalues, the solutions would diverge, and this unphysical result is a consequence of neglecting gain saturation. In a realistic laser system, gain saturation will prevent the laser intensities from growing to unphysical values, as shown later in Figure 2a.

Since the eigenvalues of the effective Hamiltonian in Equation (5) are given by $\sqrt{\kappa^2 - \Delta\omega^2}$, the eigenvalues can be swept from real to complex by sweeping κ and holding $\Delta\omega$ constant. In Figure 1b are shown the real and imaginary parts of the eigenvalues of the 2×2 effective Hamiltonian as a function of κ for a constant $\Delta\omega = 0.2$. Once again, it is seen that at the PT threshold, i.e., $\kappa = \Delta\omega$, the eigenvalues undergo a transition from real to imaginary. Thus, a pair of coupled SCLs provide multiple methods by which the PT threshold can be accessed, either by sweeping κ or by sweeping the relative detuning through either injection current modulation or temperature variation. The observations in Figure 1a,b are the characteristic behaviors for the eigenvalues of a PT-symmetric system. The regime where the time delay is non-zero leads to more complex behavior since the effective Hamiltonian now becomes infinite-dimensional instead of a simple 2×2 matrix, and this will be the subject of another article. As one illustration of the effect of time-delay, we show the real and imaginary parts of the eigenvalues of the effective Hamiltonian for a time delay $\tau = 85$ in Figure 1c, when κ is swept and $\Delta\omega$ is fixed at 0.2. Since one

cannot show all the eigenvalues of an infinite dimensional system, we show the behavior of the dominant eigenvalue, i.e., eigenvalue with the largest real part, since the real part leads to laser intensity growth. It is observed that the real part of the eigenvalues shows a growth at $\Delta\omega = \kappa$, but there are also multiple other transitions for $\kappa < \Delta\omega$. The real part of the eigenvalues changes sign at all these transitions, and so the picture is quite different from Figure 1b.

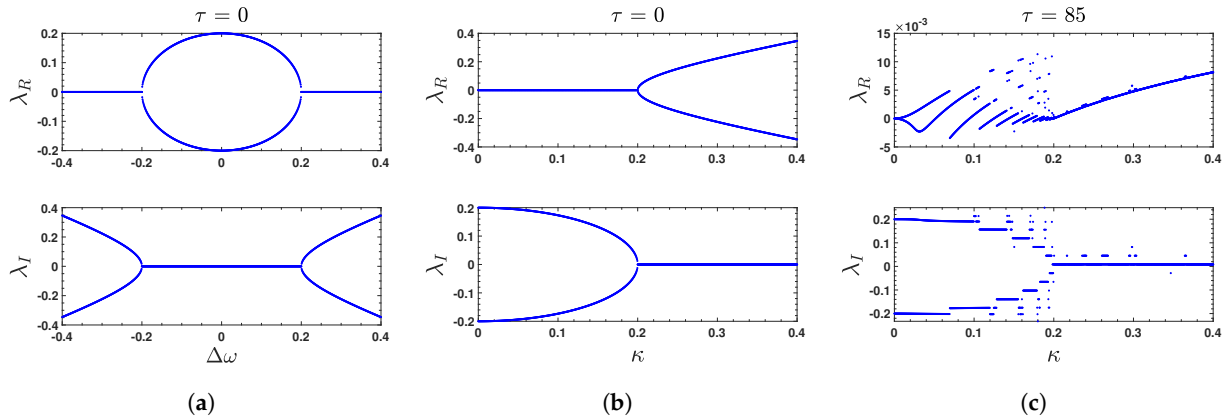


Figure 1. Real and imaginary parts of eigenvalues of the effective Hamiltonian (a) vs. $\Delta\omega$ for $\tau = 0$, (b) vs. κ for $\tau = 0$, and (c) vs. κ for $\tau = 85$. The parity–time (PT) threshold is $\Delta\omega = \kappa = 0.2$ for all three plots.

The results in Figure 1 are obtained with simplifying assumptions, including the neglect of population dynamics and gain nonlinearities. We next investigate whether the features of PT symmetry persist if the full set of rate equations is numerically solved, which then includes gain saturation and population dynamics. In the simulations, all time scales are in units of the photon lifetime, taken to be 10 ps. For all simulations, we take $\alpha = 4$, but note that the results are insensitive to the value of α . We also take the initial values for the intracavity electric fields to be the same for both lasers, chosen such that the lasers are operating at about 3%–5% above the lasing threshold. In Figure 2a are the intensities of the two lasers for a coupling strength $\kappa = 0.2$ and $\tau = 0$. The relative detuning, $\Delta\omega$, is scanned, and we observe that for $|\Delta\omega| > \kappa$, the intensities of both lasers remain bounded, and this is the regime in which the eigenvalues of the effective Hamiltonian are complex. At the PT threshold, $|\Delta\omega| = \kappa$, there is an abrupt increase in the intensities of both lasers, consistent with the simplified 2×2 model. This observation is an indicator of the robustness of the PT-symmetric behavior of this system since the PT features persist in the presence of nonlinearities and population dynamics. It is surprising and remarkable that the predictions of the rate equation model match those of the 2×2 effective Hamiltonian so well since not only does the rate equation model include gain saturation and associated nonlinearities, and population dynamics, but it also assumes each laser is operating on a single longitudinal mode. However, in practice, it is unlikely that for the coupling strengths used here, the two lasers would still be single-mode.

