COMBINATORICS PROBLEMS: A CONSTRUCTIVE RESOURCE FOR FINDING VOLUMES OF FRACTIONAL DIMENSION?

Erik Tillema
Indiana University
etillema@indiana.edu

Jinqing Liu
Indiana University
jinqliu@iu.edu

Pavneet Kaur Bharaj
Indiana University
pkbharaj@iu.edu

Volumes of fractional dimension are a 7th grade standard in the Common Core State Standards in Mathematics (CCSS-M). Relatively little research has been conducted on how students' reason about volumes with fractional dimension. To address this dearth of research, we designed a teaching experiment based on a central conjecture that combinatorics problems could be a constructive resource in the development of volumes with fractional dimension. In this paper, we outline the reason for this conjecture. In our presentation, we elaborate on this conjecture by providing four cases of how pre-service secondary teachers (PSSTs) reasoned first with combinatorics problems, and then with volumes of fractional dimension. One contribution of the study is it expands and combines in novel ways empirically grounded theoretical constructs.

Keywords: Fraction Scheme, Combinatorics Problems, Volume, Multiplicative Concepts

Fractions and rational number knowledge is regarded as one of the most challenging domains at both the elementary and secondary levels (e.g., Behr, Wachsmuth, Post, & Lesh, 1984; Hackenberg, 2013; Norton, Wilkins, & Xu, 2018; Steffe, 2003) in part because many problems involve coordinating multiple units and multiple wholes (Lamon, 2007; Steffe & Olive, 2010). At the same time, the results of large-scale assessments (e.g., NAEP, TIMSS) also indicate that geometry and measurement are complex for both elementary and secondary students (Battista, 2007; Smith & Barrett, 2018). Fractions and volume initially come together in the Common Core State Standards in Mathematics (CCSS-M) in the 7th grade where students are asked to explore the volume of rectangular prisms with fractional dimensions.

To date, relatively little research has investigated how students reason about volumes of fractional dimension. The majority of research on fractions has focused on reasoning with one-or two-dimensional models (e.g., length, number line, circles, or rectangular area) (Lamon, 2007), while the majority of research on volume has been conducted in whole number contexts (Smith & Barrett, 2018; e.g., Clements, Sarama, & Van Dine, 2017). To address the dearth of research, we designed a teaching experiment for pre-service secondary teachers (PSSTs) whose purposes were to investigate: a. how they reasoned about volumes of fractional dimension; and b. how they would use what they learned to design lessons during their student teaching placement.

A central conjecture of the study was that combinatorics problems could support the PSSTs to develop discrete three-dimensional units as the product of three one-dimensional units (Tillema & Gatza, 2016, 2017), and that doing so would serve as a constructive resource in the development of volumes of fractional dimension. This conjecture was grounded in two results from prior research: a. students find it challenging to establish a "unit cube" as a multiplicative composition of one length, one width, and one height unit (Battista, 2007; Smith & Barrett, 2018); and b. students' multiplicative schemes for discrete quantity can be a constructive resource (rather than distractor) in their construction of fraction schemes (Hackenberg & Tillema, 2009; Steffe & Olive, 2010). The following research question guides this paper: How

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do PSSTs reason about volume of fractional dimension after they have used combinatorics problems to establish 3-D arrays?

Theoretical Framework

Broadly speaking, we use operations, schemes and concepts to provide an account of students' reasoning (Piaget, 1970; Von Glasersfeld, 1995). An operation is a mental action (e.g., iterating or partitioning) where operations are the "motor" of schemes. Schemes consist of three parts, an assimilated situation, an activity, and a result. The assimilatory mechanism of a scheme involves a person making an interpretation of a problem situation; the activity of a scheme entails a person using operations that transform the assimilated situation to the result. A person has constructed a concept when he or she no longer has to implement the activity of a scheme to take the result of the scheme as given in a problem situation.

