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Harmonic mappings for which co-analytic part is a close-to-convex function of order b

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Abstract

In the present paper we investigate a class of harmonic mappings for which the second dilatation is a close-to-convex function of complex order b, $b \in \mathbb{C}/\{0\}$ (Lashin in Indian J. Pure Appl. Math. 34(7):1101-1108, 2003).

MSC: 30C45; 30C55

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1 Introduction

A planar harmonic mapping in the open unit disc $\mathbb{D} = \{z||z| < 1\}$ is a complex-valued harmonic function f which maps $\mathbb{D}$ onto some planar domain $f(\mathbb{D})$. Since $\mathbb{D}$ is a simply connected domain, the mapping f has a canonical decomposition $f = h(z) + \overline{g(z)}$, where $h(z)$ and $g(z)$ are analytic in $\mathbb{D}$ and have the following power series expansions:

$$h(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=0}^{\infty} b_n z^n, \quad z \in \mathbb{D},$$

where $a_n, b_n \in \mathbb{C}$, $n = 0, 1, 2, \ldots$. As usual, we call $h(z)$ analytic part and $g(z)$ co-analytic part of f, respectively. An elegant and complete account of the theory of planar harmonic mappings is given in Duren?s monograph [1].

Lewy [2] proved in 1936 that the harmonic mapping f is locally univalent in $\mathbb{D}$ if and only if its Jacobian $J_f = |h'(z)|^2 - |g'(z)|^2$ is different from zero in $\mathbb{D}$. In view of this result, locally univalent harmonic mappings in the open unit disc are either sense-reversing if $|g'(z)| > |h'(z)|$ or sense-preserving if $|g'(z)| < |h'(z)|$ in $\mathbb{D}$. Throughout this paper, we restrict ourselves to the study of sense-preserving harmonic mappings. We also note that $f = h(z) + \overline{g(z)}$ is sense-preserving in $\mathbb{D}$ if and only if $h'(z)$ does not vanish in the unit disc $\mathbb{D}$, and the second dilatation $w(z) = g'(z)/h'(z)$ has the property $|w(z)| < 1$ in $\mathbb{D}$.

The class of all sense-preserving harmonic mappings in the open unit disc $\mathbb{D}$ with $a_0 = b_0 = 0$ and $a_1 = 1$ is denoted by $\mathcal{S_H}$. Thus $\mathcal{S_H}$ contains the standard class $\mathcal{S}$ of analytic univalent functions.

The family of all mappings $f \in \mathcal{S_H}$ with the additional property that $g'(0) = 0$, *i.e.*, $b_1 = 0$, is denoted by $\mathcal{S}^0_{\mathcal{H}}$. Thus it is clear that $\mathcal{S} \subset \mathcal{S}^0_{\mathcal{H}} \subset \mathcal{S_H}$ [1]. Let Ω be the family of functions $\phi(z)$ regular in the open unit disc $\mathbb{D}$ and satisfying the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for all

$z \in \mathbb{D}$. We denote by $\mathcal{P}$ the family of functions $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ regular in $\mathbb{D}$ such that $p(z)$ in $\mathcal{P}$ if and only if

$$p(z) = \frac{1 + \phi(z)}{1 - \phi(z)} \tag{1.1}$$

for some $\phi(z) \in \Omega$ and every $z \in \mathbb{D}$.

Let $s_1(z) = z + c_2 z^2 + c_3 z^3 + \cdots$ and $s_2(z) = z + d_2 z^2 + d_3 z^3 + \cdots$ be analytic functions in $\mathbb{D}$. If there exists a function $\phi(z) \in \Omega$ such that $s_1(z) = s_2(\phi(z))$ for every $z \in \mathbb{D}$, then we say that $s_1(z)$ is subordinate to $s_2(z)$ and we write $s_1 \prec s_2$. We also note that if $s_1 \prec s_2$, then $s_1(\mathbb{D}) \subset s_2(\mathbb{D})$ [3, 4].

