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Notes on Starlike log-Harmonic Functions of Order α

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Abstract

For log-harmonic functions $f(z) = zh(z)\overline{g(z)}$ in the open unit disk $\mathbb{U}$, two subclasses $H^*_{LH}(\alpha)$ and $G^*_{LH}(\alpha)$ of $S^*_{LH}(\alpha)$ consisting of all starlike log-harmonic functions of order α $(0 \leq \alpha < 1)$ are considered. The object of the present paper is to discuss some coefficient inequalities for $h(z)$ and $g(z)$.

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1 Introduction

Let H be the class of functions which are analytic in the open unit disc $\mathbb{U} = \{z \in C : |z| < 1\}$. A log-harmonic function $f(z)$ is a solution of the non-linear elliptic partial differential equation

$$\frac{\overline{f_{\overline{z}}}}{\overline{f}} = w(z)\frac{f_z}{f}, \tag{1.1}$$

where $w(z) \in H$ satisfies $|w(z)| < 1$ $(z \in \mathbb{U})$ and is said to be the second dilatation, and

$$f_z = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \quad f_{\overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right). \tag{1.2}$$

Let a function $f(z)$ given by

$$f(z) = zh(z)\overline{g(z)} \tag{1.3}$$

with $0 \notin hg(\mathbb{U})$ be log-harmonic function in $\mathbb{U}$, where $h(z) \in H$ and $g(z) \in H$. Then $f(z)$ is said to be starlike log-harmonic function of order α if it satisfies

$$\frac{\partial(\arg f(re^{i\theta}))}{\partial\theta} = \operatorname{Re}\left(\frac{zf_z - \overline{z}f_{\overline{z}}}{f}\right) > \alpha \quad (z \in \mathbb{U}) \tag{1.4}$$

for some real α $(0 \leq \alpha < 1)$. We denote by $S^*_{LH}(\alpha)$ all starlike log-harmonic functions $f(z)$ of order α in $\mathbb{U}$.

The class $S^*_{LH}(\alpha)$ was studied by Abdulhadi and Muhanna [4], and Polatoğlu and Deniz [6]. Furthermore, the classes of univalent log-harmonic functions have been studied by Abdulhadi [1], [2], and Abdulhadi and Hengartner [3].

2 Coefficient Inequalities for $h(z)$

In order to consider our problem, we have to introduce the following subclass $H^*_{LH}(\alpha)$ of $S^*_{LH}(\alpha)$. A function $f(z) = zh(z)\overline{g(z)} \in S^*_{LH}(\alpha)$ is said to be in a class $H^*_{LH}(\alpha)$ if it satisfies

$$h(z) = h(0) + \sum_{n=1}^{\infty} a_n z^n \quad (h(0) > 0) \tag{2.1}$$

with $a_n = |a_n| e^{i(n\theta+\pi)}$ $(\theta \in \mathbb{R})$.

Now we derive

Theorem 2.1. *If $f = zh(z)\overline{g(z)} \in H^*_{LH}(\alpha)$ with*

$$\beta_1 < \min_{z\in\mathbb{U}} \operatorname{Re}\left(\frac{zg'(z)}{g(z)}\right) < 0, \tag{2.2}$$

then

$$\sum_{n=1}^{\infty}(n+1-\alpha-\beta_1)\,|a_n| \leq (1-\alpha-\beta_1)h(0). \tag{2.3}$$

Proof. Note that $f = zh(z)\overline{g(z)} \in H^*_{LH}(\alpha) \subset S^*_{LH}(\alpha)$ satisfies

$$\begin{aligned}\frac{\partial}{\partial\theta}(\arg f(re^{i\theta})) &= \operatorname{Re}\left(\frac{zf_z - \overline{z}f_{\overline{z}}}{f}\right)\\ &= \operatorname{Re}\left(1 + \frac{zh'(z)}{h(z)} - \frac{zg'(z)}{g(z)}\right) > \alpha \quad (z \in \mathbb{U}).\end{aligned}$$

This gives us that

$$\begin{aligned}\operatorname{Re}\left(\frac{zh'(z)}{h(z)}\right) &= \operatorname{Re}\left(\frac{\sum_{n=1}^{\infty} na_n z^n}{h(0) + \sum_{n=1}^{\infty} a_n z^n}\right)\\ &= \operatorname{Re}\left(\frac{-\sum_{n=1}^{\infty} n\,|a_n|\,e^{in\theta}z^n}{h(0) - \sum_{n=1}^{\infty}|a_n|\,e^{in\theta}z^n}\right)\\ &> \operatorname{Re}\left(\alpha - 1 + \frac{zg'(z)}{g(z)}\right)\end{aligned} \tag{2.4}$$

$$> \alpha + \beta_1 - 1$$

for all $z \in \mathbb{U}$.

