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Generalized Frames in the Space of Strong Limit Power Functions

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Abstract. By using the existence of a larger orthonormal basis, the space of strong limit power functions is extended. We use the windowed Fourier transform and wavelet transform to analyze strong limit power signals and we construct generalized frame decompositions using the discretized versions of these transforms.

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1. Introduction

For understanding the time-frequency behavior of functions in several function spaces such as Lebesgue, Hardy, Sobolev and Besov, people were interested in windowed Fourier (also known as Gabor or Weyl-Heisenberg) transform and wavelet (also known as affine) transform. Then, these transforms were used in the theory of persistent signals rather than the transient ones and people tried to find out a space different from the basic one $\mathcal{L}_2(\mathbb{R})$. One important example of such spaces is $\mathcal{AP}(\mathbb{R})$, the space of almost periodic functions (see, e.g. [1],[2],[3],[7]), which is the uniform closure of the trigonometric polynomials.

A function has limit power if

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(t)|^2 \, dt$$

exists, and the set of all such functions is denoted by H_2 [10]. $\mathcal{AP}(\mathbb{R})$ and the Besicovitch space B_2, which is the completion of $\mathcal{AP}(\mathbb{R})$, are the subsets of H_2. It is known that H_2 is not closed under addition [9] and this causes some difficulties in such areas as Robust Control; the necessity of new, larger, nice spaces arises [8].

In [11],[13], the space of uniform limit power functions, $\mathcal{ULP}(\mathbb{R}^+)$ and the space of limit power functions, $\mathcal{LP}_2$ are proposed. A function f is said to have uniform limit power if it is a uniform limit on $\mathbb{R}^+$ of a sequence of functions in the form

$$\sum_{k=1}^{n} a_k e^{i\lambda_k t^{\alpha_k}},$$

where $a_k \in \mathbb{C}$, $\lambda_k \in \mathbb{R}$, and $\lambda_k \in (0,\infty)$. If we restrict $\alpha_k = 1$ for all k, we get $f \in \mathcal{AP}(\mathbb{R}^+)$. Thus, $\mathcal{ULP}(\mathbb{R}^+)$ is a new generalization of $\mathcal{AP}(\mathbb{R}^+)$. From these works, a number of questions about the uniqueness of the Fourier series for $\mathcal{ULP}(\mathbb{R}^+)$, approximation of a function $f \in \mathcal{ULP}(\mathbb{R}^+)$ by Bochner-Fejér trigonometric polynomials and generalization of $\mathcal{ULP}(\mathbb{R}^+)$ arise. As in Gelfand's theory, a new $\mathcal{C}^*$-algebra containing $\mathcal{ULP}(\mathbb{R}^+)$ and the space of strong limit power functions $\mathcal{SLP}([a,\infty))$ are defined for the solution of this kind of problems [12]. In [14], the strong limit power functions are extended from $\mathbb{R}^+$ to $\mathbb{R}$.

In this paper we show the existence of a larger orthonormal basis and denote the existence of some limits for the space of strong limit power functions. Then generalized frame decompositions for $\mathcal{SLP}$ functions are constructed by using windowed Fourier and wavelet transforms.

2. Notations and background material

Let $Q(\mathbb{R})$ consist of functions q in the form

$$q(t) = \begin{cases} \displaystyle\sum_{l=1}^{m} \lambda_l t^{\alpha_l}, & t \geq 0 \\ \displaystyle -\sum_{l=1}^{m} \lambda_l (-t)^{\alpha_l}, & t < 0 \end{cases} \tag{2.1}$$

where $m = 1,2,\ldots,\lambda_l \in \mathbb{R}$, $l = 1,2,\ldots,m$ and $\alpha_1 > \alpha_2 > \cdots > \alpha_m > 0$. Since $q(t) = -q(-t)$ for all $t \in \mathbb{R}$, every $q \in Q(\mathbb{R})$ is odd. A function of the form

$$P(t) = \sum_{k=1}^{n} a_k e^{iq_k(t)}$$

is called a *generalized trigonometric polynomial* on $\mathbb{R}$, where $a_k \in \mathbb{C}$, $q_k(t) \in Q(\mathbb{R})$, and $k = 1,2,\ldots,n$. Denote by $\mathrm{Gtrig}(\mathbb{R})$ the set of all such polynomials.