Next, let $\mathcal{A}$ be the class of functions $s(z) = z + e_2 z^2 + \cdots$ which are analytic in $\mathbb{D}$. A function $s(z)$ in $\mathcal{A}$ is said to be a convex function of complex order b, $b \in \mathbb{C}/\{0\}$, that is, $s(z) \in \mathcal{C}(b)$ if and only if $s'(z) \neq 0$, and

$$\operatorname{Re}\left(1 + \frac{1}{b} z \frac{s''(z)}{s'(z)}\right) > 0 \quad (z \in \mathbb{D}). \tag{1.2}$$

We denote by $\mathcal{S}^*(1-b)$ the class of $\mathcal{A}$ consisting of functions which are starlike of complex order b, that is,

$$\operatorname{Re}\left(1 + \frac{1}{b}\left(z \frac{s''(z)}{s'(z)} - 1\right)\right) > 0 \quad (z \in \mathbb{D}). \tag{1.3}$$

Moreover, let $s(z)$ be an element of $\mathcal{A}$, then $s(z)$ is said to be close-to-convex of complex order b, $b \in \mathbb{C}/\{0\}$ if and only if there exists a function $\varphi(z) \in \mathcal{C}(b)$ satisfying the condition

$$\operatorname{Re}\left(1 + \frac{1}{b}\left(\frac{s'(z)}{\varphi'(z)} - 1\right)\right) > 0 \quad (z \in \mathbb{D}). \tag{1.4}$$

The class of such functions is denoted by $\mathcal{CC}(b)$.

The classes $\mathcal{C}(b)$ and $\mathcal{S}^*(1-b)$ were introduced and studied by Nasr and Aouf [5, 6], and the class $\mathcal{CC}(b)$ was introduced by Lashin [7].

Remark 1.1

(i) For $b = 1$ we obtain $\mathcal{S}^*(0) = \mathcal{S}^*$, $\mathcal{C}(1) = \mathcal{C}$, and $\mathcal{CC}(1) = \mathcal{CC}$ are well-known classes of starlike, convex and close-to-convex functions, respectively [6].

(ii) $\mathcal{S}^*(1-(1-\alpha)) = \mathcal{S}^*(\alpha)$, $\mathcal{C}(1-\alpha)$, and $\mathcal{CC}(1-\alpha)$, $0 \leq \alpha < 1$, are the classes of starlike, convex and close-to-convex functions of order α, respectively [6].

(iii) If we take $b = e^{-i\lambda}\cos\lambda$, $|\lambda| < \pi/2$, we obtain the following classes: λ-spirallike, analytic functions for which $zf'(z)$ is λ-spirallike and λ-spirallike and λ-spiral close-to-convex functions [6].

(iv) $\mathcal{S}^*(1-(1-\alpha)e^{-i\lambda}\cos\lambda)$, $\mathcal{C}^*((1-\alpha)e^{-i\lambda}\cos\lambda)$, $\mathcal{CC}^*((1-\alpha)e^{-i\lambda}\cos\lambda)$, $0 \leq \alpha < 1$, $|\lambda| < \pi/2$, are the classes of λ-spirallike functions of order α, analytic functions for which $zf'(z)$ is λ-spirallike of order α and λ-spiral close-to-convex functions of order α, respectively [6].

Finally, the aim of this investigation is to obtain some properties of the class of harmonic functions defined by

$$\mathcal{S}_{\mathcal{HCC}(b)} = \left\{ f = h(z) + \overline{g(z)} \middle| w(z) = \frac{g'(z)}{h'(z)} \prec b_1 \frac{1 + (2b-1)z}{1-z} \right.$$
$$\left. \Leftrightarrow \operatorname{Re}\left[1 + \frac{1}{b}\left(\frac{g'(z)}{h'(z)} - b_1\right)\right] > 0, b, b_1 \in \mathbb{C}/\{0\}, h(z) \in \mathcal{C}(b) \right\}$$

for all z in $\mathbb{D}$.

