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SOME INEQUALITIES WHICH HOLD FOR STARLIKE LOG-HARMONIC MAPPINGS OF ORDER α

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Abstract

Let $H(D)$ be the linear space of all analytic functions defined on the open disc $D = \{z| \quad |z| < 1\}$. A log-harmonic mappings is a solution of the nonlinear elliptic partial differential equation

$$\overline{f_{\overline{z}}} = w\frac{\overline{f}}{f}f_z$$

where $w(z) \in H(D)$ is second dilatation such that $|w(z)| < 1$ for all $z \in D$. It has been shown that if f is a non-vanishing log-harmonic mapping, then f can be expressed as

$$f(z) = h(z)\overline{g(z)}$$

where $h(z)$ and $g(z)$ are analytic function in D. On the other hand, if f vanishes at $z = 0$ but it is not identically zero then f admits following representation

$$f(z) = z\,|z|^{2\beta}\,h(z)\overline{g(z)}$$

where $Re\beta > -\frac{1}{2}$, h and g are analytic in D, $g(0) = 1$, $h(0) \neq 0$. Let $f = z\,|z|^{2\beta}\,h\overline{g}$ be a univalent log-harmonic mapping.

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We say that f is a starlike log-harmonic mapping of order α if

$$\frac{\partial(\arg f(re^{i\theta}))}{\partial\theta} = Re\frac{zf_z - \overline{z}f_{\overline{z}}}{f} > \alpha\ ,\ 0 \leq \alpha < 1.\ (\forall z \in U)$$

and denote by $S^*_{lh}(\alpha)$ the set of all starlike log-harmonic mappings of order α.

The aim of this paper is to define some inequalities of starlike log-harmonic functions of order α $(0 \leq \alpha \leq 1)$.

I. Introduction

Let Ω be the family of functions $\phi(z)$ regular in the unit disc D and satisfying the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for all $z \in D$.

Next, denote by $P(\alpha)(0 \leq \alpha < 1)$ the family of functions

$$p(z) = 1 + p_1 z + ...$$

regular in D and such that $p(z)$ in $P(\alpha)$ if and only if

$$p(z) = \frac{1 + (1 - 2\alpha)\ \phi(z)}{1 - \phi(z)}$$

for some functions $z \in \Omega$ and every $z \in D$.

Let $S_1(z)$ and $S_2(z)$ be analytic functions in the open unit disc, with $S_1(0) = S_2(0)$, if $S_1(z) = S_2(\phi(z))$ then we say that $S_1(z)$ is subordinate to $S_2(z)$, where $\phi(z) \in \Omega([4])$, and we write $S_1(z) \prec S_2(z)$.

Let $H(D)$ be the linear space of all analytic functions defined on the open disc $D = \{z|\,|z| < 1\}$. A log-harmonic mappings is a solution of the nonlinear elliptic partial differential equation

$$\overline{f_{\overline{z}}} = w\frac{\overline{f}}{f}f_z$$

where $w(z) \in H(D)$ is second dilatation such that $|w(z)| < 1$ for all $z \in D$.

It has been shown that if f is a non-vanishing log-harmonic mapping, then f can be expressed as

$$f(z) = h(z)\overline{g(z)}$$

where $h(z)$ and $g(z)$ are analytic function in D.

On the other hand, if f vanishes at $z=0$ but it is not identically zero then f admits following representation

$$f(z) = z\left|z\right|^{2\beta} h(z)\overline{g(z)}$$

where $Re\beta > -\frac{1}{2}$, h and g are analytic in D, $g(0)=1$, $h(0) \neq 0$.

Let $f = z\left|z\right|^{2\beta} h\overline{g}$ be a univalent log-harmonic mapping. We say that f is a starlike logharmonic mapping of order α if

$$\frac{\partial(\arg f(re^{i\theta}))}{\partial\theta} = Re\frac{zf_z - \overline{z}f_{\overline{z}}}{f} > \alpha \ , \ 0 \leq \alpha < 1. \ (\forall z \in U)$$

and denote by $S^*_{lh}(\alpha)$ the set of all starlike log-harmonic mappings of order α([3]).

If $\alpha = 0$, we get the class of starlike log-harmonic mappings. Also, let

$$ST(\alpha) = \left\{f \in S^*_{lh}(\alpha)\, and\, f \in H(U)\right\}.$$

If $f \in S^*_{lh}(0)$ then $F(\varsigma) = \log(f(e^{\varsigma}))$ is univalent and harmonic on the half plane $\{\varsigma \mid \ Re\{\varsigma\} < 0\}$. It is known that F is closely related with the theory of nonparametric minimal surfaces over domains of the form $-\infty < u < u_0(v)$, $u_0(v+2\pi) = u_0(v)$, see ([1],[2]).