A function f on $\mathbb{R}$ is said to have *strong limit power* if for every $\varepsilon > 0$ there exists a $P_\varepsilon \in \mathrm{Gtrig}(\mathbb{R})$ such that

$$\|f - P_\varepsilon\| = \sup\{|f(t) - P_\varepsilon(t)| : t \in \mathbb{R}\} < \varepsilon. \tag{2.2}$$

Denote by $\mathcal{SLP}(\mathbb{R})$ the set of all such functions. It is obvious that $\mathcal{AP}(\mathbb{R}) \subset \mathcal{SLP}(\mathbb{R})$. The inner product of the $\mathcal{SLP}(\mathbb{R})$ space is defined by

$$\langle f, g\rangle := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)\bar{g}(t)\,\mathrm{d}t$$

(see [11],[12],[13],[14]).

A sequence (ϕ_k) in a Hilbert space H is called a *frame* if there exist constants A, $B > 0$ such that

$$A \sum |\langle f, \phi_k \rangle|^2 \leq \|f\|^2 \leq B \sum |\langle f, \phi_k \rangle|^2 \qquad \text{for all } f \in H. \tag{2.3}$$

We shall take the following as our definition of the Fourier transform:

$$\hat{f}(\omega) := \int_{-\infty}^{\infty} f(t) e^{-2\pi i \omega t} \, \mathrm{d}t,$$

defined for $f \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$, and extended in the usual way to be an isometry from $L_2(\mathbb{R})$ onto itself.

Let $g \in L_2(\mathbb{R})$ be fixed. For $f \in L_2(\mathbb{R})$, the *windowed Fourier transform* [6, p. 45] of f can be defined by

$$\tilde{f}(\omega, t) = \int_{-\infty}^{\infty} f(s) \bar{g}(s-t) e^{-2\pi i \omega s} \, \mathrm{d}s,$$

where $\bar{g}(s-t)$ denotes $\overline{g(s-t)}$.

The windowed Fourier transform can also be regarded as an inner product

$$\tilde{f}(\omega, t) = \langle f, g_{\omega,t} \rangle,$$

where $g_{\omega,t} \in L_2(\mathbb{R})$ is defined by

$$g_{\omega,t}(s) = e^{2\pi i \omega s} g(s-t). \tag{2.4}$$

For nonseparable spaces, such as the space of almost periodic functions, it is not possible to construct countable frames, and uncountable ones are of limited use in reconstruction problems. What we would like to do is to discretize the windowed Fourier transform, and this will allow us to construct generalized frames.

The natural discretization of the window function $g_{\omega,t}$ in (2.4) is given by $\omega = m\omega_0$, $t = nt_0$, for fixed ω_0, $t_0 \geq 0$; thus

$$g_{m,n}(s) = e^{2\pi i m \omega_0 s} g(s - nt_0). \tag{2.5}$$

Let $\psi \in L_2(\mathbb{R})$ be a function that satisfies the condition

$$C_\psi := \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} \, \mathrm{d}\omega < \infty.$$

We call such a ψ an *admissible wavelet.* For $f \in L_2(\mathbb{R})$, the *wavelet transform* is defined by

$$f^\circ(x, y) = |x|^{-1/2} \int_{-\infty}^{\infty} f(t) \overline{\psi\left(\frac{t-y}{x}\right)} \, \mathrm{d}t,$$

where $x \in \mathbb{R} \setminus \{0\}$ controls the resolution (scaling), and $y \in \mathbb{R}$ controls the positioning (see, e.g. [4, p. 24], [6, p. 63]).

Let $\gamma > 1$, $\beta > 0$ be fixed and m, n range over $\mathbb{Z}$. The *natural discretization* of the wavelet $\psi_{x,y}(t) = |x|^{-1/2}\psi((t-y)/x)$ has the form

$$\psi_{m,n}(t) = \gamma^{-m/2} \psi(\gamma^{-m} t - \beta n).$$

3. Orthonormal set

It is well known that $\{e^{i\lambda t}\}$ is a complete orthonormal basis in B_2 under the norm $\|f\|^2 = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f(t)|^2\,\mathrm{d}t$. As $\{e^{i\lambda t}\}$, the set $\{e^{i\lambda t^\alpha}\}$ where $\lambda \in \mathbb{R}$ and $0 < \alpha < \infty$ is also orthonormal [13, p. 425].

