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BOUNDS FOR THE FABER COEFFICIENTS OF CERTAIN CLASSES OF FUNCTIONS ANALYTIC IN AN ELLIPSE

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ABSTRACT. Let Ω be a bounded, simply connected domain in $\mathbf{C}$ with $0 \in \Omega$ and $\partial\Omega$ analytic. Let $S(\Omega)$ denote the class of functions $F(z)$ which are analytic and univalent in Ω with $F(0) = 0$ and $F'(0) = 1$. Let $\{\Phi_n(z)\}_{n=0}^{\infty}$ be the Faber polynomials associated with Ω. If $F(z) \in S(\Omega)$, then $F(z)$ can be expanded in a series of the form

$$F(z) = \sum_{n=0}^{\infty} A_n \Phi_n(z), \quad z \in \Omega$$

in terms of the Faber polynomials. Let

$$E_r = \left\{(x,y) \in \mathbf{R}^2 : \frac{x^2}{(1+(1/r^2))^2} + \frac{y^2}{(1-(1/r^2))^2} < 1\right\},$$

where $r > 1$. In this paper we obtain sharp bounds for the Faber coefficients A_0, A_1 and A_2 of functions $F(z)$ in $S(E_r)$ and in certain related classes.

1. Introduction. Let S denote the class of functions f analytic and univalent in the unit disk $\mathbf{D} = \{z : |z| < 1\}$ such that

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

The Bieberbach conjecture [**2**] asserts that if $f \in S$, then $|a_n| \le n$, $n \ge 2$. This famous conjecture was proved by de Branges [**3**] in 1984.

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It was also shown that equality holds if and only if f is a rotation of the Koebe function

$$k(z) = \frac{z}{(1-z)^2}. \tag{1.2}$$

In this paper, we investigate bounds for the Faber coefficients in domains other than the unit disk $\mathbf{D}$, especially an elliptical domain.

Let Ω be a bounded, simply connected domain in $\mathbf{C}$ with $0 \in \Omega$. Let $g(z)$ be the unique, one-to-one, analytic mapping of $\Delta = \{z : |z| > 1\}$ onto $\mathbf{C} \setminus \overline{\Omega}$ with

$$g(z) = cz + \sum_{n=0}^{\infty} \frac{c_n}{z^n}, \quad c > 0, \quad z \in \Delta. \tag{1.3}$$

Suppose that Ω has capacity 1, so that $c = 1$ in (1.3). The *Faber polynomials*, $\{\Phi_n(z)\}_{n=0}^{\infty}$, associated with Ω, or $g(z)$, are defined by the generating function relation [**5**, p. 218]

$$\frac{\eta g'(\eta)}{g(\eta) - z} = \sum_{n=0}^{\infty} \Phi_n(z)\eta^{-n}, \quad \eta \in \Delta. \tag{1.4}$$

Faber polynomials play an important role in the theory of functions of a complex variable and in approximation theory. On a domain Ω the Faber polynomials, $\{\Phi_n(z)\}_{n=0}^{\infty}$, play a role analogous to that of $\{z^n\}_{n=0}^{\infty}$ in $\mathbf{D}$. If $\partial\Omega$ is analytic and $F(z)$ is analytic in Ω, then $F(z)$ can be expanded into a series of the form

$$F(z) = \sum_{n=0}^{\infty} A_n \Phi_n(z), \quad z \in \Omega \tag{1.5}$$

in terms of the Faber polynomials. This series is called the *Faber series*, and it converges uniformly on compact subsets of Ω. The coefficients A_n, which can be computed via the formula

$$A_n = \frac{1}{2\pi i} \int_{|z|=\rho} F(g(z)) z^{-n-1}\, dz$$

with $\rho < 1$ and close to 1 are called the *Faber coefficients* of $F(z)$ [**12**, p. 42].

Let $\phi(z)$ be the unique, one-to-one, analytic mapping of Ω onto $\mathbf{D}$ with $\phi(0) = 0$ and $\phi'(0) > 0$. Thus a function $F(z)$ which is analytic and univalent in Ω and normalized by the conditions $F(0) = 0$ and $F'(0) = 1$ may be written as

$$F(z) = \frac{f(\phi(z))}{\phi'(0)} \tag{1.6}$$

for some $f \in S$. The Faber coefficients $\{A_n\}_{n=0}^{\infty}$ of a function $F(z)$ of the form (1.6) will be denoted by $\{A_n(f)\}_{n=0}^{\infty}$ to indicate the dependence on $f \in S$.

