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Harmonic Mappings Related to Janowski Convex Functions of Complex Order b

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Abstract

Let S_H be the class of all sense-preserving harmonic mappings in the open unit disc $D = \{z \in \mathbb{C} | |z| < 1\}$.
In the present paper the authors investigate the properties of the class of harmonic mappings which is based on the generalized of R. J. Libera Theorem [7].

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1 Introduction

Let F be the class of analytic functions in D, and let S denote those functions in F that are univalent and normalized by $h(0) = 0$, $h'(0) = 1$. Furthermore, let Ω be the family of functions $\phi(z)$ regular in D and satisfying the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for every $z \in D$.
Next, for arbitrary fixed numbers A, B, $-1 < A \leq 1$, $-1 \leq B < A$ denote by $P(A, B)$, the family of functions $p(z) = 1 + p_1 z + p_2 z^2 + ...$ regular in D and such that $p(z)$ is in $P(A, B)$ if and only if

$$p(z) = \frac{1 + A\phi(z)}{1 + B\phi(z)} \tag{1}$$

for some $\phi(z) \in \Omega$ and every $z \in D$. This class was introduced by Janowski [6].
Moreover, let $S^*(A, B, b)$ be denote by the family of functions $h(z) \in S$ such that $h(z)$ is in $S^*(A, B, b)$ if and only if $(\frac{f(z)}{z}) \neq 0$,

$$1 + \frac{1}{b}(z\frac{h'(z)}{h(z)} - 1) = p(z), z \in D, (b \neq 0, complex) \tag{2}$$

for some function $p(z) \in P(A, B)$ and all $z \in D$. Let $C(A, B, b)$ denote the family of functions which are regular, such that $h(z)$ is in $C(A, B, b)$ if and only if

$$1 + \frac{1}{b}z\frac{h''(z)}{h'(z)} = p(z), z \in D, (b \neq 0, complex) \tag{3}$$

for some function $p(z) \in P(A, B)$ and all $z \in D$. Let $s_1(z) = z + d_2 z^2 + ...$ and $s_2(z) = z + e_2 z^2 + ...$ be elements of F. If there exists $\phi(z) \in \Omega$ such that $s_1(z) = s_2(\phi(z))$, then we say that $s_1(z)$ is sunbordinate to $s_2(z)$ and we write $s_1(z) \prec s_2(z)$. Specially if $s_2(z)$ is univalent in D, then $s_1(z) \prec s_2(z)$ if and only if $S_1(D) \subset S_2(D)$ and $S_1(0) = S_2(0)$ implies $S_1(D_r) \subset S_2(D_r)$ where $D_r = \{z||z| < r \ 0 < r < 1$. (Subordination and Lindelof principle [1], [5])
Finally, a planar harmonic mapping in the open disc D ia a complex-valued harmonic function f, which maps D onto some planar domain $f(D)$. Since D is simply-connected domain, the mapping f has a canonical decomposition $f = h + \overline{g}$, where $h(z)$ and $g(z)$ are analytic in D and have the following power series

$$h(z) = \sum_{n=0}^{\infty} a_n z^n, g(z) = \sum_{n=0}^{\infty} b_n z^n,$$

where a_n, $b_n \in \mathbb{C}$, $n = 0, 1, 2, 3, ...$ as usual we call $h(z)$ analytic part of $f(z)$ and $g(z)$ co-analytic part of f, an elegant and complete treatment of the theory of harmonic mapping in given Duren's monograph [4]. Lewy proved in 1936 [4] that the harmonic mapping f locally univalent in D if and only if its Jacobian $J_f = |h'(z)|^2 - |g'(z)|^2$ is different from zero in . In view of this result locally univalent harmonic mapping in the oopen unit disc D are either sense-preserving if $|h'(z)| > |g'(z)|$ in D, or sense-reversing if $|g'(z)| > |h'(z)|$ in . Through this paper we will restrict ourselves to the study sense-preserving harmonic mappings. We also note that $f = h(z) + \overline{g(z)}$ is sense-preserving in D if and only if $h'(z)$ does not vanish in D, and the second dilatation $w(z) = (\frac{h'(z)}{g'(z)})$ has the property $|w(z)| < 1$ for all $z \in D$. Therefore the class of all sense-preserving harmonic mappings in the open unit disc wit $a_0 = b_0 = 0$, and $a_1 = 1$ will be denoted by S_H. Thus S_H contains standard class S of univalent functions. The family of all mappings $f \in S_H$ wtih the additional property $g'(0) = 0$, i.e, $b_1 = 0$ is denoted by S_H^0. Thus it is clear that $S \subset S_H^0 \subset S_H$. The aim of this paper is to inveastigate some properties of the subclass

$$S_{HC(A,B,b)} = \Big\{ f = h(z) + \overline{g(z)} \mid 1 + \frac{1}{b}(\frac{g'(z)}{h'(z)} - b_1) \prec \frac{1+Az}{1+Bz}, b, b_1 \in \mathbb{C}, b \neq 0, |b_1| < 1,$$

$$h(z) \in C(A, B, b) \Big\}.$$

For the aim of this paper we will need the following lemma and theorem.