Now we'll prove that $\{e^{iq(t)}\}$ is orthonormal for every $q \in Q(\mathbb{R})$. Similar to the definition (2.1) of $q(t)$, $p(t)$ is defined as

$$p(t) = \begin{cases} \sum_{l=1}^{m} \mu_l t^{\beta_l}, & t \geq 0 \\ -\sum_{l=1}^{m} \mu_l (-t)^{\beta_l}, & t < 0 \end{cases} \tag{3.1}$$

where $m = 1, 2, \ldots, \mu_l \in \mathbb{R}$, $l = 1, 2, \ldots, m$ and $\beta_1 > \beta_2 > \cdots > \beta_m > 0$ and since $p(t) = -p(-t)$ for all $t \in \mathbb{R}$, every $p \in Q(\mathbb{R})$ is odd as well.

Theorem 3.1. *Let $q(t)$ and $p(t) \in Q(\mathbb{R})$. For $\alpha_1 \geq \beta_1 > \beta_2 > \cdots > \beta_m \geq 0$ with $\alpha_1 > \alpha_2 > \cdots > \alpha_m \geq 1$ and all nonzero λ_l's and μ_l's $\in \mathbb{R}$ where $l = 1, 2, \ldots, m$, the following limit*

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{i(q(t)-p(t))}\,\mathrm{d}t = \begin{cases} 1, & \alpha_l = \beta_l,\ \lambda_l = \mu_l\ (1 \leq l \leq m) \\ 0, & \textit{otherwise} \end{cases}$$

exists uniformly.

Proof. Since $q(t) - p(t)$ is odd, we get

$$\begin{aligned}
&\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{i(q(t)-p(t))}\,\mathrm{d}t \\
&\quad = \lim_{T\to\infty} \frac{1}{2T}\left[\int_{0}^{T} e^{i(q(t)-p(t))}\,\mathrm{d}t + \int_{-T}^{0} e^{-i(q(-t)-p(-t))}\,\mathrm{d}t\right] \\
&\quad = \lim_{T\to\infty} \frac{1}{2T}\left[\int_{0}^{T} e^{i(q(t)-p(t))}\,\mathrm{d}t + \int_{0}^{T} e^{-i(q(t)-p(t))}\,\mathrm{d}t\right] \\
&\quad = \lim_{T\to\infty} \frac{1}{2T}\left[\left(\int_{0}^{1} + \int_{1}^{T}\right) e^{i(q(t)-p(t))}\,\mathrm{d}t + \left(\int_{0}^{1} + \int_{1}^{T}\right) e^{-i(q(t)-p(t))}\,\mathrm{d}t\right]
\end{aligned}$$

for every p, $q \in Q(\mathbb{R})$.

The proof is obvious for the case $\alpha_m = \beta_m,\ \lambda_m = \mu_m$. We only need to consider the case $\alpha_m \neq \beta_m,\ \lambda_m \neq \mu_m$. Integrating by parts, we get

$$\begin{aligned}
&\int_{1}^{T} e^{i(q(t)-p(t))}\,\mathrm{d}t \\
&\qquad = \left.\frac{e^{i(q(t)-p(t))}}{i(q'(t)-p'(t))}\right|_{1}^{T} - \frac{1}{i}\int_{1}^{T} e^{i(q(t)-p(t))}\left[\frac{1}{q'(t)-p'(t)}\right]'\,\mathrm{d}t \\
&\qquad = I_1 + I_2.
\end{aligned}$$

We have

$$\begin{aligned}|I_1| &\le \frac{1}{|q'(T)-p'(T)|}+\frac{1}{|q'(1)-p'(1)|}\\ &= \frac{1}{T^{\alpha_1-1}\left|\sum_{l=1}^{m}\lambda_l\alpha_l T^{\alpha_l-\alpha_1}-\sum_{l=1}^{m}\mu_l\beta_l T^{\beta_l-\alpha_1}\right|}\\ &\quad+\frac{1}{\left|\sum_{l=1}^{m}\lambda_l\alpha_l-\sum_{l=1}^{m}\mu_l\beta_l\right|}\\ &\le M_1\end{aligned}$$

and also

$$\begin{aligned}|I_2| &\le \int_1^T\left|-\frac{q''(t)-p''(t)}{[q'(t)-p'(t)]^2}\right|\,\mathrm{d}t \le M_2\int_1^T\frac{1}{t^{\alpha_1}}\,\mathrm{d}t\\ &= \begin{cases} M_2\ln T, & \alpha_1=1\\ \\ \dfrac{M_2}{1-\alpha_1}\left[1-\dfrac{1}{T^{\alpha_1-1}}\right], & \alpha_1>1\end{cases}\end{aligned}$$

where M_1 and M_2 are constants which are independent of T.