In order to investigate the Faber coefficients $A_n(f)$, it will be convenient to work with a domain Ω for which the Faber polynomials $\Phi_n(z)$ may be computed via the formula (1.4) in terms of the *exterior* mapping $g(z)$ given by (1.3) with $c = 1$. We then express the *interior* functions $F(z)$ given by (1.6) in terms of the interior mapping $\phi(z)$. However, it is not easy to deal with both exterior and interior mappings at the same time, so we restrict our interest to the elliptical domain

$$E_r = \left\{(x, y) \in \mathbf{R}^2 : \frac{x^2}{(1 + (1/r^2))^2} + \frac{y^2}{(1 - (1/r^2))^2} < 1\right\},$$

where $r > 1$, for which both of these functions are manageable.

The function $g(z) = z + (1/r^2 z)$, $r > 1$, is analytic and univalent in Δ and maps Δ onto $\mathbf{C} \setminus \overline{E_r}$. After doing necessary calculations, we obtain from (1.4) that the Faber polynomials, $\{\Phi_n(z)\}_{n=0}^{\infty}$, associated with E_r are given by

$$\Phi_n(z) = 2^n r^{-n} P_n\left(\frac{rz}{2}\right), \quad n = 0, 1, 2, \dots .$$

Here $\{P_n(z)\}_{n=0}^{\infty}$ are the monic Chebyshev polynomials of degree n, which are given by

$$P_0(z) = 1$$

and

$$P_n(z) = 2^{-n}\left\{\left[z + \sqrt{z^2 - 1}\right]^n + \left[z - \sqrt{z^2 - 1}\right]^n\right\}, \quad n = 1, 2, 3, \dots .$$

Let $sn(z;q)$ be the Jacobi elliptic sine function with nome q, and modulus k_0, and let

$$K = \int_0^1 \frac{1}{\sqrt{1-t^2}\sqrt{1-k_0^2t^2}}\,dt,$$

[**8**, Chapter 2]. Then the function

$$\varphi(z) = \sqrt{k_0}\left(\frac{2K}{\pi}\sin^{-1}\frac{rz}{2};\frac{1}{r^4}\right)$$

is the one-to-one mapping of E_r onto $\mathbf{D}$ with $\varphi(0)=0$ and $\varphi'(0) = (rK\sqrt{k_0}/\pi) > 0$ [**10**, p. 296].

We define $S(E_r)$ as the class of functions $F(z)$ which are analytic and univalent in E_r and normalized by the conditions $F(0)=0$ and $F'(0)=1$. We define two subclasses of $S(E_r)$ as

$$C(E_r) = \{F(z)\in S(E_r) : F(E_r) \text{ is convex}\},$$

and

$$S^{(2)}(E_r) = \{F(z)\in S(E_r) : F(z) \text{ is odd}\}.$$

In addition, we let $P(E_r)$ denote the class of functions analytic in E_r and satisfying the conditions $F(0) = (1/\varphi'(0)) = (\pi/rK\sqrt{k_0})$ and $\mathrm{Re}\{F(z)\} > 0$. (The condition $F(0) = (1/\varphi'(0))$ is imposed for convenience.)

Note that if $F(z)$ is in one of the classes $S(E_r)$, $C(E_r)$, $S^{(2)}(E_r)$ or $P(E_r)$, then $F(z)$ may be written as in (1.6) for some $f(z)$ in the classes S, convex functions C, odd functions $S^{(2)}$ or functions with positive real part P defined for $\mathbf{D}$.

It has been conjectured in [**7**] that if $F(z)\in S(E_r)$, then

$$|A_0(f)| \le A_0(k) = \frac{\pi^3}{8rK^3\sqrt{k_0}(1-k_0)^2\ln r},$$

and

$$|A_n(f)| \le A_n(k) = \frac{\pi^3 n}{4rK^3\sqrt{k_0}(1-k_0)^2(1-r^{-2n})}, \quad n\ge 1,$$

whose special case for $r \to \infty$ is the famous Bieberbach conjecture.

