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HEAD-ON COLLISION OF THE SOLITARY WAVES IN FLUID-FILLED ELASTIC TUBES

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Abstract. In the present work, by employing the field equations given in [15] and the extended PLK method derived in [9], we have studied the head-on collision of solitary waves in arteries. Introducing a set of stretched coordinates which include some unknown functions characterizing the higher order dispersive effects and the trajectory functions to be determined from the removal of possible secularities that might occur in the solution. Expanding these unknown functions and the field variables into power series of the smallness parameter ϵ and introducing the resulting expansions into the field equations we obtained the sets of partial differential equations. By solving these differential equations and imposing the requirements for the removal of possible secularities we obtained the speed correction terms and the trajectory functions. The results of our calculation show that both the evolution equations and the phase shifts resulting from the head-on collision of solitary waves are quite different from those of Xue [15], who employed the incorrect formulation of Su and Mirie [4]. As opposed to the result of previous works on the same subject, in the present work the phase shifts depend on the amplitudes of both colliding waves.

Keywords: Elastic tubes, Solitary waves, Head-on collision, Extended PLK Method.

AMS Subject Classification: 35Q51, 35Q53.

1. Introduction

It is well-known that long-time asymptotic behavior of weakly nonlinear waves in various media can be characterized by the Korteweg-de Vries (KdV) equation [1]. Since the inverse scattering transform (IST) for exactly solving the KdV equation was found by Gardner, Kruskal and Miura [2], the interesting features of the collision between solitary waves had been revealed: When two solitary waves approach closely, they interact, exchange their energies and positions with one another, and, then separate off, regaining their original forms. Throughout the whole process of the collision, the solitary waves are remarkably stable entities preserving their identities throughout the interaction. The unique effect due to the collision is their phase shifts. It is believed that this striking colliding property of solitary waves can only be preserved in integrable systems.

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There are two types of collision of solitary waves: overtaking and head-on collision. According to IST, all the KdV solitary waves travel in the same direction, under the boundary conditions vanishing at infinity [2, 3]; so for overtaking collision between solitary waves, one can use the IST to obtain the overtaking colliding effect of solitary waves. However, for the head-on collision between solitary waves, one must employ the some kind of asymptotic expansion to solve the original field equations. In this regard, for the study of head-on collision problems, a comprehensive analysis had been presented by Su and Mirie [4], in which the Poincare-Lighthill-Kuo (PLK) method had been employed. In their analysis, to determine the unknown trajectory functions they made the statement that "*although certain terms do not cause any secularity at this order but they will cause secularity at the higher order expansion, therefore, those terms must vanish*". Utilizing the implications of this statement several researchers studied the head-on collision of solitary waves in various media [5-8]. Unfortunately, in our previous work [9] we showed that the terms mentioned in their work do not cause any secularity in the solution.

The measurement [10] for the simultaneous changes in amplitudes and form of the flow and pressure waves at five sites from the ascending aorta to the saphenous artery in dog shown that the pulsatile character of the blood wave is soliton-like and it suggests a possible interpretation in terms of solitons. The blood flow in arteries can be considered as an incompressible fluid flowing in a thin non-linear elastic tube. Theoretical investigations for the blood waves by weakly nonlinear theory have been developed by [11-14]. It is shown that the dynamics of the blood waves are governed by the KdV or modified KdV equations. The solitary wave model gives a reasonable explanation for the peaking and steepening of pulsatile waves in arteries. Head-on collision of solitary waves in fluid-filled elastic tubes (a model for arteries) had been studied by several researchers [15-17], in all of which the method proposed by Su and Mirie [4] have been employed. Unfortunately, as stated before, the statement made by Su and Mirie is incorrect, accordingly, all the results reported in [15-17] are not acceptable.

In the present work, by employing the field equations given in [15] and the extended PLK method developed in [9], we have studied the head-on collision of solitary waves in arteries. Introducing a set of stretched coordinates which include some unknown functions characterizing the higher order dispersive effects and the trajectory functions to be determined from the removal of possible secularities that might occur in the solution. Expanding these unknown functions and the field variables into power series of the smallness parameter ϵ and introducing the resulting expansions into the field equations we obtained the sets of partial differential equations governing the coefficients of the series. By solving these differential equations and imposing the non-secularity conditions in the solution we obtained various evolution equations. By seeking a progressive wave solution to these evolution equations we obtained the speed correction terms and the trajectory functions. The results of our calculation show that both the evolution equations and the phase shifts resulting from the head-on collision of solitary waves are quite different from those of Xue [15], who employed the incorrect formulation of Su and Mirie [4]. As opposed to the result of previous works on the same subject, in the present work the phase shifts depend on the amplitudes of both colliding waves. It is further observed that the order of the trajectory functions is ϵ^2, rather than ϵ.

2. Basic Equations

To study the head-on collision of the blood solitary waves, we assume that the blood waves propagate in a one- dimensional elastic tube, which is deemed to be a model for large artery, filled with an incompressible inviscid fluid, which is considered to be a simple model for blood. We also assume that the arteries are circularly cylindrical homogeneous tube with non-linear elasticity. Hence, the averaged non-dimensional equations of motion of the tube and the fluid may be given by [18, 19]

$$\frac{\partial S}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial}{\partial x}(Su) = 0, \tag{1}$$

$$\frac{\partial u}{\partial t} + \frac{\partial \pi}{\partial x} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0, \tag{2}$$

$$\pi = \frac{2}{2+S}\frac{\partial^2 S}{\partial t^2} + \frac{2S(2+\alpha S)}{(2+S)^2}, \tag{3}$$

where x and t are the non-dimensional space and time variables, S is the change in the cross-sectional area of the tube, u and π are the axial velocity and the pressure of the fluid body, respectively, and α characterizes the non-linearity of the tube material. For the detail of the derivation of the equations (1)-(3) the readers are referred to the references [18-19].

3. Extended PLK Method

Motivated with the results found in [9], for our future purposes, we introduce the following stretched coordinates

$$\begin{aligned} &\epsilon^{\frac{1}{2}}(x-t) = \xi + \epsilon p(\tau) + \epsilon^2 P(\xi,\eta,\tau), \\ &\epsilon^{\frac{1}{2}}(x+t) = \eta + \epsilon q(\tau) + \epsilon^2 Q(\xi,\eta,\tau), \\ &\epsilon^{3/2} t = \tau, \end{aligned} \tag{4}$$

where ϵ is the smallness parameter measuring the weakness of dispersion and nonlinearity, $p(\tau)$ and $q(\tau)$ are two unknown functions characterizing the higher order dispersive effects, $P(\xi,\eta,\tau)$ and $Q(\xi,\eta,\tau)$ are two unknown functions characterizing the trajectory functions. Then, the following differential relations hold true