It follows that

$$\left|\frac{1}{2T}\int_{-T}^{T}e^{i(q(t)-p(t))}\,\mathrm{d}t\right|\to 0$$

as $T\to\infty$. The proof is complete. □

Corollary 3.2. *Let $q(t)\in Q(\mathbb{R})$. For $\alpha_1>\alpha_2>\cdots>\alpha_m\ge 1$ and $\lambda_l\ne 0$ where $l=1,2,\ldots,m$, the following limit*

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}e^{iq(t)}\,\mathrm{d}t=0$$

exists uniformly.

Proof. If we put $\mu_l=0$ for $l=1,2,\ldots,m$ in Theorem 3.1, we get the conclusion. □

Theorem 3.3. *Let $q(t)$ and $p(t)\in Q(\mathbb{R})$. For $1>\alpha_1\ge\beta_1>\beta_2>\cdots>\beta_m\ge 0$ with $\alpha_1>\alpha_2>\cdots>\alpha_m>0$ and all nonzero λ_l's and μ_l's $\in\mathbb{R}$ where $l=1,2,\ldots,m$, the following limit*

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}e^{i(q(t)-p(t))}\,\mathrm{d}t=\begin{cases}1, & \alpha_l=\beta_l,\ \lambda_l=\mu_l\ (1\le l\le m)\\ 0, & \textit{otherwise}\end{cases}$$

exists.

Proof. Let $T>1$ be so large that

$$|\lambda_1\alpha_1|>\sum_{k=2}^{m}|\lambda_k\alpha_k|+\sum_{k=1}^{m}|\mu_k\beta_k|.$$

As in the proof of Theorem 3.1, we have

$$\int_1^T e^{i(q(t)-p(t))}\,\mathrm{d}t = I_3 + I_4.$$

Since

$$\begin{aligned} q'(t) - p'(t) &= \sum_{l=1}^m \lambda_l \alpha_l t^{\alpha_l - 1} - \sum_{l=1}^m \mu_l \beta_l t^{\beta_l - 1} \\ &= t^{\alpha_1 - 1}\left[\sum_{l=1}^m \lambda_l \alpha_l t^{\alpha_l - \alpha_1} - \sum_{l=1}^m \mu_l \beta_l t^{\beta_l - \alpha_1}\right] \end{aligned}$$

and $T > 1$, we find

$$I_3 = \frac{T^{1-\alpha_1}}{\sum_{l=1}^m \lambda_l \alpha_l T^{\alpha_l - \alpha_1} - \sum_{l=1}^m \mu_l \beta_l T^{\beta_l - \alpha_1}} - \frac{1}{\sum_{l=1}^m \lambda_l \alpha_l - \sum_{l=1}^m \mu_l \beta_l}$$

and therefore

$$\begin{aligned} |I_3| \le & \frac{T^{1-\alpha_1}}{|\lambda_1 \alpha_1| - \sum_{l=2}^m |\lambda_l \alpha_l| - \sum_{l=1}^m |\mu_l \beta_l|} \\ & + \frac{T^{1-\alpha_1}}{|\lambda_1 \alpha_1| - \sum_{l=2}^m |\lambda_l \alpha_l| - \sum_{l=1}^m |\mu_l \beta_l|} \\ \le & M_3 T^{1-\alpha_1} \end{aligned}$$

where M_3 is a constant which is independent of T.

Since

$$q''(t) - p''(t) = \sum_{l=1}^m \lambda_l \alpha_l (\alpha_l - 1) t^{\alpha_l - 2} - \sum_{l=1}^m \mu_l \beta_l (\beta_l - 1) t^{\beta_l - 2}$$

and

$$\begin{aligned} [q'(t) - p'(t)]^2 &= \left[\sum_{l=1}^m \lambda_l \alpha_l t^{\alpha_l - 1} - \sum_{l=1}^m \mu_l \beta_l t^{\beta_l - 1}\right]^2 \\ &= t^{2\alpha_1 - 2}\left[\sum_{l=1}^m \lambda_l \alpha_l t^{\alpha_l - \alpha_1} - \sum_{l=1}^m \mu_l \beta_l t^{\beta_l - \alpha_1}\right]^2, \end{aligned}$$

we get

$$\frac{q''(t) - p''(t)}{[q'(t) - p'(t)]^2} = \frac{\sum_{l=1}^m \lambda_l \alpha_l (\alpha_l - 1) t^{\alpha_l - 2\alpha_1} - \sum_{l=1}^m \mu_l \beta_l (\beta_l - 1) t^{\beta_l - 2\alpha_1}}{\left[\sum_{l=1}^m \lambda_l \alpha_l t^{\alpha_l - \alpha_1} - \sum_{l=1}^m \mu_l \beta_l t^{\beta_l - \alpha_1}\right]^2}$$