For the purpose of this paper, we need the following lemma and theorem.

Lemma 1.2 [8] *Let $\phi(z)$ be regular in the unit disc $\mathbb{D}$ with $\phi(0) = 0$. If the maximum value of $|\phi(z)|$ on the circle $|z| = r < 1$ is attained at point z_1, then we have $z_1\phi'(z_1) = k\phi(z_1)$ for some $k \geq 1$.*

Theorem 1.3 [9] *If $s(z) \in \mathcal{C}(b)$, then*

$$2\left[1 + \frac{1}{b}\left(z\frac{s'(z)}{s(z)} - 1\right)\right] - 1 = p(z) = \frac{1 + \phi(z)}{1 - \phi(z)}$$

for some $\phi(z) \in \Omega$ and every z in $\mathbb{D}$, and

$$\int_0^{2\pi} \operatorname{Re}\left(z\frac{s'(z)}{s(z)}\right) d\theta = 2pn\pi \tag{1.5}$$

for every $z \in \mathbb{D}$. A member of $\mathcal{S}^*(p,n)$ is called p-valent starlike function in the unit disc $\mathbb{D}$.

Finally, a planar harmonic mapping in the open unit disc $\mathbb{D}$ is a complex-valued harmonic function f, which maps $\mathbb{D}$ onto some planar domain $f(\mathbb{D})$. Since $\mathbb{D}$ is a simply connected domain, the mapping f has a canonical decomposition $f = h + \overline{g}$, where $h(z)$ and $g(z)$ are analytic in $\mathbb{D}$ and have the following power series expansion:

$$h(z) = z^p + a_{np+1}z^{np+1} + a_{np+2}z^{np+2} + \cdots + a_{np+m}z^{np+m} + \cdots$$

and

$$g(z) = b_{np}z^{np} + b_{np+1}z^{np+1} + b_{np+2}z^{np+2} + \cdots + b_{np+m}z^{np+m} + \cdots,$$

where $|b_{np}| < 1$, $p \geq 1$ and $n \geq 1$ are integers, $a_{np+m}, b_{np+m} \in \mathbb{C}$ and every $z \in \mathbb{D}$. As usual, we call $h(z)$ the analytic part and $g(z)$ the co-analytic part of f, respectively, and let the class of such harmonic mappings be denoted by $\mathcal{SH}(p,n)$. Lewy [2] proved in 1936 that the harmonic mapping f is locally univalent in $\mathbb{D}$ if and only if its Jacobian $J_f = |h'(z)|^2 - |g'(z)|^2$ is strictly positive in $\mathbb{D}$. In view of this result, locally univalent harmonic mappings in the open unit disc are either sense-reversing if $|g'(z)| > |h'(z)|$ or sense-preserving if $|g'(z)| < |h'(z)|$ in $\mathbb{D}$. Throughout this paper, we restrict ourselves to the study of sense-preserving harmonic mappings. We also note that an elegant and complete treatment theory of the harmonic mapping is given in Duren?s monograph [1].

The main aim of this paper is to investigate some properties of the following class:

$$\mathcal{S}^*\mathcal{H}(p,n) = \left\{ f = h + \overline{g} \in \mathcal{SH}(p,n) \middle| w(z) = \frac{g'(z)}{h'(z)} \prec b_{np}\frac{1+\phi(z)}{1-\phi(z)}, \right.$$
$$\left. \phi(z) = z^n\psi(z), \psi(z) \in \Omega_1, h(z) \in \mathcal{S}^*(p,n), z \in \mathbb{D} \right\}$$

and for this aim we need the following lemma.