$$\begin{aligned} \frac{\partial}{\partial x} =& \frac{\epsilon^{\frac{1}{2}}}{D}\left\{\left[1+\epsilon^2\left(\frac{\partial Q}{\partial \eta} - \frac{\partial P}{\partial \eta}\right)\right]\frac{\partial}{\partial \xi} + \left[1+\epsilon^2\left(\frac{\partial P}{\partial \xi} - \frac{\partial Q}{\partial \xi}\right)\right]\frac{\partial}{\partial \eta}\right\}, \\ \frac{\partial}{\partial t} =& \epsilon^{1/2}\left\{\epsilon\frac{\partial}{\partial \tau} - \frac{1}{D}\left[1+\epsilon^2\left(\frac{dp}{d\tau} + \frac{\partial P}{\partial \eta} + \frac{\partial Q}{\partial \eta}\right) + \epsilon^3\frac{\partial P}{\partial \tau} + \epsilon^4\left(\frac{dp}{d\tau}\frac{\partial Q}{\partial \eta} - \frac{dq}{d\tau}\frac{\partial P}{\partial \eta}\right)\right.\right. \\ &+\epsilon^5\left.\left(\frac{\partial P}{\partial \tau}\frac{\partial Q}{\partial \eta} - \frac{\partial Q}{\partial \tau}\frac{\partial P}{\partial \eta}\right)\right]\frac{\partial}{\partial \xi} + \frac{1}{D}\left[1+\epsilon^2\left(-\frac{dq}{d\tau} + \frac{\partial P}{\partial \xi} + \frac{\partial Q}{\partial \xi}\right) - \epsilon^3\frac{\partial Q}{\partial \tau}\right. \\ &\left.\left.+\epsilon^4\left(\frac{dp}{d\tau}\frac{\partial Q}{\partial \xi} - \frac{dq}{d\tau}\frac{\partial P}{\partial \xi}\right) + \epsilon^5\left(\frac{\partial P}{\partial \tau}\frac{\partial Q}{\partial \xi} - \frac{\partial Q}{\partial \tau}\frac{\partial P}{\partial \xi}\right)\right]\frac{\partial}{\partial \eta}\right\}, \end{aligned} \tag{5}$$

where D is defined by

$$D = \left(1+\epsilon^2\frac{\partial P}{\partial \xi}\right)\left(1+\epsilon^2\frac{\partial Q}{\partial \eta}\right) - \epsilon^4\frac{\partial P}{\partial \eta}\frac{\partial Q}{\partial \xi}. \tag{6}$$

We assume that the field quantities can be expanded into asymptotic series in ϵ as

$$\begin{aligned} &S = \epsilon S_1 + \epsilon^2 S_2 + \epsilon^3 S_3 + \dots\ , \\ &u = \epsilon u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 + \dots\ , \end{aligned}$$

$$
\begin{aligned}
&p(\tau) = p_0(\tau) + \epsilon p_1(\tau) + \epsilon^2 p_2(\tau) + \epsilon^3 p_3(\tau) + \dots \ , \\
&q(\tau) = q_0(\tau) + \epsilon q_1(\tau) + \epsilon^2 q_2(\tau) + \epsilon^3 q_3(\tau) + \dots \ , \\
&P(\xi,\eta,\tau) = P_0(\xi,\eta,\tau) + \epsilon P_1(\xi,\eta,\tau) + \dots \ , \\
&Q(\xi,\eta,\tau) = Q_0(\xi,\eta,\tau) + \epsilon Q_1(\xi,\eta,\tau) + \dots \ .
\end{aligned} \tag{7}
$$

Introducing (5) and (7) into equations (1)-(3) and setting the coefficients of like powers of ϵ equal to zero the following sets of equations are obtained:

$\mathcal{O}(\epsilon)$ *equations:*

$$
\frac{\partial S_1}{\partial \eta} - \frac{\partial S_1}{\partial \xi} + \frac{\partial u_1}{\partial \eta} + \frac{\partial u_1}{\partial \xi} = 0,
$$

$$
\frac{\partial \pi_1}{\partial \eta} + \frac{\partial \pi_1}{\partial \xi} + \frac{\partial u_1}{\partial \eta} - \frac{\partial u_1}{\partial \xi} = 0, \quad \pi_1 = S_1, \tag{8}
$$

$\mathcal{O}(\epsilon^2)$ *equations:*

$$
\frac{\partial S_2}{\partial \eta} - \frac{\partial S_2}{\partial \xi} + \frac{\partial u_2}{\partial \eta} + \frac{\partial u_2}{\partial \xi} + \frac{\partial S_1}{\partial \tau} + \frac{\partial}{\partial \eta}(S_1 u_1) + \frac{\partial}{\partial \xi}(S_1 u_1) = 0,
$$

$$
\frac{\partial \pi_2}{\partial \eta} + \frac{\partial \pi_2}{\partial \xi} + \frac{\partial u_2}{\partial \eta} - \frac{\partial u_2}{\partial \xi} + \frac{\partial u_1}{\partial \tau} + \frac{1}{2}\frac{\partial}{\partial \eta}(u_1^2) + \frac{1}{2}\frac{\partial}{\partial \xi}(u_1^2) = 0,
$$

$$
\pi_2 = S_2 - 2\frac{\partial^2 S_1}{\partial \xi \partial \eta} + \frac{\partial^2 S_1}{\partial \eta^2} + \frac{\partial^2 S_1}{\partial \xi^2} + \left(\frac{\alpha - 2}{2}\right) S_1^2, \tag{9}
$$

$\mathcal{O}(\epsilon^3)$ *equations:*

$$
\begin{aligned}
&\frac{\partial S_3}{\partial \eta} - \frac{\partial S_3}{\partial \xi} + \frac{\partial u_3}{\partial \eta} + \frac{\partial u_3}{\partial \xi} + \frac{\partial S_2}{\partial \tau} + \frac{\partial}{\partial \eta}(S_1 u_2) + \frac{\partial}{\partial \xi}(S_1 u_2) + \frac{\partial}{\partial \eta}(u_1 S_2) + \frac{\partial}{\partial \xi}(u_1 S_2) \\
&- \frac{dp_0}{d\tau}\frac{\partial S_1}{\partial \xi} - \frac{dq_0}{d\tau}\frac{\partial S_1}{\partial \eta} + \frac{\partial P_0}{\partial \xi}\frac{\partial}{\partial \eta}(u_1 + S_1) - \frac{\partial P_0}{\partial \eta}\frac{\partial}{\partial \xi}(u_1 + S_1) - \frac{\partial Q_0}{\partial \xi}\frac{\partial}{\partial \eta}(u_1 - S_1) \\
&+ \frac{\partial Q_0}{\partial \eta}\frac{\partial}{\partial \xi}(u_1 - S_1) = 0,
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
&\frac{\partial \pi_3}{\partial \eta} + \frac{\partial \pi_3}{\partial \xi} + \frac{\partial u_3}{\partial \eta} - \frac{\partial u_3}{\partial \xi} + \frac{\partial u_2}{\partial \tau} + \frac{\partial}{\partial \eta}(u_1 u_2) + \frac{\partial}{\partial \xi}(u_1 u_2) - \frac{dp_0}{d\tau}\frac{\partial u_1}{\partial \xi} - \frac{dq_0}{d\tau}\frac{\partial u_1}{\partial \eta} \\
&+ \frac{\partial P_0}{\partial \xi}\frac{\partial}{\partial \eta}(u_1 + \pi_1) - \frac{\partial P_0}{\partial \eta}\frac{\partial}{\partial \xi}(u_1 + \pi_1) + \frac{\partial Q_0}{\partial \xi}\frac{\partial}{\partial \eta}(u_1 - \pi_1) - \frac{\partial Q_0}{\partial \eta}\frac{\partial}{\partial \xi}(u_1 - \pi_1) = 0,
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
\pi_3 =& S_3 + (\alpha - 2) S_1 S_2 + \left(\frac{3 - 2\alpha}{4}\right) S_1^3 - 2\frac{\partial^2 S_2}{\partial \xi \partial \eta} + \frac{\partial^2 S_2}{\partial \eta^2} + \frac{\partial^2 S_2}{\partial \xi^2} - 2\frac{\partial^2 S_1}{\partial \xi \partial \tau} \\
&+ 2\frac{\partial^2 S_1}{\partial \eta \partial \tau} - \frac{1}{2} S_1 \left(\frac{\partial^2 S_1}{\partial \xi^2} + \frac{\partial^2 S_1}{\partial \eta^2} - 2\frac{\partial^2 S_1}{\partial \xi \partial \eta}\right).
\end{aligned} \tag{12}
$$