For the estimation of I_4, we have

$$\begin{aligned}
|I_4| &\le \int_1^T \left|\frac{q''(t)-p''(t)}{[q'(t)-p'(t)]^2}\right| \mathrm{d}t \\
&\le \int_1^T \frac{\sum_{l=1}^m \left|\lambda_l\alpha_l(\alpha_l-1)t^{\alpha_l-2\alpha_1}\right| + \sum_{l=1}^m \left|\mu_l\beta_l(\beta_l-1)t^{\beta_l-2\alpha_1}\right|}{\left(|\lambda_1\alpha_1| - \sum_{l=2}^m |\lambda_l\alpha_l| - \sum_{l=1}^m |\mu_l\beta_l|\right)^2} \mathrm{d}t \\
&\le \sum_{l=1}^m M_{3+l}(T^{\alpha_l-2\alpha_1+1}-1) + \sum_{l=1}^m M_{3+m+l}(T^{\beta_l-2\alpha_1+1}-1)
\end{aligned}$$

where the constant number M's are independent of T.

Since

$$\left|\frac{1}{2T}\int_{-T}^T e^{i(q(t)-p(t))}\,\mathrm{d}t\right| \to 0$$

as $T \to \infty$, the proof is complete. □

Theorem 3.4. *Let $q(t) \in Q(\mathbb{R})$. If $f \in \mathcal{SLP}(\mathbb{R})$, then the limits*

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^T f(t)\,\mathrm{d}t \quad \textit{and} \quad \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^T f(t)e^{-iq(t)}\,\mathrm{d}t$$

exist.

Proof. We prove the theorem in two parts:
a) Let $f \in \mathrm{Gtrig}(\mathbb{R})$. Let

$$f(t) = P(t) = x_0 + \sum_{k=1}^n a_k e^{iq_k(t)}.$$

Then by Theorems 3.1 and 3.3,

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^T f(t)\,\mathrm{d}t = x_0.$$

b) Let f be an arbitrary function in $\mathcal{SLP}(\mathbb{R})$. Then for $\varepsilon > 0$ there exists a generalized trigonometric polynomial P_ε such that (2.2) holds. Since $\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^T P_\varepsilon(t)\,\mathrm{d}t$ exists, we can find a number T_0 such that when $T_1, T_2 > T_0$,

$$\left\|\frac{1}{2T_1}\int_{-T_1}^{T_1} P_\varepsilon(t)\,\mathrm{d}t - \frac{1}{2T_2}\int_{-T_2}^{T_2} P_\varepsilon(t)\,\mathrm{d}t\right\| < \varepsilon. \tag{3.2}$$

Using (2.2) and (3.2), it is easy to show that

$$\begin{aligned}
&\left\| \frac{1}{2T_1}\int_{-T_1}^{T_1} f(t)\,\mathrm{d}t - \frac{1}{2T_2}\int_{-T_2}^{T_2} f(t)\,\mathrm{d}t \right\| \\
&\qquad \le \frac{1}{2T_1}\int_{-T_1}^{T_1} \|f(t)-P_\varepsilon(t)\|\,\mathrm{d}t \\
&\qquad\quad + \left\| \frac{1}{2T_1}\int_{-T_1}^{T_1} P_\varepsilon(t)\,\mathrm{d}t - \frac{1}{2T_2}\int_{-T_2}^{T_2} P_\varepsilon(t)\,\mathrm{d}t \right\| \\
&\qquad\quad + \frac{1}{2T_2}\int_{-T_2}^{T_2} \|f(t)-P_\varepsilon(t)\|\,\mathrm{d}t \\
&\qquad < 3\varepsilon
\end{aligned}$$

when $T_1, T_2 > T_0$.

The existence of the second limit can be shown by using the same way. □

4. Main results

Theorem 4.1. *There exist constants A, $B>0$, such that*

$$A\|f\|^2_{\mathcal{SLP}} \le \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}\sum_{m=-\infty}^{\infty}|\langle f, g_{m,n}\rangle|^2 \le B\|f\|^2_{\mathcal{SLP}}. \tag{4.1}$$

for all strong limit power functions f.