Lemma 1.4 [1] *Let $w(z) = a_n z^n + a_{n+1}z^{n+1} + a_{n+1}z^{n+2} + \cdots$ $(a_n \neq 0, n \geq 1)$ be analytic in $\mathbb{D}$. If the maximum value of $|w(z)|$ on the circle $|z| = r < 1$ is attained at $z = z_0$, then we have $z_0 w'(z_0) = pw(z_0)$, where $p \geq n$ and every $z \in \mathbb{D}$.*

2 Main results

Lemma 2.1 *Let $h(z)$ be an element of $\mathcal{C}(b)$, then*

$$\mathcal{F}_1\left(\frac{1}{2}|b|, \frac{1}{2}\operatorname{Re} b, -r\right) \leq \left|h(z)\right| \leq \mathcal{F}_1\left(\frac{1}{2}|b|, \frac{1}{2}\operatorname{Re} b, r\right) \tag{2.1}$$

and

$$\mathcal{F}_2\left(|b|, \operatorname{Re} b, -r\right) \leq \left|h'(z)\right| \leq \mathcal{F}_2\left(|b|, \operatorname{Re} b, r\right), \tag{2.2}$$

where

$$\mathcal{F}_1\left(\frac{1}{2}|b|, \frac{1}{2}\operatorname{Re} b, -r\right) = \frac{(1+r)^{|b|-\operatorname{Re} b}}{(1-r)^{|b|+\operatorname{Re} b}} \tag{2.3}$$

and

$$\mathcal{F}_2\left(|b|, \operatorname{Re} b, r\right) = \frac{(1+r)^{\frac{1}{2}|b|-\frac{1}{2}\operatorname{Re} b}}{(1-r)^{\frac{1}{2}|b|+\frac{1}{2}\operatorname{Re} b}}. \tag{2.4}$$

These inequalities are sharp because the extremal function is $h(z) = \frac{1}{(1-z)^b}$ with $z = \frac{r(r-\frac{\overline{b}}{b})^{1/2}}{1-r(\frac{\overline{b}}{b})^{1/2}}$.

Proof Using Theorem 1.3, the definition of class $\mathcal{C}(b)$ and the definition of the subordination principle, we obtain

$$z\frac{h'(z)}{h(z)} = \frac{1+(b-1)\phi(z)}{1-\phi(z)} \quad \Rightarrow \quad z\frac{h'(z)}{h(z)} \prec \frac{1+(b-1)z}{1-z}$$

or

$$\left| z\frac{h'(z)}{h(z)} - \frac{br^2}{1-r^2} \right| \leq \frac{|b|r}{1-r^2}, \tag{2.5}$$

and similarly

$$z\frac{h''(z)}{h'(z)} = \frac{2b\phi(z)}{1-\phi(z)} \quad \Rightarrow \quad z\frac{h''(z)}{h'(z)} \prec \frac{2bz}{1-z}$$

or

$$\left|z\frac{h''(z)}{h'(z)}-\frac{2br^2}{1-r^2}\right|\leq\frac{2|b|r}{1-r^2}. \tag{2.6}$$

Using (2.5) and (2.6), we get (2.1) and (2.2), respectively. □

Theorem 2.2 *Let $f=h(z)+\overline{g(z)}$ be an element of $\mathcal{S}_{\mathcal{HCC}(b)}$, then*

$$\frac{g(z)}{h(z)}\prec b_1\frac{1+(2b-1)z}{1-z}\quad(z\in\mathbb{D}).$$

Proof Since $f=h(z)+\overline{g(z)}$ is an element of $\mathcal{S}_{\mathcal{HCC}(b)}$, then we have

$$\frac{g'(z)}{h'(z)}\prec b_1\frac{1+(2b-1)z}{1-z}\quad\Leftrightarrow\quad\operatorname{Re}\left[1+\frac{1}{b}\left(\frac{g'(z)}{h'(z)}-b_1\right)\right]>0,$$

so

$$\frac{g'(z)}{h'(z)}=b_1\frac{1+(2b-1)\phi(z)}{1-\phi(z)} \tag{2.7}$$

for some $\phi(z)\in\Omega$ and every z in $\mathbb{D}$. Now, we define the function $\phi(z)$ by