In this paper we show that the conjecture is true for $n = 0, 1, 2$ and the extremal functions are given by

$$f(z) = k(z)$$

or

$$f(z) = -k(-z),$$

where $k(z)$ is the Koebe function given by (1.2). Also similar sharp upper bounds are obtained for the Faber coefficients $A_0(f)$, $A_1(f)$ and $A_2(f)$ for functions in the classes $C(E_r)$ and $P(E_r)$. In each case, there are two extremal functions that are given by

$$f(z) = c(z) = \frac{z}{1-z} \tag{1.7}$$

or

$$f(z) = -c(-z) = \frac{z}{1+z}$$

and

$$f(z) = p(z) = \frac{1+z}{1-z} \tag{1.8}$$

or

$$f(z) = p(-z) = \frac{1-z}{1+z},$$

respectively. Here it is important that the number of extremal functions is the same as the number of invariant rotations of the elliptical domain E_r. Although the results for the classes $C(E_r)$ and $P(E_r)$ are contained in [**7**], the method used in this paper is different from that of [**7**]. Moreover, the results for the class $S(E_r)$ cannot be obtained by the method of [**7**], since the extreme points of the closed convex hull of S are not known at this time. For functions in $S^{(2)}(E_r)$ a sharp bound for $A_1(f)$ is obtained and the corresponding extremal function in $S^{(2)}$ is shown to be

$$o(z) = \frac{z}{1-z^2}. \tag{1.9}$$

Similar results were obtained in [**11**], but our approach is completely different from that of [**11**].

2. Main results. We begin with a

Lemma 1. *Let $F(z)$ be analytic in E_r and the Faber series given by* (1.5). *Then the Faber coefficients $\{A_n\}_{n=0}^{\infty}$ of $F(z)$ are given by the formula*

$$A_n = \frac{r^n}{\pi}\int_0^{\pi} F\left(\frac{2\cos\theta}{r}\right)\cos n\theta\,d\theta, \quad n = 0, 1, 2, \dots .$$

Proof. Letting $z = (2\cos\theta/r)$ in (1.5) and using $\Phi_n(2\cos\theta/r) = 2r^{-n}\cos n\theta$ yields

$$F\left(\frac{2\cos\theta}{r}\right) = 2\sum_{n=0}^{\infty} A_n r^{-n}\cos n\theta. \tag{2.1}$$

Multiplying (2.1) by $\cos m\theta$ and then integrating from 0 to π gives the desired result. □

As a consequence of this representation it can be shown that if $F(z)$ has the representation (1.6) and belongs to one of the classes $S(E_r)$, $C(E_r)$, $S^{(2)}(E_r)$ and $P(E_r)$, then the Faber coefficients are given by

$$A_n(f) = \frac{r^{n-1}}{K\sqrt{k_0}}\int_0^{\pi} f\left(\varphi\left(\frac{2\cos\theta}{r}\right)\right)\cos n\theta\,d\theta, \quad n = 0, 1, 2, \dots . \tag{2.2}$$

In addition, as it was shown in [**6**] that if $F(z) \in S^{(2)}(E_r)$, then

$$A_{2n}(f) = 0, \quad n = 0, 1, 2, \dots .$$

Another representation formula for the Faber coefficients, $\{A_n(f)\}_{n=0}^{\infty}$, is given in the following corollary.

Corollary 1. *The Faber coefficients,* $\{A_n(f)\}_{n=0}^{\infty}$*, of functions in the classes* $S(E_r)$*,* $C(E_r)$*,* $S^{(2)}(E_r)$ *and* $P(E_r)$ *are given by*

$$A_n(f) = \frac{2^n n! r^{n-1}}{K\sqrt{k_0}(2n)!} \int_0^{\pi} (f(\varphi(x)))^{(n)}|_{x=(2\cos\theta/r)} \sin^{2n}\theta \, d\theta, \quad n = 0, 1, 2, \dots . \tag{2.3}$$