Multiplicative Concepts, Volume, and Combinatorial Problem Situations

To investigate students' reasoning about volume, we build on Hackenberg's (2015) work in which she identifies three qualitatively distinct multiplicative concepts students use (see also, Steffe, 1992); the primary difference among each multiplicative concept is the number of embedded units that students take as a given prior to reasoning about problem situations. Cullen et al. (2017) have used these differences in their analysis of how students reason about volume problems involving whole numbers. They found that some students conceptualize volume as composed of individual unit cubes (i.e., units of one, akin to Hackenberg's first multiplicative concept); other students conceptualize volume as composed of rows of units cubes (i.e., treat five unit cubes as one row, akin to Hackenberg's second multiplicative concept); while other students conceptualize volume as composed of layers (i.e., treat four rows of five cubes as one layer, akin to Hackenberg's third multiplicative concept). Cullen and colleagues (2017) have also stressed the importance of students establishing a spatial structure in their construction of volumes of rectangular prisms with whole number dimension (see also, Battista, 2004). Here a spatial structure means a mental organization of volume units so that there are no gaps or overlaps.

The analysis of embedded units in Cullen and colleagues as well as in Hackenberg's work takes the "basic unit" as a unit of one. Smith and Barrett (2018) have pointed out that this elides the issue that individual volume units are themselves supposed to be a multiplicative composition of one length, one width, and one height unit. This distinction is theoretical in nature, but empirical findings suggest that students may not treat volume units as if they are a multiplicative composition of one length, one width, and one height unit (Battista, 2007). The CCSS-M approaches this issue by defining one cubic unit of volume as a unit that is one length unit by one width unit by one height unit. In our work, we have contended that this definition is appropriate for people who have already constructed a cubic unit of volume but is insufficient for supporting a person's construction of it (Tillema & Gatza, 2016). Thus, for this study, we followed an approach that we have used previously with high school students in which we presented them with combinatorics problems to help them establish discrete three-dimensional units from three discrete one-dimensional units (Tillema & Gatza, 2016, 2017). We have called these discrete three-dimensional units triples where, for example, one triple might be an outfit that a student has created in the Outfits Problem.

> *Outfits Problem*. You have 4 shirts, 3 pants, and 2 belts. An outfit is one shirt, one pants, and one belt. How many outfits could you create?

In creating outfits to solve this problem, a student might first pair one shirt with one pants, and then pair the shirt-pant outcome with one belt to create what we call a triple. We call such a unit a triple because it contains three units, but is considered a single unit, one outfit. We consider these operations to entail students in establishing discrete three-dimensional units from three discrete one-dimensional units.

Moreover, in prior studies, high school students have used such problems to establish a spatial structure for 3-D arrays by locating individual triples and regions (e.g., all triples that have the first shirt) (Tillema & Gatza, 2016, 2017). The reason combinatorics problems can be conducive for establishing a spatial structure for 3-D arrays is that they can involve ordering the units within a composite unit (e.g., a first shirt, a second shirt, etc.), and ordering the triples (e.g., shirt in the first position; pants in the second position; belts in the third position). Ordering the units within a composite unit can result in an order for the units along a single axis of an array and ordering the positions in a triple can result in an order for the axes of an array.

Fraction Schemes with Length Quantity

Our analysis also uses two key constructs from research on students' fraction schemes—their construction of iterative fraction schemes and their construction of fraction composition schemes. Steffe and Olive (2010) have found that students using the third multiplicative concept can construct an iterative fraction scheme with length quantities (see also, Hackenberg, 2015). Students who have constructed an iterative fraction scheme (IFS) with length quantities can, for example, partition a length into five parts, disembed (remove from the whole) one of those parts, and iterate it four times to create four fifths as a length. IFS students interpret the relationship between one fifth of a length and four fifths of a length to be multiplicative (four fifths is four times one fifth). Moreover, IFS students are able to establish this multiplicative relationship for improper fractions like six fifths; a length that is six-fifths is six times one fifth. This conceptualization of improper fractions is not easy for students to establish and is a hallmark of students who have constructed an IFS with length quantities (Hackenberg, 2007).

Steffe (2003) identified that a central operation in students' construction of fraction composition schemes with length quantities is recursive partitioning. He defined recursive partitioning as partitioning a partition in service of a non-partitioning goal. For example, a student who engages in recursive partitioning might respond to a request to find one third of one half by partitioning one unit of length into two parts, and then partition each part into three mini-parts. She might then respond that one mini-part is one-sixth because she could iterate it six times to make one unit of length. These constructs were central to our analysis of students' reasoning about volumes of fractional dimension.