Lemma 1.1 *([2]) Let $\phi(z)$ be a non-constant analytic function in the unit disc D with $\phi(0) = 0$. If $|\phi(z)|$ attains its maximum value on the circle $|z| = r$ at the point z_0 then $z_0\phi'(z_0) = k\phi(z_0)$, $k \geq 1$.*

Theorem 1.2 *([7]) Let $h(z)$ be an element of $C(A, B, b)$, then*

$$2[1 + \frac{1}{b}(z\frac{h'(z)}{h(z)} - 1)] - 1 = \frac{1 + A\phi(z)}{1 + B\phi(z)}$$

2 Main Results

Theorem 2.1 *Let $f = (h(z) + \overline{g(z)})$ be an element of $S_{HC(A,B,b)}$, then*

$$\frac{g(z)}{h(z)} \prec b_1 \frac{1 + Az}{1 + Bz}, \tag{4}$$

Proof 2.2 *Since $f = (h(z) + \overline{g(z)})$ be an element of $S_{HC(A,B,b)}$,then*

$$1 + \frac{1}{b}(\frac{g'(z)}{h'(z)} - b_1) \prec \frac{1 + Az}{1 + Bz} \Rightarrow 1 + \frac{1}{b}(\frac{g'(z)}{h'(z)} - b_1) = \frac{1 + A\phi(z)}{1 + B\phi(z)} \Rightarrow \tag{5}$$

$$\frac{g'(z)}{h'(z)} = \frac{b_1 + (b_1B + b(A-B))\phi(z)}{1 + B\phi(z)} \Rightarrow \frac{g'(z)}{h'(z)} \prec \frac{b_1 + (b_1B + b(A-B))z}{1 + Bz} \quad (6)$$

On the other hand the transformation

$$(\frac{b_1 + (b_1B + b(A-B))z}{1 + Bz})$$

maps $|z| = r$ *onto the disc with the centre*

$$C(r) = \frac{b_1 + (b(B^2 - AB) - b_1B^2)r^2}{1 - B^2r^2}$$

and the radius $\rho(r) = \frac{|b|(A-B)r}{1-B^2r^2}$, *therefore using the subordination principle we can write*

$$\left|\frac{g'(z)}{h'(z)} - \frac{b_1 + (b(B^2 - AB) - b_1B^2)r^2}{1 - B^2r^2}\right| \leq \frac{|b|\,(A-B)r}{1 - B^2r^2} \quad (7)$$

The inequality (7) *shows that all the values of* $(\frac{g'(z)}{h'(z)})$ *are in the disc*

$$W(D_r) = \left\{z \left|\left|\frac{g'(z)}{h'(z)} - \frac{b_1 + (b(B^2 - AB) - b_1B^2)r^2}{1 - B^2r^2}\right| \leq \frac{|b|\,(A-B)r}{1 - B^2r^2}\right.\right\}. \quad (8)$$

Now, we define the function $\phi(z)$ *by,*

$$\frac{g(z)}{h(z)} = b_1\frac{1 + A\phi(z)}{1 + B\phi(z)}$$

then we see that $\phi(z)$ *is analytic in* D *and*

$$\left.\frac{g(z)}{h(z)}\right|_{z=0} = b_1 = b_1\frac{1 + A\phi(0)}{1 + B\phi(0)} \Rightarrow \phi(0) = \frac{0}{(A-B)} = 0,$$

$$w(z) = \frac{g'(z)}{h'(z)} = b_1(\frac{1 + A\phi(z)}{1 + B\phi(z)} + \frac{(A-B)z\phi'(z)}{(1 + B\phi(z))^2}\frac{h(z)}{zh'(z)}) \quad (9)$$

Using Theorem 1.2 *in the equality* (9), *this equality can be written in the following form:*

$$w(z) = \frac{g'(z)}{h'(z)} = b_1(\frac{1 + A\phi(z)}{1 + B\phi(z)} + \frac{2(A-B)z\phi'(z)}{(1 + B\phi(z))(2 + (b(A-B) + 2B))\phi(z)}) \quad (10)$$