Proof. Since $P_n(\cos\theta) = 2^{1-n}\cos n\theta$, formula (2.2) becomes

$$A_n(f) = \frac{2^{n-1} r^{n-1}}{K\sqrt{k_0}} \int_0^{\pi} f\left(\varphi\left(\frac{2\cos\theta}{r}\right)\right) \cos n\theta \, d\theta, \quad n = 0, 1, 2, \dots .$$

Making the change of variable $x = \cos\theta$ yields

$$A_n(f) = \frac{2^{n-1} r^{n-1}}{K\sqrt{k_0}} \int_{-1}^{1} f\left(\varphi\left(\frac{2x}{r}\right)\right) \frac{P_n(x)}{\sqrt{1-x^2}} \, dx.$$

Multiplying the identity

$$\frac{P_n(x)}{\sqrt{1-x^2}} = \frac{(-1)^n 2^{1-n}}{1.3\cdots(2n-1)} \frac{d^n}{dx^n}\left[(1-x^2)^{n-(1/2)}\right]$$

[**1**, p. 785] by $f(\varphi(2x/r))$ and then integrating from -1 to 1 we obtain

$$\int_{-1}^{1} \frac{P_n(x)}{\sqrt{1-x^2}} f\left(\varphi\left(\frac{2x}{r}\right)\right) dx = \frac{(-1)^n 2^{1-n}}{1.3\cdots(2n-1)} \int_{-1}^{1} \frac{d^n}{dx^n}\left[(1-x^2)^{n-(1/2)}\right] \left(f\left(\varphi\left(\frac{2x}{r}\right)\right)\right) dx. \tag{2.4}$$

Integrating the righthand side of (2.4) by parts results in

$$\int_{-1}^{1} \frac{P_n(x)}{\sqrt{1-x^2}} f\left(\varphi\left(\frac{2x}{r}\right)\right) dx = \frac{(-1)^n 2^{1-n}}{1.3\cdots(2n-1)} \int_{-1}^{1} \frac{d^{n-1}}{dx^{n-1}}\left[(1-x^2)^{n-(1/2)}\right] \left(f\left(\varphi\left(\frac{2x}{r}\right)\right)\right)' dx.$$

Continuing this process n-times yields

$$\begin{aligned}(2.5)\quad &\int_{-1}^{1} \frac{P_n(x)}{\sqrt{1-x^2}}\, f\biggl(\varphi\biggl(\frac{2x}{r}\biggr)\biggr)\, dx \\ &\quad = \frac{(-1)^n 2^{1-n}}{1.3\cdots(2n-1)} \int_{-1}^{1} \bigl(1-x^2\bigr)^{n-(1/2)} \biggl(f\biggl(\varphi\biggl(\frac{2x}{r}\biggr)\biggr)\biggr)^{(n)} dx.\end{aligned}$$

The result follows from (2.2) by letting $x = \cos\theta$ in (2.5), reverting the change of variables. $\square$

Theorem 1. *If* $k(z)$, $c(z)$ *and* $p(z)$ *are given by* (1.2), (1.8) *and* (1.9), *respectively, then*

$$(2.6)\qquad |A_0(f)| \le A_0(k), \quad f \in S,$$

$$(2.7)\qquad |A_0(f)| \le A_0(c), \quad f \in C,$$

and

$$(2.8)\qquad |A_0(f)| \le A_0(p), \quad f \in P.$$

Equalities occur in (2.6), (2.7) *and* (2.8) *if and only if* $f(z) = k(z)$ *or* $f(z) = -k(-z)$, $f(z) = c(z)$ *or* $f(z) = -c(-z)$ *and* $f(z) = p(z)$ *or* $f(z) = p(-z)$, *respectively.*

Proof. From (2.2) we have

$$A_0(f) = \frac{1}{rK\sqrt{k_0}} \int_0^{\pi} f\biggl(\varphi\biggl(\frac{2\cos\theta}{r}\biggr)\biggr)\, d\theta.$$

Since $\varphi(z)$ is an odd function we may write

(2.9)

$$A_0(f) = \frac{1}{rK\sqrt{k_0}} \int_0^{\pi/2} \biggl[f\biggl(\varphi\biggl(\frac{2\cos\theta}{r}\biggr)\biggr) + f\biggl(-\varphi\biggl(\frac{2\cos\theta}{r}\biggr)\biggr)\biggr]\, d\theta.$$