$$\frac{g(z)}{h(z)}=b_1\frac{1+\phi(z)}{1-\phi(z)}\quad(z\in\mathbb{D}),$$

then $\phi(z)$ is analytic in $\mathbb{D}$ and $\frac{g(z)}{h(z)}|_{z=0}=b_1=b_1\frac{1+\phi(0)}{1-\phi(0)}$, then $\phi(0)=0$ and

$$w(z)=\frac{g'(z)}{h'(z)}=b_1\left(\frac{1+\phi(z)}{1-\phi(z)}+\frac{2z\phi(z)}{1-\phi(z)}\cdot\frac{1}{1+(b-1)\phi(z)}\right)\quad(z\in\mathbb{D}).$$

Now it is easy to realize that the subordination $\frac{g'(z)}{h'(z)}\prec b_1\frac{1+(2b-1)z}{1-z}$ is equivalent to $|\phi(z)|<1$ for all $z\in\mathbb{D}$. Indeed, assume to the contrary, that there exists $z_1\in\mathbb{D}$ such that $|\phi(z_1)|=1$. Then by Jack?s lemma (Lemma1.4), $z_1\phi'(z_1)=k\phi(z_1)$, $k\geq 1$, for such $z_1\in\mathbb{D}$, we have

$$w(z_1)=\frac{g'(z_1)}{h'(z_1)}=b_1\left(\frac{1+\phi(z_1)}{1-\phi(z_1)}+\frac{2k\phi(z_1)}{1-\phi(z_1)}\cdot\frac{1}{1+(b-1)\phi(z_1)}\right)=w\bigl(\phi(z_1)\bigr)\notin w(\mathbb{D})$$

because $|\phi(z_1)|=1$ and $k\geq 1$. But this is a contradiction to the condition $\frac{g'(z)}{h'(z)}\prec b_1\frac{1+(2b-1)z}{1-z}$, and so assumption is wrong, *i.e.*, $|\phi(z)|<1$ for all $z\in\mathbb{D}$. □

Corollary 2.3 *Let $f=h(z)+\overline{g(z)}$ be an element of $\mathcal{S}_{\mathcal{HCC}(b)}$, then*

$$\begin{aligned}&\mathcal{F}_1\left(\frac{1}{2}|b|,\frac{1}{2}\operatorname{Re}b,-r\right)\frac{|b_1|-2|b|r-|b_1-2b|r^2}{1-r^2}\\&\quad\leq\bigl|g(z)\bigr|\leq\mathcal{F}_1\left(\frac{1}{2}|b|,\frac{1}{2}\operatorname{Re}b,r\right)\frac{|b_1|+2|b|r+|b_1-2b|r^2}{1-r^2}\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}&\mathcal{F}_2\big(|b|,\operatorname{Re} b,-r\big)\frac{|b_1|-2|b|r-|b_1-2b|r^2}{1-r^2}\\&\quad\leq\big|g'(z)\big|\leq\mathcal{F}_2\big(|b|,\operatorname{Re} b,r\big)\frac{|b_1|+2|b|r+|b_1-2b|r^2}{1-r^2}\end{aligned}\tag{2.9}$$

for all $|z|=r<1$, *where* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *are defined by* (2.3) *and* (2.4), *respectively.*

Proof Since $f=h(z)+\overline{g(z)}\in\mathcal{S}_{\mathcal{HCC}(b)}$, we have

$$\operatorname{Re}\left[1+\frac{1}{b}\left(\frac{g'(z)}{h'(z)}-b_1\right)\right]>0\quad\Leftrightarrow\quad\frac{g'(z)}{h'(z)}\prec b_1\frac{1+(2b-1)z}{1-z}$$

or

$$\left|\frac{g'(z)}{h'(z)}-\frac{b_1+(2b-b_1)r^2}{1-r^2}\right|\leq\frac{2|b|r}{1-r^2},$$