4. Solution of the field equations

From the solution of the equation set (8) we obtain

$$\begin{aligned} u_1 &= f_1(\xi,\tau) + g_1(\eta,\tau), \\ S_1 &= \pi_1 = f_1(\xi,\tau) - g_1(\eta,\tau), \end{aligned} \tag{13}$$

where $f_1(\xi,\tau)$ and $g_1(\eta,\tau)$ are two unknown functions whose governing equations will be obtained from the higher order perturbation expansion. Introducing (13) into (9) and then adding and subtracting the resulting equations side by side we obtain

$$\begin{aligned} &2\frac{\partial}{\partial\eta}(u_2+S_2) + \left[2\frac{\partial f_1}{\partial\tau} + (\alpha+1)f_1\frac{\partial f_1}{\partial\xi} + \frac{\partial^3 f_1}{\partial\xi^3}\right] + (\alpha-3)g_1\frac{\partial g_1}{\partial\eta} - (\alpha-3)f_1\frac{\partial g_1}{\partial\eta} \\ &- (\alpha-3)\frac{\partial f_1}{\partial\xi}g_1 - \frac{\partial^3 g_1}{\partial\eta^3} = 0, \end{aligned} \tag{14}$$

$$\begin{aligned} &2\frac{\partial}{\partial\xi}(u_2-S_2) - \left[2\frac{\partial g_1}{\partial\tau} + (\alpha+1)g_1\frac{\partial g_1}{\partial\eta} - \frac{\partial^3 g_1}{\partial\eta^3}\right] - (\alpha-3)f_1\frac{\partial f_1}{\partial\xi} + (\alpha-3)g_1\frac{\partial f_1}{\partial\xi} \\ &+ (\alpha-3)f_1\frac{\partial g_1}{\partial\eta} - \frac{\partial^3 f_1}{\partial\xi^3} = 0. \end{aligned} \tag{15}$$

Integrating the equation (14) with respect to η and (15) with respect to ξ we have

$$\begin{aligned} (u_2+S_2) =& -\eta\left[\frac{\partial f_1}{\partial\tau} + \left(\frac{\alpha+1}{2}\right)f_1\frac{\partial f_1}{\partial\xi} + \frac{1}{2}\frac{\partial^3 f_1}{\partial\xi^3}\right] + \frac{1}{2}\frac{\partial^2 g_1}{\partial\eta^2} + \left(\frac{\alpha-3}{2}\right) \\ &\times\left[M(\eta,\tau)\frac{\partial f_1}{\partial\xi} + f_1 g_1 - \frac{g_1^2}{2}\right] + 2f_2(\xi,\tau), \end{aligned} \tag{16}$$

$$\begin{aligned} (u_2-S_2) =& \xi\left[\frac{\partial g_1}{\partial\tau} + \left(\frac{\alpha+1}{2}\right)g_1\frac{\partial g_1}{\partial\eta} - \frac{1}{2}\frac{\partial^3 g_1}{\partial\eta^3}\right] + \frac{1}{2}\frac{\partial^2 f_1}{\partial\xi^2} + \left(\frac{\alpha-3}{2}\right) \\ &\times\left[-N(\xi,\tau)\frac{\partial g_1}{\partial\eta} - f_1 g_1 + \frac{f_1^2}{2}\right] + 2g_2(\eta,\tau), \end{aligned} \tag{17}$$

where $f_2(\xi,\tau)$ and $g_2(\eta,\tau)$ are new unknown functions, $M(\eta,\tau)$ and $N(\xi,\tau)$ are defined by

$$M(\eta,\tau) = \int^{\eta} g_1(\eta',\tau)d\eta', \qquad N(\xi,\tau) = \int^{\xi} f_1(\xi',\tau)d\xi'. \tag{18}$$

As is seen from equations (16) and (17) the terms proportional to ξ and η cause to secularity; therefore, the coefficients of them must vanish, which yields

$$\frac{\partial f_1}{\partial\tau} + \left(\frac{\alpha+1}{2}\right)f_1\frac{\partial f_1}{\partial\xi} + \frac{1}{2}\frac{\partial^3 f_1}{\partial\xi^3} = 0, \tag{19}$$

$$\frac{\partial g_1}{\partial\tau} + \left(\frac{\alpha+1}{2}\right)g_1\frac{\partial g_1}{\partial\eta} - \frac{1}{2}\frac{\partial^3 g_1}{\partial\eta^3} = 0. \tag{20}$$

Based on the remarkable statement of Su and Mirie [4], given in the Introduction of the present work, Xue [15] stated that the terms $M(\eta,\tau)\partial f_1/\partial\xi$ and $N(\xi,\tau)\partial g_1/\partial\eta$ appearing in equations (16) and (17) do not cause any secularity at this order but it will cause secularity in the next order equations; therefore, there should be some terms of order ϵ in the trajectory functions to eliminate these secularities. As will be shown in the solution of the next order equations these terms do not cause any secularity. It is that reason, in the present work we assumed that the order of the trajectory function is ϵ^2 rather than ϵ.

Then from the solution of equations (16) and (17) we obtain u_2 and S_2 as

$$\begin{aligned} u_2 =& f_2(\xi,\tau) + g_2(\eta,\tau) + \left(\frac{\alpha-3}{4}\right)\left[M(\eta,\tau)\frac{\partial f_1}{\partial \xi} - N(\xi,\tau)\frac{\partial g_1}{\partial \eta} + \frac{1}{2}\left(f_1^2 - g_1^2\right)\right] \\ &+ \frac{1}{4}\left(\frac{\partial^2 f_1}{\partial \xi^2} + \frac{\partial^2 g_1}{\partial \eta^2}\right), \\ S_2 =& f_2(\xi,\tau) - g_2(\eta,\tau) + \left(\frac{\alpha-3}{4}\right)\left[M(\eta,\tau)\frac{\partial f_1}{\partial \xi} + N(\xi,\tau)\frac{\partial g_1}{\partial \eta} + 2f_1 g_1 \right. \\ &\left. - \frac{1}{2}\left(f_1^2 + g_1^2\right)\right] - \frac{1}{4}\left(\frac{\partial^2 f_1}{\partial \xi^2} - \frac{\partial^2 g_1}{\partial \eta^2}\right). \end{aligned} \tag{21}$$

The evolution equations (19) and (20) are the conventional Korteweg-de Vries equations, which are different from those of Xue [15], who employed the same set of tube-fluid equations. These evolution equations admit the solitary wave solution of the form

$$\begin{aligned} f_1 &= A\,\mathrm{sech}^2\,\zeta_+, \quad \zeta_+ = \left[\frac{(\alpha+1)A}{12}\right]^{1/2}\left(\xi - \frac{(\alpha+1)}{6}A\tau\right), \\ g_1 &= -B\,\mathrm{sech}^2\,\zeta_-, \quad \zeta_- = \left[\frac{(\alpha+1)B}{12}\right]^{1/2}\left(\eta + \frac{(\alpha+1)}{6}B\tau\right), \end{aligned} \tag{22}$$

where A and B are constant amplitudes of the waves.