Substituting (1.1) into (2.9) yields

$$A_0(f) = \frac{2}{rK\sqrt{k_0}} \int_0^{\pi/2} \biggl(\sum_{n=1}^{\infty} a_{2n}\varphi^{2n}\biggl(\frac{2\cos\theta}{r}\biggr)\biggr)\, d\theta.$$

Thus

$$|A_0(f)| \leq \frac{2}{rK\sqrt{k_0}} \int_0^{\pi/2} \left(\sum_{n=1}^{\infty} |a_{2n}| \, \varphi^{2n}\left(\frac{2\cos\theta}{r}\right)\right) d\theta$$

since $\varphi(x) \geq 0$ for $x \in [0, (2/r)]$. Hence (2.6) follows from the proof of the de Branges' theorem [**3**] as

$$\begin{aligned} |A_0(f)| &\leq \frac{2}{rK\sqrt{k_0}} \int_0^{\pi/2} \left(\sum_{n=1}^{\infty} 2n\varphi^{2n}\left(\frac{2\cos\theta}{r}\right)\right) d\theta \\ &= A_0(k) = -A_0(-k(-z)). \end{aligned}$$

Note that the value of $A_0(k)$ is given in (1.7).

In a similar way, the proof of (2.7) follows from the coefficient estimate for the class C [**9**].

Substituting

$$f(z) = 1 + \sum_{n=1}^{\infty} b_n z^n \tag{2.10}$$

into (2.9) gives

$$A_0(f) = \frac{2}{rK\sqrt{k_0}} \int_0^{\pi/2} \left[1 + \sum_{n=1}^{\infty} b_{2n}\varphi^{2n}\left(\frac{2\cos\theta}{r}\right)\right] d\theta.$$

Thus

$$|A_0(f)| \leq \frac{2}{rK\sqrt{k_0}} \int_0^{\pi/2} \left[1 + \sum_{n=1}^{\infty} |b_{2n}| \, \varphi^{2n}\left(\frac{2\cos\theta}{r}\right)\right] d\theta, \tag{2.11}$$

since $\varphi(x) \geq 0$ for $x \in [0, (2/r)]$. Using the coefficient estimate $|b_n| \leq 2$ for the class P [**4**] in (2.11) gives (2.8) as

$$\begin{aligned} |A_0(f)| &\leq \frac{2}{rK\sqrt{k_0}} \int_0^{\pi/2} \left[1 + 2\sum_{n=1}^{\infty} \varphi^{2n}\left(\frac{2\cos\theta}{r}\right)\right] d\theta \\ &= A_0(p) = A_0(p(-z)). \qquad \square \end{aligned}$$

Theorem 2. *If $k(z), c(z)$ and $p(z)$ are defined as in Theorem* 1 *and $o(z)$ is given by* (1.9)*, then*

$$|A_1(f)| \le A_1(k), \quad f \in S, \tag{2.12}$$

$$|A_1(f)| \le A_1(c), \quad f \in C, \tag{2.13}$$

$$|A_1(f)| \le A_1(p), \quad f \in P, \tag{2.14}$$

and

$$|A_1(f)| \le A_1(o), \quad f \in S^{(2)}. \tag{2.15}$$

The extremal functions in (2.12)*,* (2.13) *and* (2.14) *are identical to those given in the statement of Theorem* 1*. In* (2.15) *equality holds if and only if $f(z) = o(z)$.*

Proof. From (2.2) we have

(2.16)
$$A_1(f) = \frac{1}{K\sqrt{k_0}} \int_0^{\pi/2} \left[f\left(\varphi\left(\frac{2\cos\theta}{r}\right)\right) - f\left(-\varphi\left(\frac{2\cos\theta}{r}\right)\right)\right] \cos\theta \, d\theta,$$

since $\varphi(x)$ is an odd function. Substituting (1.1) into (2.16) gives

$$A_1(f) = \frac{2}{K\sqrt{k_0}} \int_0^{\pi/2} \left[\varphi\left(\tfrac{2\cos\theta}{r}\right) + \sum_{n=1}^{\infty} a_{2n+1} \varphi^{2n+1}\left(\tfrac{2\cos\theta}{r}\right)\right] \cos\theta \, d\theta.$$