To ensure that the abrupt change in the lasers’ intensities is not an artifact of our simulations, we varied the relative detuning between the lasers by scanning the injection current to one of the SCLs since sweeping the pump changes the optical frequency of these lasers. Of course, varying the injection current also changes the output intensity of the laser, and so both lasers cannot be set to the same initial intensities. The dependence of the intensity and optical frequency of the lasers is given by

$$\omega(\Delta J) = \omega_o - k\Delta J, \tag{6}$$

$$I(\Delta J) = I_{thr} + \eta\Delta J, \tag{7}$$

where ω_0 is the optical frequency at the lasing threshold, I_{thr} is the lasing threshold intensity, and ΔJ is the injection current with the threshold injection current subtracted. The slopes are intrinsic characteristics of the SCLs and were taken to be $k = 1.84$ GHz/mA and $\eta = 0.55$ mW/mA.

In Figure 2b, we show a case for $\kappa = 0.0027$, where the two lasers are operated at different initial output intensities, one at 2% above threshold and the other at 30% above threshold. The injection current to this latter, higher intensity laser is varied linearly, and it is seen that at the PT threshold, i.e., when $\Delta\omega = \kappa$, there is an abrupt increase in the intensity of the other SCL. This, once again, is a clear feature of the PT-symmetric properties of this system.

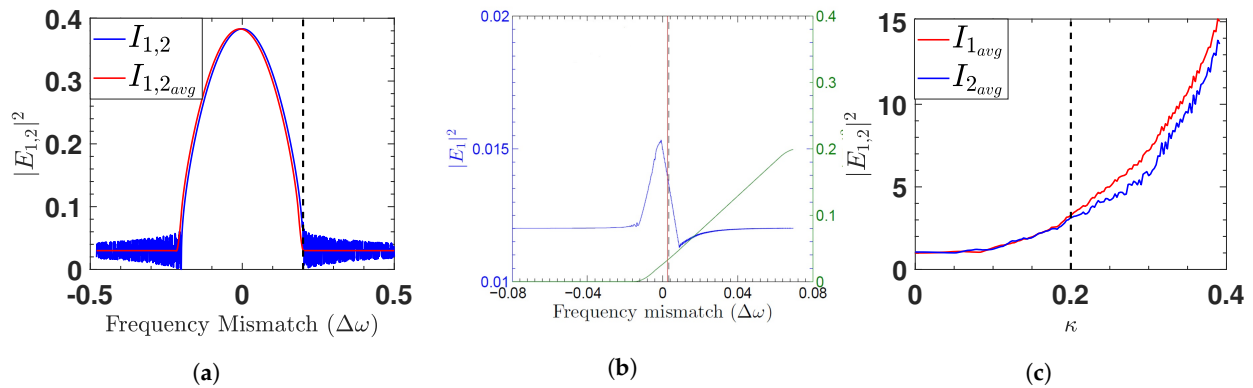


Figure 2. Intensities of the two lasers from numerical simulations of Equations (1)–(4). The vertical, dashed line indicates the PT threshold. (a) As a function of $\Delta\omega$ for $\kappa = 0.2$, $\tau = 0$. The intensities of the two lasers are indistinguishable from each other since we assume identical lasers and operating conditions. The red line is the intensity averaged over 10 ns to account for detector bandwidth; (b) as a function of $\Delta\omega$ for $\kappa = 0.0027$ when injection current to one laser (shown in green) is swept to vary its optical frequency; intensities are averaged over 10 ns; (c) as a function of κ for a $\Delta\omega = 0.2$, $\tau = 85$, intensities averaged over 10 ns.

Finally, to gain some insights into the limits of this numerical model for investigating PT symmetry, we show one example of the outcome of the numerical simulations for a non-zero time delay, when the coupling, κ , is strong so that the rate equation model is not valid. In Figure 2c, $\tau = 85$, $\Delta\omega = 0.2$, and κ is swept, and we note that the intensities of both lasers increase as the PT threshold is crossed. However, one does not observe the sharp transition that is characteristic of PT-symmetric systems, and also, there are slow oscillations in the intensities of both lasers. This behavior, for $\kappa > \Delta\omega$, is not predicted by the eigenvalues picture obtained from the 2×2 effective Hamiltonian. The principal causes for this are that the infinite dimensional nature of the system, which is not captured by the 2×2 effective Hamiltonian, and that the rate equation model assumes weak coupling, which starts to break down for the strong couplings used here. We note that the time delay can cause the intensities of the two lasers to become chaotic. However, we are only interested in the global, average behavior of the intensities and not in the dynamical regimes, and the averaging over 10 ns hides the chaotic behaviors. This average behavior is more representative of the predictions of Equation (5).