For the type of solutions given in (22) the functions $M(\eta,\tau)$ and $N(\xi,\tau)$ will be of the form $\tanh \zeta_\pm$. The integral of them leads to secularities as $\xi(\eta) \to \pm\infty$.

Substituting (13) and (21) into the set of equations (10)-(12), then adding and subtracting equations (10) and (11)side by side, we obtain

$$\begin{aligned} &2\frac{\partial}{\partial \eta}(u_3 + S_3) + 2\frac{\partial f_2}{\partial \tau} + (\alpha+1)\frac{\partial}{\partial \xi}(f_1 f_2) + \frac{\partial^3 f_2}{\partial \xi^3} + \frac{(\alpha+4)}{2}f_1\frac{\partial^3 f_1}{\partial \xi^3} + \frac{(4\alpha+11)}{2} \\ &\times \frac{\partial f_1}{\partial \xi}\frac{\partial^2 f_1}{\partial \xi^2} - \frac{3}{8}(\alpha^2 - 2\alpha + 3)f_1^2\frac{\partial f_1}{\partial \xi} + \frac{3}{4}\frac{\partial^5 f_1}{\partial \xi^5} - 2\frac{dp_0}{d\tau}\frac{\partial f_1}{\partial \xi} - \frac{(\alpha-3)^2}{4}\left(\frac{\partial f_1}{\partial \xi}\frac{\partial g_1}{\partial \eta}\right. \\ &\left. + \frac{\partial f_1^2}{\partial \xi^2}g_1\right)M - (\alpha-3)\left(\frac{\partial f_1}{\partial \xi}g_2 + f_1\frac{\partial g_2}{\partial \eta} - \frac{\partial}{\partial \eta}(g_1 g_2) + f_2\frac{\partial g_1}{\partial \eta}\right) - \frac{\partial^3 g_2}{\partial \eta^3} \\ &- \frac{(4\alpha^2 - 14\alpha + 15)}{4}\frac{\partial f_1}{\partial \xi}g_1^2 - \left((\alpha-3)\frac{\partial f_2}{\partial \xi} - \frac{(5\alpha^2 - 10\alpha + 3)}{4}f_1\frac{\partial f_1}{\partial \xi} - \frac{(\alpha-1)}{4}\right. \\ &\left. \times \frac{\partial^3 f_1}{\partial \xi^3}\right)g_1 - \frac{1}{2}\frac{\partial^5 g_1}{\partial \eta^5} - 4\frac{\partial P_0}{\partial \eta}\frac{\partial f_1}{\partial \xi} - \frac{(\alpha-3)}{4}\left((\alpha-3)\frac{\partial}{\partial \eta}\left(g_1\frac{\partial g_1}{\partial \eta}\right) - (\alpha-3)\frac{\partial f_1}{\partial \xi}\right. \\ &\left. \times \frac{\partial g_1}{\partial \eta} - \frac{\partial^4 g_1}{\partial \eta^4} - (\alpha-3)f_1\frac{\partial^2 g_1}{\partial \eta^2}\right)N + \frac{(5\alpha^2 - 10\alpha + 3)}{8}g_1^2\frac{\partial g_1}{\partial \eta} + \frac{(3\alpha-3)}{4}f_1\frac{\partial^3 g_1}{\partial \eta^3} \\ &- \frac{(7\alpha^2 - 22\alpha + 21)}{8}\left(2f_1 g_1\frac{\partial g_1}{\partial \eta} - f_1^2\frac{\partial g_1}{\partial \eta}\right) - \frac{(\alpha-5)}{4}\frac{\partial^2 f_1}{\partial \xi^2}\frac{\partial g_1}{\partial \eta} + \frac{3}{2}\frac{\partial f_1}{\partial \xi}\frac{\partial^2 g_1}{\partial \eta^2} \\ &+ \frac{5}{2}\frac{\partial}{\partial \eta}\left(g_1\frac{\partial^2 g_1}{\partial \eta^2}\right) + \frac{(5\alpha+9)}{8}\frac{\partial}{\partial \eta}\left[\left(\frac{\partial g_1}{\partial \eta}\right)^2\right] = 0, \end{aligned} \tag{23}$$

$$2\frac{\partial}{\partial \xi}(u_3 - S_3) - 2\frac{\partial g_2}{\partial \tau} - (\alpha+1)\frac{\partial}{\partial \eta}(g_1 g_2) + \frac{\partial^3 g_2}{\partial \eta^3} - \frac{(\alpha+4)}{2}g_1\frac{\partial^3 g_1}{\partial \eta^3} - \frac{(4\alpha+11)}{2}$$

$$\times \frac{\partial g_1}{\partial \eta}\frac{\partial^2 g_1}{\partial \eta^2} - \frac{3}{8}(\alpha^2-2\alpha+3)g_1^2\frac{\partial g_1}{\partial \eta} + \frac{3}{4}\frac{\partial^5 g_1}{\partial \eta^5} + 2\frac{dq_0}{d\tau}\frac{\partial g_1}{\partial \eta} - \frac{(\alpha-3)^2}{4}\left(\frac{\partial f_1}{\partial \xi}\frac{\partial g_1}{\partial \eta}\right.$$
$$\left.+\frac{\partial g_1^2}{\partial \eta^2}f_1\right)N + (\alpha-3)\left(\frac{\partial g_1}{\partial \eta}f_2 + g_1\frac{\partial f_2}{\partial \xi} - \frac{\partial}{\partial \xi}(f_1 f_2) + g_2\frac{\partial f_1}{\partial \xi}\right) - \frac{\partial^3 f_2}{\partial \xi^3}$$
$$-\frac{(4\alpha^2-14\alpha+15)}{4}\frac{\partial g_1}{\partial \eta}f_1^2 + \left((\alpha-3)\frac{\partial g_2}{\partial \eta} + \frac{(5\alpha^2-10\alpha+3)}{4}g_1\frac{\partial g_1}{\partial \eta} - \frac{(\alpha-1)}{4}\right.$$
$$\left.\times\frac{\partial^3 g_1}{\partial \eta^3}\right)f_1 - \frac{1}{2}\frac{\partial^5 f_1}{\partial \xi^5} - 4\frac{\partial Q_0}{\partial \xi}\frac{\partial g_1}{\partial \eta} - \frac{(\alpha-3)}{4}\left((\alpha-3)\frac{\partial}{\partial \xi}\left(f_1\frac{\partial f_1}{\partial \xi}\right) - (\alpha-3)\frac{\partial f_1}{\partial \xi}\right.$$
$$\left.\times\frac{\partial g_1}{\partial \eta} + \frac{\partial^4 f_1}{\partial \xi^4} - (\alpha-3)g_1\frac{\partial^2 f_1}{\partial \xi^2}\right)M + \frac{(5\alpha^2-10\alpha+3)}{8}f_1^2\frac{\partial f_1}{\partial \xi} - \frac{(3\alpha-3)}{4}g_1\frac{\partial^3 f_1}{\partial \xi^3}$$
$$-\frac{(7\alpha^2-22\alpha+21)}{8}\left(2g_1 f_1\frac{\partial f_1}{\partial \xi} - g_1^2\frac{\partial f_1}{\partial \xi}\right) + \frac{(\alpha-5)}{4}\frac{\partial^2 g_1}{\partial \eta^2}\frac{\partial f_1}{\partial \xi} - \frac{3}{2}\frac{\partial g_1}{\partial \eta}\frac{\partial^2 f_1}{\partial \xi^2}$$
$$-\frac{5}{2}\frac{\partial}{\partial \xi}\left(f_1\frac{\partial^2 f_1}{\partial \xi^2}\right) - \frac{(5\alpha+9)}{8}\frac{\partial}{\partial \xi}\left[\left(\frac{\partial f_1}{\partial \xi}\right)^2\right] = 0. \tag{24}$$