Proof. Let us begin with the case when f is a generalized trigonometric polynomial $f(t)=\sum_{k=1}^{K} a_k e^{iq_k(t)}$ where $q_k(t)=\lambda_k\sum_{l=1}^{m'} t^{\alpha_{kl}}$. Take

$$\begin{aligned}
&\hat{g}\left(m\omega_0-\frac{\lambda_k}{2\pi}\right)\\
&\quad = \int_{-\infty}^{\infty} g\left(\sum_{l=1}^{m'} t^{\alpha_{kl}} - nt_0\right) e^{-2\pi i\left(m\omega_0-\frac{\lambda_k}{2\pi}\right)\left(\sum_{l=1}^{m'} t^{\alpha_{kl}}-nt_0\right)}\,\mathrm{d}\left(\sum_{l=1}^{m'} t^{\alpha_{kl}}\right).
\end{aligned}$$

We calculate

$$\begin{aligned}
&\frac{1}{2N+1}\sum_{n=-N}^{N}\sum_{m=-\infty}^{\infty}|\langle f, g_{m,n}\rangle|^2 \\
&\qquad = \frac{1}{2N+1}\sum_{n=-N}^{N}\sum_{m=-\infty}^{\infty}\sum_{k=1}^{K}\sum_{\ell=1}^{K} a_k\bar{a}_\ell e^{i(\lambda_k-\lambda_\ell)nt_0} \\
&\qquad\qquad \times \hat{g}\left(\frac{\lambda_k}{2\pi}-m\omega_0\right)\overline{\hat{g}\left(\frac{\lambda_\ell}{2\pi}-m\omega_0\right)}.
\end{aligned}$$

As $N \to \infty$ this tends to

$$\sum_{k=1}^{K} |a_k|^2 h(\lambda_k) + \sum{}' a_k \bar{a}_\ell j(\lambda_k, \lambda_\ell), \tag{4.2}$$

where

$$h(\lambda_k) = \sum_{m=-\infty}^{\infty} \left| \hat{g}\left(\frac{\lambda_k}{2\pi} - m\omega_0\right) \right|^2, \tag{4.3}$$

the second sum in (4.2) is taken over those k, ℓ such that $\lambda_k - \lambda_\ell$ is a (nonzero) multiple of $2\pi/t_0$, and

$$j(\lambda_k, \lambda_\ell) = \sum_{m=-\infty}^{\infty} \hat{g}\left(\frac{\lambda_k}{2\pi} - m\omega_0\right) \overline{\hat{g}\left(\frac{\lambda_\ell}{2\pi} - m\omega_0\right)}. \tag{4.4}$$

In this case,

$$\begin{aligned}
&\left|\sum{}' a_k \bar{a}_\ell j(\lambda_k, \lambda_\ell)\right| \\
&\quad = \left|\sum{}' a_k \bar{a}_\ell \sum_{m=-\infty}^{\infty} \hat{g}\left(\frac{\lambda_k}{2\pi} - m\omega_0\right) \overline{\hat{g}\left(\frac{\lambda_\ell}{2\pi} - m\omega_0\right)}\right| \\
&\quad = \left|\sum_{\lambda\in\mathbb{R}} \sum_{s\in\mathbb{Z}\setminus\{0\}} a_\lambda \bar{a}_{\lambda+\frac{2\pi s}{t_0}} \sum_{m=-\infty}^{\infty} \hat{g}\left(\frac{\lambda}{2\pi} - m\omega_0\right) \overline{\hat{g}\left(\frac{\lambda}{2\pi} + \frac{s}{t_0} - m\omega_0\right)}\right| \\
&\quad \le \sum_{s\in\mathbb{Z}\setminus\{0\}} \left(\sum_{m=-\infty}^{\infty} \sum_{\lambda\in\mathbb{R}} |a_\lambda|^2 \left|\hat{g}\left(\frac{\lambda}{2\pi} - m\omega_0\right)\right| \left|\hat{g}\left(\frac{\lambda}{2\pi} + \frac{s}{t_0} - m\omega_0\right)\right|\right)^{1/2} \\
&\qquad \times \left(\sum_{m=-\infty}^{\infty} \sum_{\lambda\in\mathbb{R}} \left|a_{\lambda+\frac{2\pi s}{t_0}}\right|^2 \left|\hat{g}\left(\frac{\lambda}{2\pi} - m\omega_0\right)\right| \left|\hat{g}\left(\frac{\lambda}{2\pi} + \frac{s}{t_0} - m\omega_0\right)\right|\right)^{1/2} \\
&\quad \le \sum_{s\in\mathbb{Z}\setminus\{0\}} \left(\sum_{m=-\infty}^{\infty} \sum_{\lambda\in\mathbb{R}} |a_\lambda|^2 \left|\hat{g}\left(\frac{\lambda}{2\pi} - m\omega_0\right)\right| \left|\hat{g}\left(\frac{\lambda}{2\pi} + \frac{s}{t_0} - m\omega_0\right)\right|\right)^{1/2} \\
&\qquad \times \left(\sum_{m=-\infty}^{\infty} \sum_{\lambda\in\mathbb{R}} |a_\lambda|^2 \left|\hat{g}\left(\frac{\lambda}{2\pi} - \frac{s}{t_0} - m\omega_0\right)\right| \left|\hat{g}\left(\frac{\lambda}{2\pi} - m\omega_0\right)\right|\right)^{1/2} \\
&\quad \le \sum_{\lambda\in\mathbb{R}} |a_\lambda|^2 \sum_{s\in\mathbb{Z}\setminus\{0\}} \left(\Gamma(s)\Gamma(-s)\right)^{1/2},
\end{aligned}$$

