

POSITIVE SOLUTIONS FOR A SUM-TYPE SINGULAR FRACTIONAL INTEGRO-DIFFERENTIAL EQUATION WITH m -POINT BOUNDARY CONDITIONS

M. Aydoğan¹, S. Z. Nazemi², Sh. Rezapour³

We study the existence and uniqueness of positive solutions for a sum-type singular fractional integro-differential equation with m -point boundary condition. Also, we provide an example to illustrate our main result.

Keywords: Fixed point, Fractional derivative, Riemann-Liouville integral, Singular integro-differential equation.

MSC2010: 34A08, 34B16.

1. Introduction and Preliminaries

Many researchers have been investigated distinct fractional differential equations and inclusions which have been applied in modeling of different problems in some sciences. There are many papers on the existence of solutions (or positive solution) for some singular fractional differential equations (see for example, [1]-[3], [10], [12], [14], [17], [18] and [19]). In 2013, the existence and uniqueness of positive solutions for the fractional differential equation $-D^\alpha u(t) = f(t, u(t)) + g(t, u(t))$ with the boundary conditions $u(0) = u'(0) = u''(0) = u''(1) = 0$ or $u(0) = u'(0) = u''(0) = 0, u''(1) = \beta u''(\eta)$ investigated, where $0 < t < 1, 3 < \alpha \leq 4$ and D^α is the Riemann-Liouville fractional derivative ([21]). By using idea of the work and some another published works such [4], [8], [9] and [16], we investigate the singular fractional integro-differential equation

$$\begin{aligned} D^\alpha u(t) + f(t, u(t), \int_0^t \gamma(t, s)h(t, s, u(s), D^{\beta_1} u(s), \dots, D^{\beta_L} u(s))ds, \\ D^{\mu_1} u(t), \dots, D^{\mu_N} u(t)) \\ + g(t, u(t), \int_0^t \gamma(t, s)h(t, s, u(s), D^{\beta_1} u(s), \dots, D^{\beta_L} u(s))ds, \\ D^{\mu_1} u(t), \dots, D^{\mu_N} u(t)) = 0 \quad (1.1) \end{aligned}$$

with m -point boundary conditions $D^{\mu_i} u(0) = D^{\beta_j} u(0) = 0$ for $1 \leq i \leq N$ and $1 \leq j \leq L$, $D^{\mu_{N+1}} u(0) = \dots = D^{\mu_{N+3}} u(0) = 0$ and $D^{\mu_{N+3}} u(1) = \sum_{j=1}^{m-2} a_j D^{\mu_{N+3}} u(\xi_j)$, where $0 < t < 1, n \geq 5, n-1 < \alpha \leq n, 0 < \mu_1 < \dots < \mu_N, 0 < \beta_1 < \dots < \beta_L \leq \mu_N, 4 < \alpha - \mu_N \leq 5, a_j \in (0, \infty), 0 < \xi_1 < \dots < \xi_{m-2} < 1, \sum_{j=1}^{m-2} a_j \xi_j^{\alpha - \mu_N - 4} < 1, \gamma : [0, 1] \times [0, 1] \rightarrow [0, \infty), h : [0, 1] \times [0, 1] \times \mathbb{R}^{L+1} \rightarrow [0, \infty)$ and $f, g : (0, 1] \times \mathbb{R}^{N+2} \rightarrow [0, \infty)$ are continuous mappings, D is the Riemann-Liouville fractional derivative, $\lim_{t \rightarrow 0^+} f(t, \dots, \dots) = +\infty$

¹ Department of Mathematics, Isik University, Mesrutiyet koyu, Sile Kampusu, 34980 Istanbul, Turkey, e-mail: melike.aydogan@isikun.edu.tr

² Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran, e-mail: szahranazemi@gmail.com

³ Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran, & Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan, e-mail: sh.rezapour@azaruniv.edu & sh.rezapour@mail.cmuh.org.tw (Corresponding author)

and $\lim_{t \rightarrow 0^+} g(t, \dots, \dots) = +\infty$, that is, f, g are singular at $t = 0$. As you know, the Riemann-Liouville fractional integral of order $\alpha > 0$ for a function $f : [0, \infty) \rightarrow \mathbb{R}$ is defined by $I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds$ for $t > 0$ provided the integral exists ([7], [13] and [15]). Also, the Riemann-Liouville fractional derivative of order $\alpha > 0$ for a continuous function $f : [0, \infty) \rightarrow \mathbb{R}$ is defined by $D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} f(s) ds$ for $t > 0$, where n is the smallest integer greater than or equal α ([7], [13] and [15]).