In order to gain some further insight into the properties of this system, we re-write the rate equations for the field in terms of equations for the time evolution of the intensity and phase of the two lasers. The complex electric fields are written in terms of a real amplitude and a real phase modulation term as $E_{1,2}(t) = A_{1,2}(t)e^{i\phi_{1,2}(t)}$. Inserting this into the rate equation model and separating the real and imaginary components, the time evolution of the phases is given by

$$\dot{\phi}_{1,2}(t) = \alpha N_{1,2}(t) \pm \Delta\omega + \kappa \frac{A_{2,1}(t-\tau)}{A_{1,2}(t)} \sin(\phi_{2,1}(t-\tau) - \phi_{1,2}(t) - \theta\tau). \quad (8)$$

The time evolution of the phase difference, $\Delta\phi = \phi_1 - \phi_2$, is given by

$$\begin{aligned} \dot{\Delta\phi}(t) = & \alpha(N_1 - N_2) + 2\Delta\omega + \kappa \left(\frac{A_2(t-\tau)}{A_1(t)} \sin(\phi_2(t-\tau) - \phi_1(t) - \theta\tau) \right. \\ & \left. - \kappa \left(\frac{A_1(t-\tau)}{A_2(t)} \sin(\phi_1(t-\tau) - \phi_2(t) - \theta\tau) \right) \right). \end{aligned} \quad (9)$$

To establish the connection to PT symmetry in the SCL system, we follow an approach similar to the one above and assume that carrier inversion is negligible, i.e., $N_{1,2} = 0$, and that the time delay is zero, i.e., $\tau = 0$. For identical lasers, with $A_1 = A_2$, the time evolution of the phase difference simplifies to the Adler equation [33],

$$\dot{\Delta\phi}(t) = 2\Delta\omega - 2\kappa \sin(\Delta\phi(t)). \quad (10)$$

The above equation suggests that the phase locking condition is given by $|\Delta\omega| < \kappa$, which is the exact same condition that governs the PT threshold for a system with zero time-delay. This analysis establishes that the regime where the phase locking condition is satisfied is also the regime wherein the eigenvalues of the effective Hamiltonian are real, while the regime in which the phases are unlocked is the regime where the eigenvalues of the effective Hamiltonian are complex. This analysis, within the assumptions of neglecting population dynamics and zero time-delay, establishes the equivalence of the PT threshold and the phase locking condition [34].

4. Discussion

We have shown in this work that a pair of mutually coupled semiconductor lasers can serve as a template for investigating parity–time symmetry. The advantage of this system is that one does not need to fabricate a system in which the gain and loss are exactly balanced. As shown in this paper, the gain/loss terms are now replaced by the relative detuning between the lasers, which are identically equal and opposite in sign for a pair of SCLs. Since the optical frequency of an SCL is easily controlled via temperature or injection current, and as the coupling between the two lasers can also be easily controlled and measured, the coupled SCL system offers advantages over other PT-symmetric systems where to alter the coupling, one needs to fabricate a new system. Our work has shown how the rate equation model that describes the coupled SCL system can be reduced to a simple 2×2 effective Hamiltonian, which is identical to the typical PT-symmetric dimer. We then showed that the predictions of this simple effective Hamiltonian are reproduced by the full rate equation model, despite the latter having additional complexities such as population dynamics and gain nonlinearities. The model has some limitations, such as the assumption that both lasers operate on a single longitudinal model and that the coupling is weak. However, our results indicate that the features of PT symmetry are very robust and still evident in the rate equation model.

Among possibilities for further exploration, the SCL model for PT symmetry allows the inclusion of quantum noise due to spontaneous emission, as well shot noise in the carrier inversion. In the context of SCL dynamics, there are instances where noise plays a critical role, and it is, a priori, difficult to know when it will be important. Typically, it is only during a comparison of experiments and numerical simulations that the influence of noise is revealed. The effect of noise can be studied by augmenting Equation (2) with appropriate Langevin terms to account for spontaneous emission and shot noise in the inversion. The PT-symmetric SCL system is different from other PT systems since population dynamics are inextricably intertwined with the intensity dynamics. To get a handle on the properties of the system, it is instructive

to look at the populations and how they influence the intensities of the lasers. Our system allows us to make the coupling term purely real, complex, or purely imaginary, thereby offering further richness in the parameter space for probing the properties of the system. In summary, the use of commercially available, low-cost SCLs means that our PT system is flexible enough to add additional oscillators, whereas other systems do not offer this simplicity of extension since each additional oscillator requires fabrication of the entire array from scratch.

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