Let us consider a point z such that $z = |z|\, e^{-i\theta} \in \mathbb{U}$. Then (2.4) becomes that

$$\text{Re}\left(\frac{zh'(z)}{h(z)}\right) = \frac{-\sum_{n=1}^{\infty} n|a_n||z|^n}{h(0) - \sum_{n=1}^{\infty} |a_n||z|^n} > \alpha + \beta_1 - 1 \quad (z \in \mathbb{U}). \tag{2.5}$$

Letting $|z| \to 1^-$, we obtain that

$$-\sum_{n=1}^{\infty} n\,|a_n| \geq (\alpha + \beta_1 - 1)\left(h(0) - \sum_{n=1}^{\infty} |a_n|\right),$$

that is, that

$$\sum_{n=1}^{\infty} (n + 1 - \alpha - \beta_1)\,|a_n| \leq (1 - \alpha - \beta_1) h(0).$$

□

Example 2.2. Let us consider a function $f(z) = zh(z)\overline{g(z)} \in H^*_{LH}(\alpha)$ with

$$h(z) = h(0) + \sum_{n=1}^{\infty} \frac{(1 - \alpha - \beta_1) h(0) e^{in\theta}}{n(n+1)(n + 1 - \alpha - \beta_1)} z^n$$

and

$$g(z) = \frac{2\beta_1}{1 - z} \quad (\beta_1 < 0).$$

Then

$$0 > \min_{z \in \mathbb{U}} \text{Re}\left(\frac{zg'(z)}{g(z)}\right) > \beta_1$$

and

$$\sum_{n=1}^{\infty} (n + 1 - \alpha - \beta_1)\,|a_n| = (1 - \alpha - \beta_1) h(0) \left(\sum_{n=1}^{\infty} \frac{1}{n(n+1)}\right)$$
$$= (1 - \alpha - \beta_1) h(0).$$

Theorem 2.1 gives us the following corollary.

Corollary 2.3. *If* $f(z) = zh(z)\overline{g(z)} \in H^*_{LH}(\alpha)$ *with* (2.2) *then*

$$|a_n| \leq \frac{1 - \alpha - \beta_1}{n + 1 - \alpha - \beta_1} h(0) \quad (n = 1, 2, 3, \cdots).$$

Next, we show

Theorem 2.4. *If* $f(z) = zh(z)\overline{g(z)} \in H^*_{LH}(\alpha)$ *with* (2.2) *then*

$$\left(1 - \frac{1-\alpha-\beta_1}{2-\alpha-\beta_1}|z|\right)h(0) \leq |h(z)| \leq \left(1 + \frac{1-\alpha-\beta_1}{2-\alpha-\beta_1}|z|\right)h(0) \tag{2.6}$$

and

$$|a_1| - ((1-\alpha-\beta)h(0) - (2-\alpha-\beta)|a_1|)|z| \tag{2.7}$$

$$\leq |h'(z)| \leq |a_1| + ((1-\alpha-\beta)h(0) - (2-\alpha-\beta)|a_1|)|z|$$

for $z \in \mathbb{U}$. *The equality in* (2.6) *holds for* $f(z) = zh(z)\overline{g(z)}$ *with*

$$h(z) = h(0) + \frac{1-\alpha-\beta_1}{2-\alpha-\beta_1}h(0)e^{i\theta}z.$$

Proof. We note that the inequality (2.3) gives us that

$$\sum_{n=1}^{\infty} |a_n| \leq \frac{1-\alpha-\beta_1}{2-\alpha-\beta_1}h(0)$$

and

$$\sum_{n=2}^{\infty} n|a_n| \leq (1-\alpha-\beta_1)h(0) - (2-\alpha-\beta_1)|a_1|.$$

Thus, we have that

$$|h(z)| \leq h(0) + |z|\sum_{n=1}^{\infty} |a_n| \leq \left(1 + \frac{1-\alpha-\beta_1}{2-\alpha-\beta_1}|z|\right)h(0)$$

and

$$|h(z)| \geq h(0) - |z|\sum_{n=1}^{\infty} |a_n| \geq \left(1 - \frac{1-\alpha-\beta_1}{2-\alpha-\beta_1}|z|\right)h(0).$$

Furthermore, we have that

$$\begin{aligned}|h'(z)| &\leq |a_1| + |z| \sum_{n=2}^{\infty} n\,|a_n| \\ &\leq |a_1| + ((1-\alpha-\beta_1)h(0) - (2-\alpha-\beta_1)\,|a_1|)\,|z|\end{aligned}$$

and

$$\begin{aligned}|h'(z)| &\geq |a_1| - |z| \sum_{n=2}^{\infty} n\,|a_n| \\ &\geq |a_1| - ((1-\alpha-\beta_1)h(0) - (2-\alpha-\beta_1)\,|a_1|)\,|z|\,.\end{aligned}$$