Methods and Methodology

The data for this study was collected using teaching experiment methodology (Confrey & LaChance, 2000; Steffe & Thompson 2000). This methodology involves a researcher engaging students in problematic situations aimed at helping them in the construction of schemes and operations (Hackenberg, 2010). A researcher designs initial problems for students based on her prior experience working with students in the domain, and her understanding of prior relevant research. This initial design work is supported by guiding conjectures. Once a researcher begins interactions with students her goal is to develop working models of students' reasoning (Steffe & Weigel, 1994). Throughout the experiment, the researcher refines these working models by establishing conjectures about the affordances and constraints of students' current reasoning and designing problem situations to test these conjectures. A researcher can establish these conjectures both in between teaching episodes and in the moment of interacting with students.

The broad goal of the teaching experiment was to investigate how the PSSTs developed a combinatorial understanding of common algebraic identities (e.g., $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$). As part of supporting this work the researchers used a quantitative approach in which students represented sets of outcomes in combinatorics problems as 3-D arrays, and subsequently represented volumes of fractional dimension for these algebraic identities. The PSSTs used snap cubes as a representational tool throughout the teaching episodes (Figure 1).

The four participants for the study were enrolled in the second semester of a secondary mathematics methods class at a Midwestern university during the fall of 2018. They were selected based on willingness to participate in regular meetings over the course of the 2018-2019 academic year. During Fall 2018, they worked as a pair in 13 teaching episodes; each episode lasted for 60-90 minutes and was video recorded using three cameras—one to capture the interaction between the researcher and the PSSTs and two to capture student work.

For data analysis, all three authors watched relevant video from the entire data set. They used a four-column table as a guide for analysis. The four-column table included timestamp, brief characterization of 3-D reasoning, low inference description of what happened, and inferential analysis of the PSSTs' reasoning (Saldana, 2013). Each row in the table was a single utterance, a turn of talk, or a period of talk focused around a specific idea. The authors met weekly to discuss ongoing conjectures and reconcile differences in their interpretations of the data. These discussions were aimed at developing internal consistency and coherence with the data, and for consistency and coherence with prior research findings.


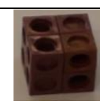

Figure 1: A Single Snap Cube (left) and Eight Snap Cubes Configured as a Volume (right)

Findings

The PSSTs' solution of combinatorics problems revealed differences in the way they established discrete three-dimensional units. All four of PSSTs entered the teaching experiment having constructed a recursive scheme for the solution of three-dimensional combinatorics problems. The recursive nature of their schemes meant that they could take the result of their scheme as input for a further application of their scheme; for example, they were able to take each pants and pair it with all shirts to produce all shirt-pants outcomes, and then take each shirt-pants outcome and pair it with all belts to produce the set of all possible triples.

However, not all of them produced the same multiplicative relationships in this process. One PSST had interiorized a multiplicative relationship that can be symbolized as, for example, 1 x 4 = 4. This can be thought about as one discrete 1-dimensional unit times four discrete 1-dimensional units produces four discrete 2-dimensional units. She could then iterate these operations to produce all of the possible 2-dimensional units in her solution of a problem. Once she produced all possible 2-dimensional units, she took each individual 2-dimensional unit (i.e., a 1 x 1), and paired it with all 1-dimensional discrete units from the third composite unit. This can be symbolized as (1 x 1) x 5 = 5, where the 1 x 1 in the statement symbolizes a pair. These operations were sufficient to establish a row in a 3-D array as a unit of three dimensional discrete

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units. The other three PSSTs established different multiplicative relationships as they created three-dimensional discrete units, which will be discussed in the presentation.

During their solution of combinatorics problems all four PSSTs worked to coordinate the multiplicative relationships that they produced with a spatial structuring for a 3-D array. None of the four entered the teaching experiment able to locate points and regions in three-dimensional space in situations where they needed to coordinate orthogonal “planes” to determine the location of a point or region. This issue, along with how these problems supported students to construct volumes of fractional dimension, will be discussed in the presentation.

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