Now, it is easy to realize that the subordination (4) *is equivalent to* $|\phi(z)| < 1$ *for all* $z \in D$. *Indeed we assume the contrary; then* *there is a* $z_0 \in D$ *such that* $|\phi(z_0)| = 1$ *so by I. S. Jack's Lemma (Lemma* 1.1*),* $z_0\phi'(z_0) = k\phi(z_0)$ *for some* $k \geq 1$ *and for such* z_0 *we have*

$$w(z) = \frac{g'(z_0)}{h'(z_0)} = b_1(\frac{1 + A\phi(z_0)}{1 + B\phi(z_0)} + \frac{2k(A - B)\phi(z_0)}{(1 + B\phi(z_0))(2 + (b(A - B) + 2B))\phi(z_0)})$$

$$= w(\phi(z_0)) \notin w(D_r)$$

but this contradicts to (4)*; so our assumption is wrong, i. e,* $|\phi(z)| < 1$ *for all* $z \in D$*.*

Lemma 2.3 *Let* $h(z)$ *be an element of* $C(A, B, b)$ *then*

$$rF(A, B, |b|, Reb, -r) \leq |h(z)| \leq rF(A, B, |b|, Reb, r), B \neq 0;$$

$$re^{-A|b|r} \leq |h(z)| \leq re^{A|b|r}, B = 0.$$

$$G(A, B, |b|, Reb, -r) \leq |h'(z)| \leq G(A, B, |b|, Reb, r), B \neq 0;$$

$$e^{-Abr} \leq |h'(z)| \leq e^{Abr}, B = 0.$$

where

$$F(A, B, |b|, Reb, r) = \frac{(1 + Br)^{\frac{A-B}{4B}(|b|+Reb)}}{(1 - Br)^{\frac{A-B}{4B}(|b|-Reb)}}$$

$$G(A, B, |b|, Reb, r) = \frac{(1 + Br)^{\frac{A-B}{2B}(|b|+Reb)}}{(1 - Br)^{\frac{A-B}{2B}(|b|-Reb)}}$$

These inequalities are sharp.

Proof 2.4 *Using Theorem* 1.2 *and subordination principle, then we write*

$$\left|(2\left(1 + \frac{1}{b}(z\frac{h'(z)}{h(z)} - 1)\right) - 1) - \frac{1 + ABr^2}{1 - B^2r^2}\right| \leq \frac{(A - B)r}{1 - B^2r^2}, B \neq 0;$$

$$\left|(2\left(1 + \frac{1}{b}(z\frac{h'(z)}{h(z)} - 1)\right) - 1) - 1\right| \leq Ar, B = 0.$$

After the straightforward calculations, we obtain

$$\left|z\frac{h'(z)}{h(z)} - \frac{2 - (2B^2 - b(B^2 - AB))r^2)}{2(1 - B^2r^2)}\right| \leq \frac{|b|(A - B)r}{2(1 - B^2r^2)}, B \neq 0;$$

$$\left|z\frac{h'(z)}{h(z)} - 1\right| \leq \frac{|b|Ar}{2}, B = 0.$$

These inequalities can be written in the following form,

$$\frac{2-|b|(A-B)r-(M-Nx)r^2}{2r(1+Br)(1-Br)} \leq \frac{\partial}{\partial r}\log|h(z)| \leq \frac{2+|b|(A-B)r+(M-Nx)r^2}{2r(1+Br)(1-Br)}, B\neq 0;$$

$$\frac{1}{r}-\frac{A|b|}{2} \leq \frac{\partial}{\partial r}\log|h(z)| \leq \frac{1}{r}+\frac{A|b|}{2}, B=0.$$

where $M=2B^2$, $N=(B^2-AB)$, $x=Reb$. *Then after integration from the last inequality, we get the result.*

Similarly, using subordination principle nad the definition of the class $C(A,B,b)$, *then we write*

$$\left|(1+\frac{1}{b}\frac{h''(z)}{h'(z)})-\frac{1-ABr^2}{1-Br^2}\right| \leq \frac{(A-B)r}{1-B^2r^2}, B\neq 0;$$

$$\left|\frac{1}{b}\frac{h''(z)}{h'(z)}\right| \leq Ar, B=0.$$

These inequalities can be written in the following form,

$$\frac{rK_1-L_1}{(1-Br)(1+Br)} \leq \frac{\partial}{\partial r}\log|h'(z)| \leq \frac{rK_1+L_1}{(1-Br)(1+Br)}, B\neq 0;$$

$$-|b|A \leq \frac{\partial}{\partial r}\log|h'(z)| \leq |b|A, B=0.$$

where $K_1=(B^2-AB)Reb$, $L_1=|b|(A-B)$. *After the integration from here, we get the result.*