□

Next, we consider

Theorem 2.5. *Let* $f(z) = zh(z)\overline{g(z)}$*,where* $h(z)$ *is given by* (2.1) *and* $a_n = |a_n|\,e^{i(n\theta+\pi)}$ $(\theta \in \mathbb{R})$*. If* $f(z)$ *satisfies*

$$\beta_2 > \max_{z\in\mathbb{U}} \operatorname{Re}\left(\frac{zg'(z)}{g(z)}\right) > 0 \tag{2.8}$$

and

$$\sum_{n=1}^{\infty}(n+1-\alpha-\beta_2)|a_n| \leq (1-\alpha-\beta_2)h(0), \tag{2.9}$$

then $f(z) \in H^*_{LH}(\alpha)$*, where* $0 < \beta_2 < 1-\alpha$*.*

Proof. Note that if $f(z)$ satisfies

$$\left|\frac{zh'(z)}{h(z)}\right| < 1-\alpha-\beta_2 \quad (z \in \mathbb{U}), \tag{2.10}$$

then we have that

$$\operatorname{Re}\left(\frac{zh'(z)}{h(z)}\right) > \alpha+\beta_2-1 \quad (z \in \mathbb{U}).$$

This implies that

$$\operatorname{Re}\left(1+\frac{zh'(z)}{h(z)} - \frac{zg'(z)}{g(z)}\right) > \alpha \quad (z \in \mathbb{U}).$$

Therefore, if $f(z)$ satisfies the inequality (2.10), then $f(z) \in H^*_{LH}(\alpha)$. Indeed we see that

$$\left|\frac{zh'(z)}{h(z)}\right| = \left|\frac{-\sum_{n=1}^{\infty} n\,|a_n|\,e^{in\theta} z^n}{h(0) - \sum_{n=1}^{\infty} |a_n|\,e^{in\theta} z^n}\right| < \frac{\sum_{n=1}^{\infty} n\,|a_n|}{h(0) - \sum_{n=1}^{\infty} |a_n|}.$$

Thus, if $f(z)$ satisfies (2.9), then we have the inequality (2.10). □

3 Coefficient Inequalities for $g(z)$

Let $f(z) = zh(z)\overline{g(z)}$ be in the class $S^*_{LH}(\alpha)$. If $f(z)$ satisfies

(3.1) $$g(z) = g(0) + \sum_{n=1}^{\infty} b_n z^n \quad (g(0) > 0)$$

with $b_n = |b_n|\,e^{in\theta}$ $(\theta \in \mathbb{R})$, then we say that $f(z) \in G^*_{LH}(\alpha)$.

Theorem 3.1. *If $f(z) = zh(z)\overline{g(z)} \in G^*_{LH}(\alpha)$ with*

(3.2) $$\gamma_1 > \max_{z \in \mathbb{U}} \operatorname{Re}\left(\frac{zh'(z)}{h(z)}\right) > 0,$$

then

(3.3) $$\sum_{n=1}^{\infty} (n - 1 + \alpha - \gamma_1)\,|b_n| \leq (1 - \alpha + \gamma_1) g(0).$$

Proof. Note that if $f(z) \in G^*_{LH}(\alpha) \subset S^*_{LH}(\alpha)$, then

$$\operatorname{Re}\left(\frac{zg'(z)}{g(z)}\right) < \operatorname{Re}\left(1 - \alpha + \frac{zh'(z)}{h(z)}\right) \quad (z \in \mathbb{U}),$$

which implies that

$$\operatorname{Re}\left(\frac{zg'(z)}{g(z)}\right) < 1 - \alpha + \gamma_1 \quad (z \in \mathbb{U}).$$

Therefore, we see that

$$\begin{aligned}\operatorname{Re}\left(\frac{zg'(z)}{g(z)}\right) &= \operatorname{Re}\left(\frac{\sum_{n=1}^{\infty} nb_n z^n}{g(0)+\sum_{n=1}^{\infty} b_n z^n}\right)\\ &= \operatorname{Re}\left(\frac{\sum_{n=1}^{\infty} n\left|b_n\right| e^{in\theta} z^n}{g(0)+\sum_{n=1}^{\infty} \left|b_n\right| e^{in\theta} z^n}\right)\\ &< 1-\alpha+\gamma_1 \quad (z\in\mathbb{U}).\end{aligned}$$

Let us consider a point z such that $z=|z|\,e^{-i\theta}\in\mathbb{U}$. Then, we have that

$$\operatorname{Re}\left(\frac{zg'(z)}{g(z)}\right)=\frac{\sum_{n=1}^{\infty} n\left|b_n\right| |z|^n}{g(0)+\sum_{n=1}^{\infty} \left|b_n\right| |z|^n}<1-\alpha+\gamma_1 \quad (z\in\mathbb{U}).$$