then

$$\frac{|b_1|-2|b|r-|b_1-2b|r^2}{1-r^2}\leq\frac{|g'(z)|}{|h'(z)|}\leq\frac{|b_1|+2|b|r+|b_1-2b|r^2}{1-r^2},\tag{2.10}$$

and using Theorem 2.2 we obtain

$$\left|\frac{g(z)}{h(z)}-\frac{b_1+(2b-b_1)r^2}{1-r^2}\right|\leq\frac{2|b|}{1-r^2}$$

or

$$\frac{|b_1|-2|b|r-|b_1-2b|r^2}{1-r^2}\leq\frac{|g(z)|}{|h(z)|}\leq\frac{|b_1|+2|b|r+|b_1-2b|r^2}{1-r^2}\tag{2.11}$$

for all $|z|=r<1$. Considering Lemma 2.1, (2.10) and (2.11) together, we obtain (2.8) and (2.9). □

Lemma 2.4 *If* $f=h(z)+\overline{g(z)}\in\mathcal{S}_{\mathcal{HCC}(b)}$, *then*

$$\frac{|b_1|-r}{1+|b_1|r}\leq\big|w(z)\big|\leq\frac{|b_1|+r}{1+|b_1|r},\tag{2.12}$$

$$\frac{(1-r^2)(1-|b_1|^2)}{(1+|b_1|r)^2}\leq1-\big|w(z)\big|^2\leq\frac{(1-r^2)(1-|b_1|^2)}{(1-|b_1|r)^2},\tag{2.13}$$

$$\frac{(1-r)(1+|b_1|)}{1-|b_1|r}\leq1+\big|w(z)\big|\leq\frac{(1+r)(1+|b_1|)}{1+|b_1|r}\tag{2.14}$$

and

$$\frac{(1-r)(1-|b_1|)}{1+|b_1|r}\leq1-\big|w(z)\big|\leq\frac{(1+r)(1-|b_1|)}{1-|b_1|r}\tag{2.15}$$

for all $|z|=r<1$.

Proof Since $f = h(z) + \overline{g(z)} \in \mathcal{S}_{\mathcal{HCC}(b)}$, it follows that

$$w(z) = \frac{g'(z)}{h'(z)} = \frac{b_1 + 2b_2 z + \cdots}{1 + 2a_2 z + \cdots} \quad \text{so } w(0) = b_1 \text{ and } \left|w(z)\right| < 1.$$

So, the function

$$\phi(z) = \frac{w(z) - w(0)}{1 - \overline{w(0)}w(z)} = \frac{w(z) - b_1}{1 - \overline{b}_1 w(z)} \quad (z \in \mathbb{D})$$

satisfies the conditions of Schwarz lemma. Therefore, we have

$$w(z) = \frac{b_1 + \phi(z)}{1 + \overline{b}_1 \phi(z)} \quad \text{if and only if} \quad w(z) \prec \frac{b_1 + z}{1 + \overline{b}_1 z} \quad (z \in \mathbb{D}).$$

On the other hand, the linear transformation $\frac{b_1+z}{1+\overline{b}_1 z}$ maps $|z| = r$ onto the disc with the center $C(r) = (\frac{(1-r^2)\operatorname{Re} b_1}{1-|b_1|^2 r^2}, \frac{(1-r^2)\operatorname{Im} b_1}{1-|b_1|^2 r^2})$ and the radius $\rho(r) = \frac{(1-|b_1|^2)r}{1-|b_1|r^2}$. Then we have (2.12), which gives (2.13), (2.14) and (2.15). □