In this paper, we obtain Marx-Strohhacker Inequality and new distortion theorems using the subordination prinsiple for the starlike log-harmonic mappings of order α, previously studied by Z. Abdulhadi and Y. Abu Muhanna [3] who obtained the representation theorem and a different distortion theorem for the same class.

II. Main Results

Theorem 2.1. Let $f(z) = zh(z)\overline{g(z)}$ be an analytic logaritmic harmonic function in the open unit disc U. If $f(z)$ satisfies the condition

$$z\frac{h'(z)}{h(z)} - z\frac{g'(z)}{g(z)} \prec \frac{2(1-\alpha)z}{1-z} = F(z) \tag{1}$$

then $f \in S^*_{lh}(\alpha)$.

Proof. We define the function by

$$\frac{h}{g} = (1-\phi(z))^{-2(1-\alpha)} \tag{2}$$

where $(1-\phi(z))^{-2(1-\alpha)}$ has the value 1 at $z = 0$. Then $w(z)$ is analytic and $\phi(0) = 0$. If we take the logarithmic derivative of (2) and the after brief calculations, we get

$$z\frac{h'(z)}{h(z)} - z\frac{g'(z)}{g(z)} \prec \frac{2(1-\alpha)z\phi'(z)}{1-\phi(z)}$$

Now it is easy to realize that the subordination (1) is equivalent to $|\phi(z)| < 1$ for all $z \in U$. Indeed assume the contrary: then there is a $z_1 \in U$ such that $|\phi(z_1)| = 1$, so by I.S. Jack Lemma $z_1\phi'(z_1) = k\phi(z_1)$ for some $k \geq 1$ and for such $z_1 \in U$, we have

$$z_1\frac{h'(z_1)}{h(z_1)} - z_1\frac{g'(z_1)}{g(z_1)} \prec \frac{2(1-\alpha)k\phi(z_1)}{1-\phi(z_1)} = F(\phi(z_1)) \notin F(U)$$

but this contradicts (1); so our assumption is wrong, i.e, $|\phi(z)| < 1$ for all $z \in u$. By using condition (1) we get

$$1 + z\frac{h'(z)}{h(z)} - z\frac{g'(z)}{g(z)} = \frac{1+(1-2\alpha)\phi(z)}{1-\phi(z)} \quad . \tag{3}$$

The equality (3) shows that $f(z) \in S^*_{lh}(\alpha)$.

Corollary 2.2. For the starlike logharmonic functions of order α, we have Marx-Strohhacker Inequality is

$$\left|1-\left(\frac{g}{h}\right)^{\frac{1}{2(1-\alpha)}}\right|<1$$

g and $h$are analytic in u and $0 \notin hg(u)$.

Proof. Using theorem 2.1 we have

$$(1-\phi(z))^{\frac{1-2\alpha+1}{-1}}=\tfrac{h}{g} \;\Rightarrow (1-\phi(z))^{-2(1-\alpha)}=\tfrac{h}{g} \;\Rightarrow \tfrac{1}{(1-\phi(z))^{2(1-\alpha)}}=\tfrac{h}{g} \;\Rightarrow \tfrac{1}{1-\phi(z)}=\left(\tfrac{h}{g}\right)^{\frac{1}{2(1-\alpha)}} \Rightarrow$$
$$1-\phi(z)=\left(\tfrac{g}{h}\right)^{\frac{1}{2(1-\alpha)}} \;\Rightarrow 1-\left(\tfrac{g}{h}\right)^{\frac{1}{2(1-\alpha)}}=\phi(z) \;\Rightarrow \left|1-\left(\tfrac{g}{h}\right)^{\frac{1}{2(1-\alpha)}}\right|=|\phi(z)|<1.$$

Theorem 2.3. If $f \in S^*_{lh}(\alpha)$ then

$$\frac{1}{(1+r)^{2(1-\alpha)}} \leq \left|\frac{h}{g}\right| < \frac{1}{(1-r)^{2(1-\alpha)}} \tag{4}$$

Proof. The set of the values of the function $\left(\frac{2(1-\alpha)z}{(1-z)}\right)$ is the closed disc with the centre C and the radius ρ**,** where

$$C=C(r)=\left(\frac{2(1-\alpha)r^2}{1-r^2},0\right) \quad , \quad \rho=\rho(r)=\frac{2(1-\alpha)r}{1-r^2}.$$