Lemma 1.1. ([7], [13]) *If $u \in C(0, 1) \cap L^1(0, 1)$ with $D^\alpha u \in C(0, 1) \cap L^1(0, 1)$, then*

$$I^\alpha D^\alpha u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \dots + c_n t^{\alpha-n},$$

where c_1, \dots, c_n are some real numbers and n is the smallest integer greater than or equal α .

Lemma 1.2. ([7], [13]) *If $u \in L^1[0, 1]$ and $\rho > \sigma > 0$, then $I^\rho I^\sigma u(t) = I^{\rho+\sigma} u(t)$, $D^\sigma I^\rho u(t) = I^{\rho-\sigma} u(t)$ and $D^\sigma I^\sigma u(t) = u(t)$. If $\rho > 0$ and $\nu > -1$, then $D^\rho t^\nu = \frac{\Gamma(\nu+1)}{\Gamma(\nu-\rho+1)} t^{\nu-\rho}$. Also, $I^\alpha u \in C[0, 1]$ for all $\alpha > 0$ and $u \in C[0, 1]$.*

By using Lemma 1.2, one can easily conclude next result.

Lemma 1.3. *Let $v \in C([0, 1])$ and $u(t) = I^{\mu_N} v(t)$. Then (1.1) reduces to the problem*

$$\begin{aligned} & D^{\alpha-\mu_N} v(t) + f(t, I^{\mu_N} v(t), \int_0^t \gamma(t, s) h(t, s, I^{\mu_N} v(s), I^{\mu_N-\beta_1} v(s), \dots, I^{\mu_N-\beta_L} v(s)) ds, \\ & \quad I^{\mu_N-\mu_1} v(t), I^{\mu_N-\mu_2} v(t), \dots, I^{\mu_N-\mu_{N-1}} v(t), v(t)) \\ & + g(t, I^{\mu_N} v(t), \int_0^t \gamma(t, s) h(t, s, I^{\mu_N} v(s), I^{\mu_N-\beta_1} v(s), \dots, I^{\mu_N-\beta_L} v(s)) ds, \\ & \quad I^{\mu_N-\mu_1} v(t), I^{\mu_N-\mu_2} v(t), \dots, I^{\mu_N-\mu_{N-1}} v(t), v(t)) = 0 \end{aligned} \quad (1.2)$$

with boundary conditions $v'''(1) = \sum_{j=1}^{m-2} a_j v'''(\xi_j)$ and $v(0) = v'(0) = v''(0) = v'''(0) = 0$, where $0 < t < 1$. Moreover if $v \in C([0, 1])$ is a positive solution of the problem (1.2), then $u(t) = I^{\mu_N} v(t)$ is a positive solution for the problem (1.1).

Lemma 1.4. *Let $4 < \alpha - \mu_N \leq 5$ and $\sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\mu_N-4} \neq 1$. If $y \in C(0, 1]$, then the problem $D^{\alpha-\mu_N} w(t) + y(t) = 0$ with boundary conditions $w(0) = w'(0) = w''(0) = w'''(0) = 0$ and $w'''(1) = \sum_{j=1}^{m-2} a_j w'''(\xi_j)$ has the unique solution*

$$\begin{aligned} w(t) &= \int_0^1 G(t, s) y(s) ds \\ &+ \frac{t^{\alpha-\mu_N-1} \sum_{j=1}^{m-2} a_j}{(\alpha - \mu_N - 1)(\alpha - \mu_N - 2)(\alpha - \mu_N - 3)(1 - \sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\mu_N-4})} \int_0^1 H(\xi_j, s) y(s) ds, \end{aligned}$$

where $G(t, s) = \frac{1}{\Gamma(\alpha-\mu_N)} t^{\alpha-\mu_N-1} (1-s)^{\alpha-\mu_N-4} - (t-s)^{\alpha-\mu_N-1}$ whenever $0 \leq s \leq t \leq 1$, $G(t, s) = \frac{1}{\Gamma(\alpha-\mu_N)} t^{\alpha-\mu_N-1} (1-s)^{\alpha-\mu_N-4}$ whenever $0 \leq t \leq s \leq 1$, $H(t, s) = \frac{\partial^3 G(t, s)}{\partial t^3} = \frac{(\alpha-\mu_N-1)(\alpha-\mu_N-2)(\alpha-\mu_N-3)}{\Gamma(\alpha-\mu_N)} (t^{\alpha-\mu_N-4} (1-s)^{\alpha-\mu_N-4} - (t-s)^{\alpha-\mu_N-4})$ whenever $0 \leq s \leq t \leq 1$ and $H(t, s) = \frac{\partial^3 G(t, s)}{\partial t^3} = \frac{(\alpha-\mu_N-1)(\alpha-\mu_N-2)(\alpha-\mu_N-3)}{\Gamma(\alpha-\mu_N)} t^{\alpha-\mu_N-4} (1-s)^{\alpha-\mu_N-4}$ whenever $0 \leq t \leq s \leq 1$.