Hence

$$|A_1(f)| \le \frac{2}{K\sqrt{k_0}} \int_0^{\pi/2} \left[\varphi\left(\tfrac{2\cos\theta}{r}\right) + \sum_{n=1}^{\infty} |a_{2n+1}| \, \varphi^{2n+1}\left(\tfrac{2\cos\theta}{r}\right)\right] \cos\theta \, d\theta,$$

since $\varphi(x) \ge 0$ for $x \in [0, (2/r)]$. As in the proof of Theorem 1, inequalities (2.12) and (2.13) result from applying the coefficient estimates for the classes S and C, respectively. Similarly, if $f(z) \in P$ is given by (2.10), then

$$|A_1(f)| \le \frac{2}{K\sqrt{k_0}} \int_0^{\pi/2} \left[\sum_{n=1}^{\infty} |b_{2n-1}| \, \varphi^{2n-1}\left(\frac{2\cos\theta}{r}\right)\right] \cos\theta \, d\theta, \tag{2.17}$$

since $\varphi(x) \geq 0$ for $x \in [0, (2/r)]$. Hence, using the coefficient estimate for the class P in (2.17) results in (2.14).

For $f \in S^{(2)}$, (2.16) gives

$$A_1(f) = \frac{2}{K\sqrt{k_0}} \int_0^{\pi/2} f\left(\varphi\left(\frac{2\cos\theta}{r}\right)\right) \cos\theta \, d\theta.$$

Thus

$$|A_1(f)| \leq \frac{2}{K\sqrt{k_0}} \int_0^{\pi/2} \left| f\left(\varphi\left(\frac{2\cos\theta}{r}\right)\right)\right| \cos\theta \, d\theta. \tag{2.18}$$

By the distortion theorem

$$|f(z)| \leq \frac{|z|}{1-|z|^2}, \qquad f \in S^{(2)},$$

[**5**, p. 70], it follows from (2.18) that

$$|A_1(f)| \leq \frac{2}{K\sqrt{k_0}} \int_0^{\pi/2} \frac{\varphi\big((2\cos\theta)/r\big)}{1-\varphi^2\big((2\cos\theta)/r\big)} \cos\theta \, d\theta = A_1(o),$$

because $0 \leq \varphi(x) < 1$ for $x \in [0, (2/r)]$. □

Remark 1. We can also obtain (2.12), (2.13) and (2.14) in the same way, by using (2.3) instead of (2.2) and noting that $\varphi'(x) \geq 0$, since $\varphi(x)$ is increasing for $x \in [-(2/r), (2/r)]$. Also using $\varphi'(x) \geq 0$ for $x \in [-(2/r), (2/r)]$ and the distortion theorem,

$$|f'(z)| \leq \frac{1+|z|^2}{\big(1-|z|^2\big)^2}, \quad f \in S^{(2)}$$

[**5**, p. 70] in (2.3) leads to (2.15).

Theorem 3. *If $k(z)$, $c(z)$ and $p(z)$ are defined as in Theorem* 1, *then*

$$|A_2(f)| \leq A_2(k), \quad f \in S, \tag{2.19}$$

$$|A_2(f)| \leq A_2(c), \quad f \in C, \tag{2.20}$$

and

$$|A_2(f)| \leq A_2(p), \quad f \in P. \tag{2.21}$$

The extremal functions are identical to those given in the statement of Theorem 1.

Proof. From (2.2) we have

$$A_2(f) = \frac{r}{K\sqrt{k_0}} \int_0^{\pi} f\left(\varphi\left(\frac{2\cos\theta}{r}\right)\right) \cos 2\theta \, d\theta$$

or

$$A_2(f) = \frac{r}{K\sqrt{k_0}} \int_0^{\pi/2} \left[f\left(\varphi\left(\frac{2\cos\theta}{r}\right)\right) + f\left(-\varphi\left(\frac{2\cos\theta}{r}\right)\right)\right] \cos 2\theta \, d\theta$$

since $\varphi(z)$ is an odd function. Then

$$\begin{aligned} A_2(f) = \frac{r}{K\sqrt{k_0}} \int_0^{\pi/4} \Bigg\{ &\left[f\left(\varphi\left(\frac{2\cos\theta}{r}\right)\right) + f\left(-\varphi\left(\frac{2\cos\theta}{r}\right)\right)\right] \\ &- \left[f\left(\varphi\left(\frac{2\sin\theta}{r}\right)\right) + f\left(-\varphi\left(\frac{2\sin\theta}{r}\right)\right)\right]\Bigg\} \cos 2\theta \, d\theta. \end{aligned} \tag{2.22}$$