Integrating (23) with respect to η and (24) with respect to ξ we obtain

$$2(u_3+S_3) + \eta\left(2\frac{\partial f_2}{\partial \tau} + (\alpha+1)\frac{\partial}{\partial \xi}(f_1 f_2) + \frac{\partial^3 f_2}{\partial \xi^3} + \frac{(\alpha+4)}{2}f_1\frac{\partial^3 f_1}{\partial \xi^3} + \frac{(4\alpha+11)}{2}\right.$$
$$\left.\times\frac{\partial f_1}{\partial \xi}\frac{\partial^2 f_1}{\partial \xi^2} - \frac{3}{8}(\alpha^2-2\alpha+3)f_1^2\frac{\partial f_1}{\partial \xi} + \frac{3}{4}\frac{\partial^5 f_1}{\partial \xi^5} - 2\frac{dp_0}{d\tau}\frac{\partial f_1}{\partial \xi}\right) - \frac{(\alpha-3)^2}{4}$$
$$\times\left(\frac{\partial f_1}{\partial \xi}\int^{\eta}\left(\frac{\partial g_1}{\partial \eta}M\right)d\eta' + \frac{\partial f_1^2}{\partial \xi^2}\int^{\eta}(g_1 M)d\eta'\right) - (\alpha-3)\frac{\partial f_1}{\partial \xi}\int^{\eta} g_2 d\eta' - (\alpha-3)$$
$$\times(f_1 g_2 - g_1 g_2 + f_2 g_1) - \frac{\partial^2 g_2}{\partial \eta^2} - \frac{(4\alpha^2-14\alpha+15)}{4}\frac{\partial f_1}{\partial \xi}\int^{\eta} g_1^2 d\eta' - \left((\alpha-3)\frac{\partial f_2}{\partial \xi}\right.$$
$$\left.-\frac{(\alpha-1)}{4}\frac{\partial^3 f_1}{\partial \xi^3} - \frac{(5\alpha^2-10\alpha+3)}{4}f_1\frac{\partial f_1}{\partial \xi}\right)M - \frac{1}{2}\frac{\partial^4 g_1}{\partial \eta^5} - 4P_0\frac{\partial f_1}{\partial \xi}$$
$$-\frac{(\alpha-3)}{4}\left((\alpha-3)g_1\frac{\partial g_1}{\partial \eta} - \frac{\partial^3 g_1}{\partial \eta^3} - (\alpha-3)\frac{\partial f_1}{\partial \xi}g_1 - (\alpha-3)f_1\frac{\partial g_1}{\partial \eta}\right)N$$
$$+\frac{(5\alpha^2-10\alpha+3)}{24}g_1^3 + \frac{(3\alpha-3)}{4}f_1\frac{\partial^2 g_1}{\partial \eta^2} - \frac{(7\alpha^2-22\alpha+21)}{8}\left(f_1 g_1^2 - f_1^2 g_1\right)$$
$$-\frac{(\alpha-5)}{4}\frac{\partial^2 f_1}{\partial \xi^2}g_1 + \frac{3}{2}\frac{\partial f_1}{\partial \xi}\frac{\partial g_1}{\partial \eta} + \frac{5}{2}g_1\frac{\partial^2 g_1}{\partial \eta^2} + \frac{(5\alpha+9)}{8}\left(\frac{\partial g_1}{\partial \eta}\right)^2 = 2f_3(\xi,\tau), \tag{25}$$

$$2(u_3-S_3) + \xi\left(-2\frac{\partial g_2}{\partial \tau} - (\alpha+1)\frac{\partial}{\partial \eta}(g_1 g_2) + \frac{\partial^3 g_2}{\partial \eta^3} - \frac{(\alpha+4)}{2}g_1\frac{\partial^3 g_1}{\partial \eta^3} - \frac{(4\alpha+11)}{2}\right.$$
$$\left.\times\frac{\partial g_1}{\partial \eta}\frac{\partial^2 g_1}{\partial \eta^2} - \frac{3}{8}(\alpha^2-2\alpha+3)g_1^2\frac{\partial g_1}{\partial \eta} + \frac{3}{4}\frac{\partial^5 g_1}{\partial \eta^5} + 2\frac{dq_0}{d\tau}\frac{\partial g_1}{\partial \eta}\right) - \frac{(\alpha-3)^2}{4}$$
$$\times\left(\frac{\partial g_1}{\partial \eta}\int^{\xi}\left(\frac{\partial f_1}{\partial \xi}N\right)d\xi' + \frac{\partial g_1^2}{\partial \eta^2}\int^{\xi}(f_1 N)\,d\xi'\right) + (\alpha-3)\frac{\partial g_1}{\partial \eta}\int^{\xi} f_2 d\xi' + (\alpha-3)$$