Proof. By using Lemma 1.1, the solution of this problem is

$$w(t) = -I^{\alpha-\mu_N} y(t) + c_1 t^{\alpha-\mu_N-1} + c_2 t^{\alpha-\mu_N-2} + c_3 t^{\alpha-\mu_N-3} + c_4 t^{\alpha-\mu_N-4} + c_5 t^{\alpha-\mu_N-5},$$

where $c_1, c_2, c_3, c_4, c_5 \in \mathbb{R}$ are some constants. By using the boundary conditions, we get $c_2 = c_3 = c_4 = c_5 = 0$ and

$$c_1 = \frac{1}{\Gamma(\alpha - \mu_N)(1 - \sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\mu_N-4})} \times \left[\int_0^1 (1-s)^{\alpha-\mu_N-4} y(s) ds - \sum_{j=1}^{m-2} a_j \int_0^{\xi_j} (\xi_j - s)^{\alpha-\mu_N-4} y(s) ds \right].$$

Thus, the unique solution of this problem is

$$\begin{aligned} w(t) &= - \int_0^t \frac{(t-s)^{\alpha-\mu_N-1}}{\Gamma(\alpha - \mu_N)} y(s) ds + \frac{t^{\alpha-\mu_N-1}}{\Gamma(\alpha - \mu_N)(1 - \sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\mu_N-4})} \\ &\quad \times \left[\int_0^1 (1-s)^{\alpha-\mu_N-4} y(s) ds - \sum_{j=1}^{m-2} a_j \int_0^{\xi_j} (\xi_j - s)^{\alpha-\mu_N-4} y(s) ds \right] \\ &= - \int_0^t \frac{(t-s)^{\alpha-\mu_N-1}}{\Gamma(\alpha - \mu_N)} y(s) ds + \left(\frac{t^{\alpha-\mu_N-1}}{\Gamma(\alpha - \mu_N)} \right. \\ &\quad \left. + \frac{t^{\alpha-\mu_N-1} \sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\mu_N-4}}{\Gamma(\alpha - \mu_N)(1 - \sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\mu_N-4})} \right) \int_0^1 (1-s)^{\alpha-\mu_N-4} y(s) ds \\ &\quad - \frac{t^{\alpha-\mu_N-1} \sum_{j=1}^{m-2} a_j}{\Gamma(\alpha - \mu_N)(1 - \sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\mu_N-4})} \int_0^{\xi_j} (\xi_j - s)^{\alpha-\mu_N-4} y(s) ds \\ &= \frac{1}{\Gamma(\alpha - \mu_N)} \int_0^t \left(t^{\alpha-\mu_N-1} (1-s)^{\alpha-\mu_N-4} - (t-s)^{\alpha-\mu_N-1} \right) y(s) ds \\ &\quad + \frac{1}{\Gamma(\alpha - \mu_N)} \int_t^1 t^{\alpha-\mu_N-1} (1-s)^{\alpha-\mu_N-4} y(s) ds \\ &\quad + \frac{t^{\alpha-\mu_N-1} \sum_{j=1}^{m-2} a_j}{\Gamma(\alpha - \mu_N)(1 - \sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\mu_N-4})} \\ &\quad \times \left[\int_0^1 \xi_j^{\alpha-\mu_N-4} (1-s)^{\alpha-\mu_N-4} y(s) ds - \int_0^{\xi_j} (\xi_j - s)^{\alpha-\mu_N-4} y(s) ds \right] \\ &= \int_0^1 G(t, s) y(s) ds \\ &\quad + \frac{t^{\alpha-\mu_N-1} \sum_{j=1}^{m-2} a_j}{(\alpha - \mu_N - 1)(\alpha - \mu_N - 2)(\alpha - \mu_N - 3)(1 - \sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\mu_N-4})} \\ &\quad \times \int_0^1 H(\xi_j, s) y(s) ds. \end{aligned}$$