Corollary 2.5 *Let* $f=(h(z)+\overline{g(z)})$ *be an element of* $S_{HC(A,B,b)}$, *then*

$$rF(A,B,|b|,Reb,-r)\frac{|b_1|-|b|(A-B)r-|b_1B^2-bB^2+bAB|r^2}{1-B^2r^2} \leq |g(z)| \leq$$

$$rF(A,B,|b|,Reb,r)\frac{|b_1|+|b|(A-B)r+|b_1B^2-bB^2+bAB|r^2}{1-B^2r^2}, B\neq 0;$$

$$re^{-A|b|r}[|b_1|-|b|Ar] \leq |g(z)| \leq re^{A|b|r}[|b_1|+|b|Ar], B=0.$$

$$G(A,B,|b|,Reb,-r)\frac{|b_1|-|b|(A-B)r-|b_1B^2-bB^2+bAB|r^2}{1-B^2r^2} \leq |g'(z)| \leq$$

$$G(A,B,|b|,Reb,r)\frac{|b_1|+|b|(A-B)r+|b_1B^2-bB^2+bAB|r^2}{1-B^2r^2}, B\neq 0;$$

$$G(A,B,|b|,Reb,-r)[|b_1|-|b|Ar] \leq |g'(z)| \leq G(A,B,|b|,Reb,r)[|b_1|+|b|Ar], B=0.$$

Proof 2.6 *This corollary is a simple consequence of Theorem* 1.2 *and Lemma* 2.2.

Lemma 2.7 *Let* $f = (h(z) + \overline{g(z)})$ *be an element of* $S_{HC(A,B,b)}$*, then*

$$\frac{|b_1| - r}{1 + |b_1|\, r} \leq |w(z)| = \left|\frac{g'(z)}{h'(z)}\right| \leq \frac{|b_1| + r}{1 + |b_1|\, r} \tag{11}$$

$$\frac{(1 - r^2)(1 - |b_1|^2)}{(1 + |b_1|\, r)^2} \leq (1 - |w(z)|^2) \leq \frac{(1 - r^2)(1 - |b_1|^2)}{(1 - |b_1|\, r)^2} \tag{12}$$

$$\frac{(1 - r)(1 + |b_1|)}{(1 - |b_1|\, r)} \leq (1 + |w(z)|) \leq \frac{(1 + r)(1 + |b_1|)}{(1 + |b_1|\, r)} \tag{13}$$

$$\frac{(1 - r)(1 - |b_1|)}{(1 + |b_1|\, r)} \leq (1 - |w(z)|) \leq \frac{(1 + r)(1 - |b_1|)}{(1 - |b_1|\, r)} \tag{14}$$

Proof 2.8 *Since* $f = (h(z) + \overline{g(z)}) \in S_{HC(A,B,b)}$*, then*

$$w(z) = \frac{g'(z)}{h'(z)} = \frac{b_1 + 2b_2 z + ...}{1 + 2a_2 z + ...} \Rightarrow w(0) = b_1, |w(z)| < 1$$

so the function

$$\phi(z) = \frac{w(z) - w(0)}{1 - \overline{w(0)}w(z)} = \frac{w(z) - b_1}{1 - \overline{b_1}w(z)}$$

satisfies the conditions of Schwarz Lemma. Therefore we have

$$w(z) = \frac{g'(z)}{h'(z)} = \frac{b_1 + \phi(z)}{1 + \overline{b_1}\phi(z)} \tag{15}$$

This shows that,

$$w(z) = \frac{g'(z)}{h'(z)} \prec \frac{b_1 + z}{1 + \overline{b_1}z}$$

On the other hand, the linear transformation $(\frac{b_1+z}{1+\overline{b_1}z})$ *maps* $|z| = r$ *onto the disc with the center*

$$C(r) = (\frac{(1 - r^2)Reb_1}{1 - r^2}, \frac{(1 - r^2)Imb_1}{1 - r^2})$$

at the radius

$$\rho(r) = \frac{(1 - |b_1|)^2 r}{1 - r^2}$$

then we have the results, easily.