Thus, letting $|z|\to 1^-$, we obtain that

$$\sum_{n=1}^{\infty}(n-1+\alpha-\gamma_1)\left|b_n\right| \leq (1-\alpha+\gamma_1)g(0).$$

□

Example 3.2. If $f(z)=zh(z)\overline{g(z)}\in G_{LH}^{*}(\alpha)$ with

$$h(z)=\frac{2\gamma_1}{1-z} \quad (\gamma_1>0)$$

and

$$g(z)=g(0)+\sum_{n=1}^{\infty}\frac{(1-\alpha+\gamma_1)g(0)e^{in\theta}}{n(n+1)(n-1+\alpha-\gamma_1)}z^n,$$

then

$$0<\max_{z\in\mathbb{U}}\operatorname{Re}\left(\frac{zh'(z)}{h(z)}\right)<\gamma_1.$$

It follows that $f(z)$ satisfies

$$\sum_{n=1}^{\infty}(n-1+\alpha-\gamma_1)\left|b_n\right| = (1-\alpha+\gamma_1)g(0).$$

Applying Theorem 3.1, we have the following result.

Theorem 3.3. *If* $f(z)=zh(z)\overline{g(z)}$ *with* (3.2), *then*

$$\left(1-\frac{1-\alpha+\gamma_1}{\alpha-\gamma_1}\,|z|\right)g(0)\leq |g(z)|\leq \left(1+\frac{1-\alpha+\gamma_1}{\alpha-\gamma_1}\,|z|\right)g(0) \tag{3.4}$$

and

$$(3,5) \qquad |b_1| - ((1-\alpha+\gamma_1)g(0) - (\alpha-\gamma_1)|b_1|)|z|$$
$$\leq |g'(z)| \leq |b_1| + ((1-\alpha+\gamma_1)g(0) - (\alpha-\gamma_1)|b_1|)|z|$$

for $z \in \mathbb{U}$, *where* $0 < \gamma_1 < \alpha$.

Proof. Since

$$\sum_{n=1}^{\infty} |b_n| \leq \frac{1-\alpha+\gamma_1}{\alpha-\gamma_1} g(0)$$

and

$$\sum_{n=2}^{\infty} n|b_n| \leq (1-\alpha+\gamma_1)g(0) - (\alpha-\gamma_1)|b_1|$$

for $f(z) \in G^*_{LH}(\alpha)$, we prove the inequalities (3.4) and (3.5). □

Finally, we derive

Theorem 3.4. *Let* $f(z) = zh(z)\overline{g(z)}$, *where* $g(z)$ *is given by* (3.1) *and* $b_n = |b_n| e^{in\theta}$ $(\theta \in \mathbb{R})$. *If* $f(z)$ *satisfies*

$$\gamma_2 < \min_{z\in\mathbb{U}} \operatorname{Re}\left(\frac{zh'(z)}{h(z)}\right) < 0 \tag{3.6}$$

and

$$\sum_{n=1}^{\infty} (n-1+\alpha-\gamma_2)|b_n| \leq (1-\alpha+\gamma_2)g(0) \tag{3.7}$$

then $f(z) \in G^*_{LH}(\alpha)$, *where* $\alpha - 1 < \gamma_2 < 0$.

Proof. Note that if $f(z)$ satisfies

$$\left|\frac{zg'(z)}{g(z)}\right| < 1-\alpha+\gamma_2 \quad (z \in \mathbb{U}),$$

then

$$\operatorname{Re}\left(\frac{zg'(z)}{g(z)}\right) < 1-\alpha+\gamma_2 \leq 1-\alpha+\operatorname{Re}\left(\frac{zh'(z)}{h(z)}\right) \quad (z \in \mathbb{U}),$$

which shows that $f(z) \in S^*_{LH}(\alpha)$.

It follows that

$$\left|\frac{zg'(z)}{g(z)}\right| = \left|\frac{\sum_{n=1}^{\infty} n\,|b_n|\,e^{in\theta}z^n}{g(0)+\sum_{n=1}^{\infty}|b_n|\,e^{in\theta}z^n}\right| < \frac{\sum_{n=1}^{\infty} n\,|b_n|}{g(0)-\sum_{n=1}^{\infty}|b_n|} \leq 1-\alpha+\gamma_2 \tag{3.8}$$

if the inequality (3.7) holds true. Therefore, we see that $f(z) \in G^*_{LH}(\alpha)$. □

4 Open questions

We know that Jahangiri [5] has showed the coefficient inequality which is the necessary and sufficient condition for harmonic convex functions $f(z)$ of order α in $\mathbb{U}$. There are many necessary and sufficient inequalities for some classes of analytic functions in $\mathbb{U}$. We hope we will discuss some necessary and sufficient conditions for starlike log-harmonic functions $f(z)$ in $\mathbb{U}$.

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