Corollary 2.5 *Let $f(z)$ be an element of $\mathcal{S}_{\mathcal{HCC}(b)}$, then*

$$\frac{(1-r^2)(1-|b_1|)^2}{(1+|b_1|r)^2}\bigl(\mathcal{F}_2\bigl(|b|, \operatorname{Re} b, -r\bigr)\bigr)^2 \le J_f \le \frac{(1-r^2)(1-|b_1|)^2}{(1-|b_1|r)^2}\bigl(\mathcal{F}_2\bigl(|b|, \operatorname{Re} b, r\bigr)\bigr)^2$$

and

$$\begin{aligned}&\bigl(1 - |b_1|\bigr)\int_0^r \frac{1-\rho}{1+|b_1|\rho}\mathcal{F}_2\bigl(|b|, \operatorname{Re} b, -\rho\bigr)\,d\rho \\ &\quad \le |f| \le \bigl(1 + |b_1|\bigr)\int_0^r \frac{1+\rho}{1+|b_1|\rho}\mathcal{F}_2\bigl(|b|, \operatorname{Re} b, \rho\bigr)\,d\rho\end{aligned}$$

for all $|z| = r < 1$, where $\mathcal{F}_2$ is defined by (2.4).

Proof Since

$$\bigl(1 - \left|w(z)\right|^2\bigr)\left|h'(z)\right|^2 \le J_j \le \bigl(1 + \left|w(z)\right|^2\bigr)\left|h'(z)\right|$$

and

$$\bigl(1 - \left|w(z)\right|\bigr)\left|h'(z)\right||dz| \le |df| \le \bigl(1 + \left|w(z)\right|\bigr)\left|h'(z)\right||dz|,$$

thus using Lemma 2.1 and Lemma 2.4 in the last two inequalities we obtain the desired result. □

Theorem 2.6 *Let $f(z)$ be an element of $\mathcal{S}_{\mathcal{HCC}(b)}$, then*

$$\sum_{k=2}^{n} |b_k - b_1 a_k|^2 \le \sum_{k=1}^{n-1} \left|b_k + b_1(2b-1)a_k\right|^2.$$

Proof Using Theorem 2.2, we obtain the following relation:

$$\frac{g(z)}{h(z)} \prec b_1\frac{1+(2b-1)z}{1-z} \quad \Rightarrow \quad \frac{g(z)}{h(z)} = \frac{b_1 + b_1(2b-1)\phi(z)}{1-\phi(z)}$$

or

$$g(z) - b_1h(z) = \bigl(g(z) + b_1(2b-1)h(z)\bigr)\phi(z) \quad \bigl(z \in \mathbb{D}, \phi(z) \in \Omega\bigr). \tag{2.16}$$

Equality (2.16) can be written in the following form:

$$\sum_{k=2}^{n}(b_k - b_1a_k)z^k + \sum_{k=n+1}^{\infty} d_kz^k = \left(\sum_{k=1}^{n-1}\bigl(b_k + b_1(2b-1)a_k\bigr)z^k\right)\phi(z) \quad (z \in \mathbb{D}). \tag{2.17}$$

Since the last equality has the form $f_1(z) = f_2(z)\phi(z)$ with $|\phi(z)| < 1$, it follows that

$$\frac{1}{2\pi}\int_0^{2\pi}\bigl|f_1\bigl(re^{i\theta}\bigr)\bigr|^2\,d\theta \le \frac{1}{2\pi}\int_0^{2\pi}\bigl|f_2\bigl(re^{i\theta}\bigr)\bigr|^2\,d\theta \tag{2.18}$$

for each r $(0 < r < 1)$. Expressing (2.18) in terms of the coefficients in (2.17), we obtain the inequality

$$\sum_{k=2}^{n}|b_k - b_1a_k|^2r^{2k} + \sum_{k=n+1}^{\infty}|d_k|^2r^{2k} \le \sum_{k=1}^{n-1}\bigl|b_k + b_1(2b-1)a_k\bigr|^2r^{2k}, \tag{2.19}$$

where $d_k = (b_k - b_1a_k) - (b_k + b_1(2b-1)a_k)\phi(z)$. By letting $r \to 1^-$ in (2.19) we obtain the desired result. The proof of this method is due to Clunie [10]. □

Competing interests
The authors declare that they have no competing interests.

Authors? contributions
All authors contributed equally and significantly to writing this paper. All authors read and approved the final manuscript.

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