This completes the proof. \square

One can check that G is a continuous function on $[0, 1] \times [0, 1]$, $G(t, s) \geq 0$ and $H(t, s) \geq 0$ for all $t, s \in [0, 1]$ and $G(t, s) > 0$ for all $t, s \in (0, 1)$. Also,

$$\frac{1}{\Gamma(\alpha - \mu_N)} s(s^2 - 3s + 3)(1-s)^{\alpha-\mu_N-4} t^{\alpha-\mu_N-1} \leq G(t, s)$$

$$\leq \frac{1}{\Gamma(\alpha - \mu_N)} (1-s)^{\alpha - \mu_N - 4} t^{\alpha - \mu_N - 1}$$

for all $t, s \in [0, 1]$. Also, $\sup_{0 \leq t \leq 1} \int_0^1 G(t, s) s^{-\sigma} ds = \frac{1}{\Gamma(\alpha - \mu_N)} (\beta(1 - \sigma, \alpha - \mu_N - 3) - \beta(1 - \sigma, \alpha - \mu_N))$ and

$$\int_0^1 H(\eta, s) s^{-\sigma} ds = \frac{(\alpha - \mu_N - 1)(\alpha - \mu_N - 2)(\alpha - \mu_N - 3)}{\Gamma(\alpha - \mu_N)} (\eta^{\alpha - \mu_N - 4} - \eta^{\alpha - \mu_N - \sigma - 3}) \beta(1 - \sigma, \alpha - \mu_N - 3)$$

for all $0 < \sigma < 1$ and $0 < \eta < 1$. This conclude that

$$K := \sup_{0 \leq t \leq 1} \int_0^1 \left[G(t, s) + \frac{t^{\alpha - \mu_N - 1} \sum_{j=1}^{m-2} a_j}{(\alpha - \mu_N - 1)(\alpha - \mu_N - 2)(\alpha - \mu_N - 3)(1 - \sum_{j=1}^{m-2} a_j \xi_j^{\alpha - \mu_N - 4})} \times H(\xi_j, s) \right] s^{-\sigma} ds$$

$$= \frac{1}{\Gamma(\alpha - \mu_N)} \left[\left(1 + \frac{\sum_{j=1}^{m-2} a_j (\xi_j^{\alpha - \mu_N - 4} - \xi_j^{\alpha - \mu_N - \sigma - 3})}{1 - \sum_{j=1}^{m-2} a_j \xi_j^{\alpha - \mu_N - 4}} \right) \beta(1 - \sigma, \alpha - \mu_N - 3) - \beta(1 - \sigma, \alpha - \mu_N) \right].$$

By using some calculations, one can prove next key result.

Theorem 1.1. *Let $0 < \sigma < 1$, $4 < \alpha - \mu_N \leq 5$, $\sum_{j=1}^{m-2} a_j \xi_j^{\alpha - \mu_N - 4} \neq 1$ and $F : (0, 1] \rightarrow \mathbb{R}$ be a continuous function with $\lim_{t \rightarrow 0^+} F(t) = \infty$. Suppose that $t^\sigma F(t)$ is a continuous function on $[0, 1]$. Then the functions defined by $L(t) = \int_0^1 G(t, s) F(s) ds$ and $W(t) =$*

$$\frac{t^{\alpha - \mu_N - 1} \sum_{j=1}^{m-2} a_j}{(\alpha - \mu_N - 1)(\alpha - \mu_N - 2)(\alpha - \mu_N - 3)(1 - \sum_{j=1}^{m-2} a_j \xi_j^{\alpha - \mu_N - 4})} \int_0^1 H(\xi_j, s) F(s) ds$$

are continuous on $[0, 1]$.

Let $(X, \|\cdot\|)$ be a real Banach space which has a partially order by using a cone $P \subset X$. A nonempty closed convex set $P \subset X$ is a cone whenever $x \in P$ and $\lambda \geq 0$ implies $\lambda x \in P$ and $P \cap (-P) = \{0\}$ ([5] and [20]). A cone P is called solid whenever interior of P is nonempty. Each cone P defines the order \leq on X by $x \leq y$ if and only if $y - x \in P$ ([5] and [20]). A cone P is called normal if there exists a constant $N > 0$ such that $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$ ([5] and [20]). In this case, least number N is called the normal constant of P ([5] and [20]). Define $x \sim y$ whenever there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda x \leq y \leq \mu x$. Then, \sim is an equivalence relation on X ([5] and [20]). For each $k \geq \theta$ with $k \neq \theta$, define $P_k = \{x \in X : x \sim k\}$. One can check that $P_k \subset P$ for all $k \in P$ ([5] and [20]).