$$\times (g_1 f_2 - f_1 f_2 + g_2 f_1) - \frac{\partial^2 f_2}{\partial \xi^2} - \frac{(4\alpha^2 - 14\alpha + 15)}{4}\frac{\partial g_1}{\partial \eta}\int^{\xi} f_1^2 d\xi' + \Bigg((\alpha-3)\frac{\partial g_2}{\partial \eta}$$
$$-\frac{(\alpha-1)}{4}\frac{\partial^3 g_1}{\partial \eta^3} + \frac{(5\alpha^2 - 10\alpha + 3)}{4} g_1 \frac{\partial g_1}{\partial \eta}\Bigg) N - \frac{1}{2}\frac{\partial^4 f_1}{\partial \xi^4} - 4Q_0 \frac{\partial g_1}{\partial \eta}$$
$$-\frac{(\alpha-3)}{4}\left((\alpha-3) f_1 \frac{\partial f_1}{\partial \xi} + \frac{\partial^3 f_1}{\partial \xi^3} - (\alpha-3) f_1 \frac{\partial g_1}{\partial \eta} - (\alpha-3)\frac{\partial f_1}{\partial \xi} g_1\right) M$$
$$+\frac{(5\alpha^2 - 10\alpha + 3)}{24} f_1^3 - \frac{(3\alpha-3)}{4} g_1 \frac{\partial^2 f_1}{\partial \xi^2} - \frac{(7\alpha^2 - 22\alpha + 21)}{8}\left(f_1^2 g_1 - f_1 g_1^2\right)$$
$$+\frac{(\alpha-5)}{4}\frac{\partial^2 g_1}{\partial \eta^2} f_1 - \frac{3}{2}\frac{\partial g_1}{\partial \eta}\frac{\partial f_1}{\partial \xi} - \frac{5}{2} f_1 \frac{\partial^2 f_1}{\partial \xi^2} - \frac{(5\alpha+9)}{8}\left(\frac{\partial f_1}{\partial \xi}\right)^2 = 2 g_3(\eta,\tau), \tag{26}$$

where $f_3(\xi,\tau)$ and $g_3(\eta,\tau)$ are two unknown functions whose evolution equations will be obtained from the next order equations. In order to remove the secularity caused by the terms proportional to ξ and η, the coefficient of η in (25) and the coefficient of ξ in (26) must vanish, which yields

$$\frac{\partial f_2}{\partial \tau} + \frac{(\alpha+1)}{2}\frac{\partial}{\partial \xi}(f_1 f_2) + \frac{1}{2}\frac{\partial^3 f_2}{\partial \xi^3} = \frac{3}{16}(\alpha^2 - 2\alpha + 3) f_1^2 \frac{\partial f_1}{\partial \xi} - \frac{(4\alpha+11)}{4}\frac{\partial f_1}{\partial \xi}\frac{\partial^2 f_1}{\partial \xi^2}$$
$$-\frac{(\alpha+4)}{4} f_1 \frac{\partial^3 f_1}{\partial \xi^3} - \frac{3}{8}\frac{\partial^5 f_1}{\partial \xi^5} + \frac{dp_0}{d\tau}\frac{\partial f_1}{\partial \xi}, \tag{27}$$

$$\frac{\partial g_2}{\partial \tau} + \frac{(\alpha+1)}{2}\frac{\partial}{\partial \eta}(g_1 g_2) - \frac{1}{2}\frac{\partial^3 g_2}{\partial \eta^3} = -\frac{3}{16}(\alpha^2 - 2\alpha + 3) g_1^2 \frac{\partial g_1}{\partial \eta} - \frac{(4\alpha+11)}{4}\frac{\partial g_1}{\partial \eta}\frac{\partial^2 g_1}{\partial \eta^2}$$
$$-\frac{(\alpha+4)}{4} g_1 \frac{\partial^3 g_1}{\partial \eta^3} + \frac{3}{8}\frac{\partial^5 g_1}{\partial \eta^5} + \frac{dq_0}{d\tau}\frac{\partial g_1}{\partial \eta}. \tag{28}$$

As is seen from the equations (25) and (26) the other terms in the expression of u_3 and S_3 do not cause any secularity of the type $\int M(\eta')d\eta'$ and $\int N(\xi')d\xi'$ for this order, but it might have secularities in the next order.

Seeking a progressive wave solution for the equations (27) and (28) of the form $f_2 = f_2(\zeta_+)$, $g_2 = g_2(\zeta_-)$, one obtains

$$f_2 = \frac{1}{24(\alpha+1)}\left[(5\alpha^2 + 11\alpha + 33)2A f_1 - (7\alpha+16)3 f_1^2\right],$$
$$g_2 = \frac{1}{24(\alpha+1)}\left[(5\alpha^2 + 11\alpha + 33)2B g_1 + (7\alpha+16)3 g_1^2\right],$$
$$p_0(\tau) = \frac{A^2}{24}(\alpha+1)^2\tau, \quad q_0(\tau) = -\frac{B^2}{24}(\alpha+1)^2\tau. \tag{29}$$

Here $A^2(\alpha+1)^2/24$ and $-B^2(\alpha+1)^2/24$ correspond to the speed correction terms for the right and left going waves, respectively. By using the above results the following identities can be obtained for the terms involving f_1, N,

$$\frac{\partial^2 f_1}{\partial \xi^2} = \frac{(\alpha+1)}{6}(2A f_1 - 3 f_1^2), \quad \left(\frac{\partial f_1}{\partial \xi}\right)^2 = \frac{(\alpha+1)}{3}(A f_1^2 - f_1^3),$$
$$\int f_1^2 d\xi' = \frac{N}{3}(f_1 + 2A), \quad \int f_1 N d\xi' = -\left(\frac{6}{\alpha+1}\right) f_1,$$
$$\int \left(\frac{\partial f_1}{\partial \xi} N\right) d\xi' = \frac{2N}{3}(f_1 - A), \tag{30}$$

and also similar identities are valid for the terms involving g_1 and M. Then the equations (25) and (26) can be written in the following form

$$\begin{aligned} u_3 + S_3 =& \frac{(7\alpha+16)}{4(\alpha+1)} g_1^3 + \frac{(43\alpha^2+103\alpha+276)}{48(\alpha+1)} B g_1^2 + \frac{(7\alpha^2+15\alpha+35)}{72} B^2 g_1 \\ &+ \frac{(12\alpha^3-37\alpha^2+9\alpha-50)}{24(\alpha+1)} A f_1 g_1 + \frac{(2\alpha^3-7\alpha^2+3\alpha-96)}{24(\alpha+1)} B f_1 g_1 \\ &+ \frac{(4\alpha^3-11\alpha^2-3\alpha-24)}{16(\alpha+1)} f_1 g_1^2 - \frac{(14\alpha^3-41\alpha^2+3\alpha+22)}{16(\alpha+1)} f_1^2 g_1 \\ &+ \Bigg(\frac{(4\alpha^3-5\alpha^2+\alpha-98)}{24(\alpha+1)} A \frac{\partial f_1}{\partial \xi} - \frac{(\alpha^3+\alpha^2-9\alpha+63)}{24(\alpha+1)} B \frac{\partial f_1}{\partial \xi} \\ &- \frac{(4\alpha^3+\alpha^2-11\alpha-44)}{8(\alpha+1)} f_1 \frac{\partial f_1}{\partial \xi} + \frac{(4\alpha^3-11\alpha^2+3\alpha+6)}{16(\alpha+1)} g_1 \frac{\partial f_1}{\partial \xi} \Bigg) M \\ &- \frac{(\alpha-3)}{2} \left(\frac{(\alpha+1)}{12} B \frac{\partial g_1}{\partial \eta} + \frac{(\alpha-3)}{4} \left(f_1 \frac{\partial g_1}{\partial \eta} + g_1 \frac{\partial f_1}{\partial \xi} \right) + g_1 \frac{\partial g_1}{\partial \eta} \right) N \\ &- \frac{3}{4} \frac{\partial f_1}{\partial \xi} \frac{\partial g_1}{\partial \eta} + 2P_0 \frac{\partial f_1}{\partial \xi} + 2 f_3(\xi,\tau), \end{aligned} \tag{31}$$