Corollary 2.9 *Let $f = (h(z) + \overline{g(z)})$ be an element of $S_{HC(A,B,b)}$, then*

$$(G(A,B,|b|,Reb,-r))^2 \frac{(1-r^2)(1-|b_1|^2)}{(1+|b_1|r)^2} \leq J_f \leq (G(A,B,|b|,Reb,r))^2 \frac{(1-r^2)(1-|b_1|^2)}{(1-|b_1|r)^2},$$

$$e^{-2A|b|r} \frac{(1-r^2)(1-|b_1|^2)}{(1+|b_1|r)^2} \leq J_f \leq e^{2A|b|r} \frac{(1-r^2)(1-|b_1|^2)}{(1-|b_1|r)^2}, B = 0$$

Proof 2.10 *Since*

$$J_f = |h'(z)|^2 - |g'(z)|^2 = |h'(z)|^2 - |h'(z)|^2 |w(z)|^2$$
$$= |h'(z)|^2 (1 - |w(z)|^2)$$

Using Lemma 2.7, we get the result.

Corollary 2.11 *Let $f = (h(z) + \overline{g(z)})$ be an element of $S_{HC(A,B,b)}$, then*

$$\int_0^r G(A,B,|b|,Reb,-r) \frac{(1-r)(1-|b_1|)}{(1+|b_1|r)} dr \leq |f| \leq \int_0^r G(A,B,|b|,Reb,r) \frac{(1+r)(1+|b_1|)}{(1+|b_1|r)},$$

$$\int_0^r e^{-A|b|r} \frac{(1-r)(1-|b_1|)}{(1+|b_1|r)} \leq |f| \leq \int_0^r e^{A|b|r} \frac{(1+r)(1+|b_1|)}{(1+|b_1|r)}, B = 0.$$

Proof 2.12 *Since*

$$(|h'(z)| - |g'(z)|)\,|dz| \leq |df| \leq (|h'(z)| + |g'(z)|)\,|dz| \Rightarrow$$

$$|h'(z)|\,(1 - |w(z)|)\,|dz| \leq |df| \leq |h'(z)|\,(1 + |w(z)|)\,|dz| \qquad (16)$$

Using Lemma 2.7 and after integration we obtain the result.

Theorem 2.13 *Let $f = (h(z) + \overline{g(z)})$ be an element of $S_{HC(A,B,b)}$, then*

$$\sum_{k=2}^{n} k^2 |b_k - b_1 a_k|^2 \leq \left|1 - b_1^2\right|^2 + \sum_{k=2}^{n+1} k^2 |a_k - b_1 k|^2 \qquad (17)$$

Proof 2.14 *Using Lemma 2.7 we can write*

$$w(z) = \frac{g'(z)}{h'(z)} \prec \frac{b_1 + z}{1 + \overline{b_1} z} \Rightarrow \frac{g'(z)}{h'(z)} = \frac{b_1 + \phi(z)}{1 + b_1 \phi(z)} \Rightarrow$$

$$g'(z)(1+\overline{b_1}\phi(z)) = h'(z)(b_1+\phi(z)) \Rightarrow (g'(z) - b_1 h'(z)) = (h'(z) - \overline{b_1} g'(z))\phi(z) \Rightarrow$$

$$(\sum_{n=1}^{\infty} b_n z^n)' - b_1(z + \sum_{n=2}^{\infty} a_n z^n)' = [(z + \sum_{n=2}^{\infty} a_n z^n)' - b_1(\sum_{n=1}^{\infty} b_n z^n)']\phi(z) \Rightarrow$$

$$\sum_{k=2}^{n} k(b_k - b_1 a_k) z^{k-1} + \sum_{k=n+1}^{\infty} d_k z^{k-1} = [(1 - b_1^2) + \sum_{k=2}^{n} k(a_k - b_1 b_k) z^{k-1}]\phi(z) \tag{18}$$

Since the last inequality has the form $f_1(z) = f_2(z)\phi(z)$ with $|\phi(z)| < 1$, it follows that

$$\frac{1}{2\pi}\int_0^{2\pi} \left|f_1(re^{i\theta})\right|^2 d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} \left|f_2(re^{i\theta})\right|^2 d\theta \tag{19}$$

for each r, $(0 < r < 1)$. Expressing (19) *in terms of coefficient in* (17) *we obtain the inequality,*

$$\sum_{k=2}^{n} k \left|b_k - b_1 a_k\right|^2 r^{2k} + \sum_{k=n+1}^{\infty} \left|d_k\right|^2 r^{2k} \leq [\left|1 - b_1^2\right|^2 + \sum_{k=2}^{n+1} k^2 \left|a_k - b_1 b_k\right|^2] r^{2k} \tag{20}$$

By letting $r \to 1^-$ in (20)*, we obtain desired result. The Proof of this method is due to Clunie [1].*

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