where $\Gamma(s) = \sup_{\lambda\in\mathbb{R}} \sum_{m\in\mathbb{Z}} \left|\hat{g}\left(\frac{\lambda}{2\pi} - m\omega_0\right)\right| \left|\hat{g}\left(\frac{\lambda}{2\pi} + \frac{s}{t_0} - m\omega_0\right)\right|$. If we assume

$$A = \inf_{0\le\lambda<\omega_0} \sum_{m\in\mathbb{Z}} \left|\hat{g}\left(\frac{\lambda}{2\pi} - m\omega_0\right)\right|^2 - \sum_{s\in\mathbb{Z}\setminus\{0\}} \left(\Gamma(s)\Gamma(-s)\right)^{1/2} > 0$$

$$B = \sup_{0\le\lambda<\omega_0} \sum_{m\in\mathbb{Z}} \left|\hat{g}\left(\frac{\lambda}{2\pi} - m\omega_0\right)\right|^2 + \sum_{s\in\mathbb{Z}\setminus\{0\}} \left(\Gamma(s)\Gamma(-s)\right)^{1/2} < \infty,$$

hence we get the inequality (4.1) for the generalized trigonometric polynomials. A standard approximation argument completes the proof for general strong limit power functions. □

We can define the wavelet transform of a strong limit power function, just as we can for a function in $L_2(\mathbb{R})$, at least assuming that $\psi \in L_1(\mathbb{R})$. Then we construct generalized frames for strong limit power functions using the discretized form of the wavelet transform and the ideas in the windowed Fourier transform case. We write $\psi_{m,n}(t) = 2^{m/2}\psi(2^m t - n)$, for $m,\ n \in \mathbb{Z}$.

Lemma 4.2. *Let* $f(t) = \sum_{k=1}^{K} a_k e^{iq_k(t)}$ *where* $q_k(t) = \lambda_k \sum_{l=1}^{m'} t^{\alpha_{kl}}$ *be a generalized trigonometric polynomial , and let* $\psi \in L_1(\mathbb{R})$*. Then*

$$f^\circ(x,y) = |x|^{1/2} \sum_{k=1}^{K} a_k e^{i\lambda_k y} \overline{\hat{\psi}\left(\frac{-x\lambda_k}{2\pi}\right)}, \tag{4.5}$$

and is hence a trigonometric polynomial in y*, for fixed* $x \neq 0$*.*

Proof. We calculate:

$$\begin{aligned} f^\circ(x,y) &= |x|^{-1/2} \int_{-\infty}^{\infty} f(t) \overline{\psi\left(\frac{t-y}{x}\right)} \,\mathrm{d}t \\ &= |x|^{-1/2} \sum_{k=1}^{K} a_k \int_{-\infty}^{\infty} e^{i\lambda_k \sum_{l=1}^{m'} t^{\alpha_{kl}}} \\ &\qquad \times \overline{\psi\left(\frac{\sum_{l=1}^{m'} t^{\alpha_{kl}} - y}{x}\right)} \,\mathrm{d}\left(\sum_{l=1}^{m'} t^{\alpha_{kl}}\right) \\ &= |x|^{1/2} \sum_{k=1}^{K} a_k \int_{-\infty}^{\infty} e^{i\lambda_k (xu+y)} \overline{\psi(u)} \,\mathrm{d}u \\ &= |x|^{1/2} \sum_{k=1}^{K} a_k e^{i\lambda_k y} \overline{\hat{\psi}\left(\frac{-x\lambda_k}{2\pi}\right)}, \end{aligned}$$

as asserted. □

Theorem 4.3. *Let* f *be a strong limit power function. Then, for fixed* $x \neq 0$*,* $f^\circ(x,y)$ *is a strong limit power function in* y*.*