$$\begin{aligned} u_3 - S_3 =& \frac{(7\alpha+16)}{4(\alpha+1)} f_1^3 - \frac{(43\alpha^2+103\alpha+276)}{48(\alpha+1)} A f_1^2 + \frac{(7\alpha^2+15\alpha+35)}{72} A^2 f_1 \\ &- \frac{(12\alpha^3-37\alpha^2+9\alpha-50)}{24(\alpha+1)} B f_1 g_1 - \frac{(2\alpha^3-7\alpha^2+3\alpha-96)}{24(\alpha+1)} A f_1 g_1 \\ &+ \frac{(4\alpha^3-11\alpha^2-3\alpha-24)}{16(\alpha+1)} f_1^2 g_1 - \frac{(14\alpha^3-41\alpha^2+3\alpha+22)}{16(\alpha+1)} f_1 g_1^2 \\ &+ \Bigg(-\frac{(4\alpha^3-5\alpha^2+\alpha-98)}{24(\alpha+1)} B \frac{\partial g_1}{\partial \eta} + \frac{(\alpha^3+\alpha^2-9\alpha+63)}{24(\alpha+1)} A \frac{\partial g_1}{\partial \eta} \\ &- \frac{(4\alpha^3+\alpha^2-11\alpha-44)}{8(\alpha+1)} g_1 \frac{\partial g_1}{\partial \eta} + \frac{(4\alpha^3-11\alpha^2+3\alpha+6)}{16(\alpha+1)} f_1 \frac{\partial g_1}{\partial \eta} \Bigg) N \\ &+ \frac{(\alpha-3)}{2} \left(\frac{(\alpha+1)}{12} A \frac{\partial f_1}{\partial \xi} - \frac{(\alpha-3)}{4} \left(f_1 \frac{\partial g_1}{\partial \eta} + g_1 \frac{\partial f_1}{\partial \xi} \right) - f_1 \frac{\partial f_1}{\partial \xi} \right) M \\ &+ \frac{3}{4} \frac{\partial f_1}{\partial \xi} \frac{\partial g_1}{\partial \eta} + 2Q_0 \frac{\partial g_1}{\partial \eta} + 2 g_3(\eta,\tau). \end{aligned} \tag{32}$$

As might be seen from equations (31) and (32) these terms appearing in the expressions of u_2 and S_2 do not cause any secularity in the solution of u_3 and S_3. Therefore the statement by Su and Mirie [4] is incorrect. However as we stated before, some of the terms appearing in the expressions of u_3 and S_3 (The equations (31) and (32)) may cause additional secularity in the expressions of u_4 and S_4. There appears to be two types of secularity in the solution of $\mathcal{O}(\epsilon^4)$ equation. As was seen before, the first type of secularity results from the terms proportional to ξ and η which will not be studied more. The second type of secularity occurs from the terms proportional $\int^{\xi} N(\xi',\tau)d\xi'$ and $\int^{\eta} M(\eta',\tau)d\eta'$ as $\xi(\eta) \to \pm\infty$. Here we shall only consider the parts of $\mathcal{O}(\epsilon^4)$ equations leading to $\int^{\eta} M(\eta',\tau)d\eta'$ type of secularity. Similar expressions may be valid for $\int^{\xi} N(\xi',\tau)d\xi'$ type of secularity.

For this purpose we consider the following part of the $\mathcal{O}(\epsilon^4)$ equation

$$\begin{aligned}
&2\frac{\partial}{\partial\eta}(u_4+S_4)+\frac{\partial}{\partial\tau}(u_3+S_3)+(\alpha-2)\frac{\partial}{\partial\xi}(S_1S_3)+\frac{\partial}{\partial\xi}(u_1S_3)+\frac{\partial}{\partial\xi}\left[u_3(u_1+S_1)\right]\\
&+\frac{\partial^3 S_3}{\partial\xi^3}+(\alpha-2)S_2\frac{\partial S_2}{\partial\xi}+u_2\frac{\partial}{\partial\xi}(u_2+S_2)+S_2\frac{\partial u_2}{\partial\xi}-\frac{dp_0}{d\tau}\frac{\partial}{\partial\xi}(u_2+S_2)\\
&-\frac{(6\alpha-9)}{4}\frac{\partial}{\partial\xi}\left(S_1^2S_2\right)-2\frac{\partial^3 S_2}{\partial\xi^2\partial\tau}-\frac{1}{2}\frac{\partial}{\partial\xi}\left(S_1\frac{\partial^2 S_2}{\partial\xi^2}+\frac{\partial^2 S_1}{\partial\xi^2}S_2\right)+\frac{(3\alpha-4)}{2}S_1^3\frac{\partial S_1}{\partial\xi}\\
&+\frac{\partial^3 S_1}{\partial\xi\partial\tau^2}+\frac{1}{4}\frac{\partial}{\partial\xi}\left(S_1^2\frac{\partial^2 S_1}{\partial\xi^2}\right)+\frac{\partial}{\partial\xi}\left(S_1\frac{\partial^2 S_1}{\partial\xi\partial\tau}\right)+2\frac{dp_0}{d\tau}\frac{\partial^3 S_1}{\partial\xi^3}-\frac{dp_1}{d\tau}\frac{\partial}{\partial\xi}(u_1+S_1)=0. \quad (33)
\end{aligned}$$

A similar expression may be given for $2\frac{\partial}{\partial\xi}(u_4-S_4)$ equation. We split (33) into two parts which contain the variables u_3+S_3 and $(u_2,\, S_2,\, u_3-S_3)$, respectively. Then, we obtain:

$$\begin{aligned}
&\frac{\partial}{\partial\tau}(u_3+S_3)+\frac{(\alpha+1)}{4}\frac{\partial}{\partial\xi}[(u_1+S_1)(u_3+S_3)]+\frac{1}{2}\frac{\partial^3}{\partial\xi^3}(u_3+S_3)=\\
&\frac{-(4\alpha^3+\alpha^2-11\alpha-44)}{8}(\alpha+1)\left[\frac{7}{8}f_1^4-\frac{7}{6}Af_1^3+\frac{1}{3}A^2f_1^2\right]M, \quad (34)
\end{aligned}$$

$$\begin{aligned}
&(\alpha-2)S_2\frac{\partial S_2}{\partial\xi}+u_2\frac{\partial}{\partial\xi}(u_2+S_2)+S_2\frac{\partial u_2}{\partial\xi}-\frac{dp_0}{d\tau}\frac{\partial}{\partial\xi}(u_2+S_2)-\frac{(6\alpha-9)}{4}\frac{\partial}{\partial\xi}\left(S_1^2S_2\right)\\
&-2\frac{\partial^3 S_2}{\partial\xi^2\partial\tau}-\frac{1}{2}\frac{\partial}{\partial\xi}\left(S_1\frac{\partial^2 S_2}{\partial\xi^2}+\frac{\partial^2 S_1}{\partial\xi^2}S_2\right)+\frac{(3\alpha-9)}{4}\frac{\partial}{\partial\xi}(S_1S_3)-\frac{1}{2}\frac{\partial^3}{\partial\xi^3}(u_3-S_3)\\
&-\frac{(\alpha-3)}{2}\frac{\partial}{\partial\xi}[(u_1-S_1)S_3]-\frac{(\alpha-3)}{4}\frac{\partial}{\partial\xi}[(u_1+S_1)(u_3-S_3)]=\\
&\frac{(7\alpha^2-5\alpha-48)}{8}(\alpha+1)\left[\frac{7}{8}f_1^4-\frac{7}{6}Af_1^3+\frac{1}{3}A^2f_1^2\right]M. \quad (35)
\end{aligned}$$