Substituting (1.1) into (2.22) gives

$$A_2(f) = \tfrac{2r}{K\sqrt{k_0}} \int_0^{\pi/4} \left[\sum_{n=1}^{\infty} a_{2n} \left(\varphi^{2n}\left(\tfrac{2\cos\theta}{r}\right) - \varphi^{2n}\left(\tfrac{2\sin\theta}{r}\right)\right)\right] \cos 2\theta \, d\theta.$$

Since $\varphi(x) \geq 0$ and $\varphi(x)$ is increasing for $x \in [0, (2/r)]$, we have

$$\varphi^{2n}\left(\frac{2\cos\theta}{r}\right) - \varphi^{2n}\left(\frac{2\sin\theta}{r}\right) \geq 0, \quad n = 1, 2, 3, \dots, \; 0 \leq \theta \leq \frac{\pi}{4}.$$

Thus

$$|A_2(f)| \leq \tfrac{2r}{K\sqrt{k_0}} \int_0^{\pi/4} \left[\sum_{n=1}^{\infty} |a_{2n}| \left(\varphi^{2n}\left(\tfrac{2\cos\theta}{r}\right) - \varphi^{2n}\left(\tfrac{2\sin\theta}{r}\right)\right)\right] \cos 2\theta \, d\theta.$$

Hence (2.19) and (2.20) are obtained from the coefficient estimates for the classes S and C, respectively.

If $f(z) \in P$ is given by (2.10), then (2.21) follows from

$$|A_2(f)| \leq \frac{4}{K\sqrt{k_0}} \int_0^{\pi/4} \left[\sum_{n=1}^{\infty} |b_{2n}| \left(\varphi^{2n}\left(\frac{2\cos\theta}{r}\right) - \varphi^{2n}\left(\frac{2\sin\theta}{r}\right) \right) \right] \cos 2\theta \, d\theta$$

by using the coefficient estimate for the class P.

REFERENCES

1. M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, Dover Publ., Inc., New York, 1972.

2. L. Bieberbach, *Über die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln*, S.-B. Preuss. Akad. Wiss. (1916), 940–955.

3. L. de Branges, *A proof of the Bieberbach conjecture*, Acta Math. **154** (1985), 137–152.

4. C. Carathéodory, *Über den Variabilitätsbereich der Koeffizienten von Pontenzreihen, die gegebene Werte nicht annehmen*, Math. Ann. **64** (1907), 95–115.

5. P.L. Duren, *Univalent functions*, Springer-Verlag, New York, 1983.

6. E. Haliloğlu, *On the Faber coefficients of functions univalent in an ellipse*, Trans. Amer. Math. Soc. **349** (1997), 2901–2916.

7. ———, *Generalizations of coefficient estimates for certain classes of analytic functions*, Proc. Japan Acad. Ser. A Math. Sci. **73** (1997), 116–121.

8. D.F. Lawden, *Elliptic functions and applications*, Springer-Verlag, New York, 1989.

9. K. Loewner, *Untersuchungen über die Verzerrung bei konformen Abbildungen des Einheitskreises $|z| < 1$, die durch Funktionen mit nichtverschwindender Ableitung geliefert werden*, S.-B. Sächs. Akad. Wiss. **69** (1917), 89–106.

10. Z. Nehari, *Conformal mapping*, McGraw-Hill, New York, 1952.

11. W. C. Royster, *Coefficient problems for functions regular in an ellipse*, Duke Math. J. **26** (1959), 361–371.

12. G. Schober, *Univalent functions* – Selected topics, Lecture Notes in Math., vol. 478, Springer-Verlag, New York, 1975.

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