Using the subordination, we can write

$$\left|\left(z\frac{h'(z)}{h(z)}-z\frac{g'(z)}{g(z)}\right)-\frac{2(1-\alpha)r^2}{1-r^2}\right| \leq \frac{2(1-\alpha)r}{1-r^2}. \tag{5}$$

Therefore we have

$$-\frac{2(1-\alpha)r}{1+r} \leq Re\left(z\frac{h'(z)}{h(z)}-z\frac{g'(z)}{g(z)}\right) \leq \frac{2(1-\alpha)r}{1-r}. \tag{6}$$

On the other hand

$$Re\left(z\frac{h'}{h}\right)-Re\left(z\frac{g'}{g}\right)=r\frac{\partial}{\partial r}\left(\log|h|-\log|g|\right). \tag{7}$$

If we consider the relations (5), (6), (7) together we obtain

$$-\frac{2(1-\alpha)}{1+r} \leq \frac{\partial}{\partial r}\left(\log|h|-\log|g|\right) \leq \frac{2(1-\alpha)}{1-r} \tag{8}$$

After the integrating we obtain (4).

Theorem 2.4. If $f \in S^*_{lh}(\alpha)$ then

$$\frac{|b_1| - |a_1|r}{|a_1| - |b_1|r}(1-r)^{2(1-\alpha)} \leq \frac{|g'(z)|}{|h'(z)|} \leq \frac{|b_1| + |a_1|r}{|a_1| + |b_1|r}(1+r)^{2(1-\alpha)}. \tag{9}$$

Proof. Using theorem 2.3 we can write

$$(1-r)^{2(1-\alpha)} \leq \frac{|g(z)|}{|h(z)|} \leq (1+r)^{2(1-\alpha)} \tag{10}$$

On the other hand, since f is solution of the non-linear elliptic partial differential equation

$$\overline{f_{\overline{z}}} = w\frac{\overline{f}}{f}f_z$$

then we obtain

$$w(z) = \frac{\frac{g'(z)}{h'(z)}}{\frac{g(z)}{h(z)}} = \frac{b_1}{a_1} + \ldots \tag{11}$$

Now we define the function

$$\phi(z) = \frac{w(z) - w(0)}{1 - \overline{w(0)}w(z)}, z \in D. \tag{12}$$

Therefore $\phi(z)$ satisfies the condition of Schwarz lemma. Using the estimate the Schwarz lemma $|\phi(z)| \leq r$,which given

$$|\phi(z)| = \left|\frac{w(z) - w(0)}{1 - \overline{w(0)}w(z)}\right| \leq r \tag{13}$$

The inequality (13) can be written in the following form

$$\left|\frac{w(z) - \frac{b_1}{a_1}}{1 - \overline{\frac{b_1}{a_1}}}w(z)\right| \leq r \Rightarrow \left|w(z) - \frac{b_1}{a_1}\right| \leq r\left|1 - \frac{\overline{b_1}}{\overline{a_1}}w(z)\right| \tag{14}$$

The inequality (14) is equivalent

$$\left|w(z) - \frac{(1-r^2)\left|\frac{b_1}{a_1}\right|}{1 - (\frac{b_1}{a_1})^2 r^2}\right| \leq \frac{\left(1 - \left|\frac{b_1}{a_1}\right|^2\right)r}{1 - \left|\frac{b_1}{a_1}\right|^2 r^2} \tag{15}$$

The equality holds in the inequality (15) only for the function

$$w(z) = \frac{\frac{g'(z)}{h'(z)}}{\frac{g(z)}{h(z)}} \tag{16}$$

From the inequality (15) we have

$$\frac{\left|\frac{b_1}{a_1}\right| - r}{1 - \left|\frac{b_1}{a_1}\right| r} \leq |w(z)| \leq \frac{\left|\frac{b_1}{a_1}\right| + r}{1 + \left|\frac{b_1}{a_1}\right| r} \tag{17}$$

Considering the relation (10), (17) together, end after the simple calculations,

$$\frac{|b_1| - |a_1| r}{|a_1| - |b_1| r} \left|\frac{g(z)}{h(z)}\right| \leq \left|\frac{g'(z)}{h'(z)}\right| \leq \frac{|b_1| + |a_1| r}{|a_1| + |b_1| r} \left|\frac{g(z)}{h(z)}\right| \tag{18}$$

using inequality (4) in the inequality (18) we get (8).

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