As is seen from the last part the integration of equations (34) and (35) with respect to η leads to secularity. In order to remove the secularity, we should set the coefficient of the term $f_1\frac{\partial f_1}{\partial\xi}M$ in u_3+S_3 equal to $-\frac{(7\alpha^2-5\alpha-48)}{8(\alpha+1)}$. Similar expression may be given for $\int^{\xi} N(\xi',\tau)d\xi'$ type of secularities. In order to remove these secularities the trajectory functions should have the following form:

$$\begin{aligned}
P_0&=\frac{(2\alpha^2-5\alpha+2)}{8}f_1(\xi,\tau)M(\eta,\tau),\\
Q_0&=\frac{(2\alpha^2-5\alpha+2)}{8}g_1(\eta,\tau)N(\xi,\tau). \quad (36)
\end{aligned}$$

To obtain the secularities of type η (or ξ) we use the equation (33) to obtain the governing equation for $f_3(\xi,\tau)$. We substitute the field variables into (33) then the terms proportional to η in this equation cause to secularity. In order to remove secularity, the coefficient of η in (33) must vanish, that is

$$\frac{\partial f_3}{\partial\tau}+\frac{(\alpha+1)}{2}\frac{\partial}{\partial\xi}(f_1f_3)+\frac{1}{2}\frac{\partial^3 f_3}{\partial\xi^3}=\frac{\partial S(f_1)}{\partial\xi} \quad (37)$$

where $S(f_1)$ is defined as follows

$$\begin{aligned}S(f_1) =&\frac{(-341\alpha^2-1472\alpha-1536)}{256(\alpha+1)}f_1^4+\left(\frac{(134\alpha^3+1471\alpha^2+4862\alpha+5712)}{576(\alpha+1)}A\right.\\&\left.-\frac{(\alpha+1)(\alpha-3)^2}{16}B\right)f_1^3+\left(\frac{(23\alpha^4-100\alpha^3-479\alpha^2-1700\alpha-2073)}{576(\alpha+1)}A^2\right.\\&\left.+\frac{(\alpha+1)(\alpha-3)^2}{16}AB\right)f_1^2-\frac{5(\alpha+1)^3}{432}A^3f_1+\frac{dp_1}{d\tau}f_1\end{aligned} \tag{38}$$

Seeking a progressive wave solution for the equation (37) of the form $f_3 = f_3(\zeta_+)$, the speed correction term $p_1(\tau)$ for right going wave is found to be

$$p_1(\tau) = \frac{5(\alpha+1)^3}{432}A^3\tau. \tag{39}$$

Similar solution may be given for left going wave and speed correction term may be given by

$$q_1(\tau) = -\frac{5(\alpha+1)^3}{432}B^3\tau. \tag{40}$$

Thus, for this order, the trajectories of the solitary waves become

$$\begin{aligned}\epsilon^{\frac{1}{2}}(x-t) &= \xi+\epsilon p_0(\tau)+\epsilon^2p_1(\tau)+\epsilon^2P_0+\mathcal{O}(\epsilon^3),\\ \epsilon^{\frac{1}{2}}(x+t) &= \eta+\epsilon q_0(\tau)+\epsilon^2q_1(\tau)+\epsilon^2Q_0+\mathcal{O}(\epsilon^3).\end{aligned} \tag{41}$$

To obtain the phase shifts after a head-on collision of solitary waves characterized by A and B are asymptotically far from each other at the initial time ($t = -\infty$), the solitary wave A is at $\xi = 0$, $\eta = -\infty$, and the solitary wave B is at $\eta = 0$, $\xi = +\infty$, respectively. After the collision ($t = +\infty$), the solitary wave B is far to the right of solitary wave A, i.e., the solitary wave A is at $\xi = 0$, $\eta = +\infty$, and the solitary wave B is at $\eta = 0$, $\xi = -\infty$. Using the equations (22) and (36) one can obtain the corresponding phase shifts Δ_A and Δ_B as follows:

$$\begin{aligned}\Delta_A &= \epsilon^{1/2}(x-t)\mid_{\xi=0,\eta=\infty} - \epsilon^{1/2}(x-t)\mid_{\xi=0,\eta=-\infty}\\ &= \epsilon^2\left(\frac{2\alpha^2-5\alpha+2}{8}\right)f_1(0)\int_{-\infty}^{+\infty}g_1(\eta^{'})d\eta^{'} = \epsilon^2\left(\frac{2\alpha^2-5\alpha+2}{8}\right)A\int_{-\infty}^{+\infty}g_1(\eta^{'})d\eta^{'}\\ &= -\epsilon^2\left(\frac{2\alpha^2-5\alpha+2}{4}\right)\left(\frac{12}{\alpha+1}\right)^{1/2}AB^{1/2},\end{aligned} \tag{42}$$

$$\begin{aligned}\Delta_B &= \epsilon^{1/2}(x+t)\mid_{\eta=0,\xi=-\infty} - \epsilon^{1/2}(x+t)\mid_{\eta=0,\xi=\infty}\\ &= -\epsilon^2\left(\frac{2\alpha^2-5\alpha+2}{8}\right)g_1(0)\int_{-\infty}^{+\infty}f_1(\xi^{'})d\xi^{'} = \epsilon^2\left(\frac{2\alpha^2-5\alpha+2}{8}\right)B\int_{-\infty}^{+\infty}f_1(\xi^{'})d\xi^{'}\\ &= \epsilon^2\left(\frac{2\alpha^2-5\alpha+2}{4}\right)\left(\frac{12}{\alpha+1}\right)^{1/2}A^{1/2}B.\end{aligned} \tag{43}$$

Here, as opposed to the results of previous works on the same subject the phase shifts depend on the amplitudes of both waves.

5. Conclusion

Employing the field equations given in [15] and the extended PLK method derived in [9], we have studied the head-on collision of solitary waves in arteries. Introducing a set of stretched coordinates that include some unknown functions characterizing the higher order dispersive effects and the trajectory functions, which are to be determined from the removal of possible secularities that might occur in the solution. Expanding these unknown functions and the field variables into power series of the smallness parameter ϵ and introducing the resulting expansions into the field equations we obtained the sets of partial differential equations. By solving these differential equations and imposing the requirements for the removal of possible secularities we obtained the speed correction terms and the trajectory functions. The results of our calculation show that both the evolution equations and the phase shifts resulting from the head-on collision of solitary waves are quite different from those of Xue [15], who employed the incorrect formulation of Su and Mirie [4]. As opposed to the result of previous works on the same subject, in the present work the phase shifts depend on the amplitudes of both colliding waves.

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