Proof. Let $f(t) = \sum_{k=1}^{K} a_k e^{iq_k(t)}$ where $q_k(t) = \lambda_k \sum_{l=1}^{m'} t^{\alpha_{kl}}$ be a generalized trigonometric polynomial. Then, by Theorem 4.2, $f^\circ(x,y)$ is a generalized trigonometric polynomial in y, for fixed $x \neq 0$. Thus it is sufficient to verify that, if $\|f_n - f\|_\infty \to 0$, then $\|f_n^\circ(x,y) - f^\circ(x,y)\|_{L_\infty(y)} \to 0$. Since

$$f_n^\circ(x,y) - f^\circ(x,y) = |x|^{-1/2} \int_{-\infty}^{\infty} \left(f_n\left(\sum_{l=1}^{m'} t^{\alpha_{kl}}\right) - f\left(\sum_{l=1}^{m'} t^{\alpha_{kl}}\right)\right) \times \overline{\psi\left(\frac{\sum_{l=1}^{m'} t^{\alpha_{kl}} - y}{x}\right)} \,\mathrm{d}\left(\sum_{l=1}^{m'} t^{\alpha_{kl}}\right),$$

we get

$$\begin{aligned}
\|f_n^\circ(x,y) - f^\circ(x,y)\|_{L_\infty(y)} \\
&= \operatorname*{ess.\,sup}_y |f_n^\circ(x,y) - f^\circ(x,y)| \\
&= \operatorname*{ess.\,sup}_y \left| |x|^{1/2} \int_{-\infty}^{\infty} (f_n(xu+y) - f(xu+y))\overline{\psi(u)}\,\mathrm{d}u \right| \\
&\le |x|^{1/2}\|f_n - f\|_\infty \|\psi\|_1,
\end{aligned}$$

which gives the result. □

Theorem 4.4. *If f is a strong limit power function, then there exist constants A, $B > 0$, such that*

$$A\sum_{\lambda\neq 0}|a_\lambda|^2 \le \sum_{m\in\mathbb{Z}} \frac{1}{\gamma^m} \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |\langle f, \psi_{m,n}\rangle|^2 \le B\sum_{\lambda\neq 0}|a_\lambda|^2. \quad (4.6)$$

Proof. Since $\langle f, \psi_{m,n}\rangle = \gamma^{-m/2}\int_{-\infty}^{\infty} f(t)\psi(\gamma^{-m}t - \beta n)\,\mathrm{d}t = f^\circ(\gamma^m, \beta n \gamma^m)$, for every $m \in \mathbb{Z}$ we have that $\{\langle f, \psi_{m,n}\rangle\}_{n=-\infty}^{\infty}$ is a sequence of strong limit power functions. Moreover, if f is a generalized trigonometric polynomial, we see that

$$\begin{aligned}
\langle f, \psi_{m,n}\rangle &= \gamma^{-m/2} \int_{-\infty}^{\infty} \sum_{k=1}^{K} a_k e^{i\lambda_k \sum_{l=1}^{m'} t^{\alpha_{kl}}} \times \overline{\psi\left(\gamma^{-m}\sum_{l=1}^{m'} t^{\alpha_{kl}} - \beta n\right)} \,\mathrm{d}\left(\sum_{l=1}^{m'} t^{\alpha_{kl}}\right) \\
&= \gamma^{m/2} \sum_{\lambda\in\mathbb{R}} a_\lambda \overline{\hat{\psi}(\lambda\gamma^m)} e^{i\lambda\beta n\gamma^m} \\
&= \gamma^{m/2} \sum_{0\le\lambda<\frac{1}{\beta\gamma^m}} \left(\sum_{k\in\mathbb{Z}} a_{\lambda+\frac{k}{\beta\gamma^m}} \overline{\hat{\psi}(\lambda\gamma^m + k/\beta)}\right) e^{i\lambda\beta n\gamma^m}.
\end{aligned}$$

By using the same method in [5, Theorem 3], we get the generalized frame (4.6) for the strong limit power functions. □

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