Theorem 1.2. ([6]) *Let (X, d) be a complete metric space, \leq an order on X , $T : X \rightarrow X$ an increasing map and $x_n \leq x$ for all n whenever $\{x_n\}$ is an increasing sequence in X with $x_n \rightarrow x$. Suppose that there exists a continuous and increasing function $\psi : [0, \infty) \rightarrow [0, \infty)$ such that ψ is positive on $(0, \infty)$, $\psi(0) = 0$ and $d(T(x), T(y)) \leq d(x, y) - \psi(d(x, y))$ for all $x \geq y$. If there exists $x_0 \in X$ with $x_0 \leq Tx_0$, then T has a fixed point. If for each $x, y \in X$, there exists $z \in X$ which is comparable to x and y , then T has a unique fixed point.*

Let X be a real Banach space, P a cone in X and $0 \leq \gamma < 1$ a real number. An operator $A : P \rightarrow P$ is said to be γ -concave whenever $A(tx) \geq t^\gamma Ax$ for all $t \in (0, 1)$ and $x \in P$ ([20]). Also, $A : P \rightarrow P$ is called homogeneous whenever $A(\lambda x) = \lambda Ax$ for all $\lambda > 0$ and $x \in P$ ([20]). Finally, $A : P \rightarrow P$ is said to be sub-homogeneous whenever $A(tx) \geq tAx$ for all $t \in (0, 1)$ and $x \in P$ ([20]). In 2011, Zhai and Anderson proved next result.

Theorem 1.3. ([20]) *Let P be a normal cone in a real Banach space X , $A : P \rightarrow P$ an increasing γ -concave map and $B : P \rightarrow P$ an increasing sub-homogeneous operator. Assume that there is $h > \theta$ such that $Ah \in P_h$ and $Bh \in P_h$. Also, there exists $\delta_0 > 0$ such that*

$Ax \geq \delta_0 Bx$ for all $x \in P$. Then the operator equation $Ax + Bx = x$ has a unique solution x^* in P_k . Moreover, the sequence $y_n = Ay_{n-1} + By_{n-1}$ ($n \geq 1$) with initial value $y_0 \in P_k$ converges to x^* .

Note that last result holds whenever B is a null operator. In this paper, we use the Banach space $X = C([0, 1])$ with the partial order $x \leq y$ if and only if $x(t) \leq y(t)$ for all $t \in [0, 1]$ and $x, y \in C([0, 1])$. It has been proved $(C([0, 1]), \leq)$ has this property that $x_n \leq x$ for all n whenever $\{x_n\}$ is an increasing sequence in $C([0, 1])$ with $x_n \rightarrow x$ ([11]). Moreover, $\max\{x, y\} \in C([0, 1])$ for all $x, y \in C([0, 1])$, that is, for each $x, y \in C([0, 1])$ there exists $z \in C([0, 1])$ which is comparable to x and y . We consider the normal cone $P = \{x \in C([0, 1]) : x(t) \geq 0 \text{ for all } t \in [0, 1]\}$ with normal constant 1.

2. Main Results

Now, we are ready to state and prove our main result.

Theorem 2.1. *Suppose that $f, g : (0, 1) \times \mathbb{R}^{N+2} \rightarrow [0, \infty)$ are continuous mappings with $f(t, \dots, \dots, \dots), g(t, \dots, \dots, \dots) \rightarrow +\infty$ as $t \rightarrow 0^+, 0 < \sigma < 1$, the maps $t^\sigma f(t, x_1, x_2, \dots, x_{N+2})$ and $t^\sigma g(t, x_1, x_2, \dots, x_{N+2})$ are continuous on $[0, 1] \times \mathbb{R}^{N+2}$, $t^\sigma f(t, \dots, \dots, \dots), t^\sigma g(t, \dots, \dots, \dots)$ and $h(t, s, \dots, \dots, \dots)$ are increasing with respect to their components on $[0, \infty)$ for each fixed t and s in $[0, 1]$ and also $t^\sigma g(t, 0, 0, \dots, 0) \neq 0$. Also, assume that*

$$t^\sigma g(t, \lambda x_1, \lambda x_2, \dots, \lambda x_{N+2}) \geq \lambda t^\sigma g(t, x_1, x_2, \dots, x_{N+2}),$$

$h(t, s, \lambda y_1, \lambda y_2, \dots, \lambda y_{L+1}) \geq \lambda h(t, s, y_1, y_2, \dots, y_{L+1})$ and there exists a constant $\gamma \in [0, 1)$ such that $t^\sigma f(t, \lambda x_1, \lambda x_2, \dots, \lambda x_{N+2}) \geq \lambda^\gamma t^\sigma f(t, x_1, x_2, \dots, x_{N+2})$ for all $t, s \in [0, 1], \lambda \in (0, 1)$ and $x_i, y_j \in [0, \infty)$ ($1 \leq i \leq N + 2, 1 \leq j \leq L + 1$). If there exists $\delta_0 > 0$ such that $t^\sigma f(t, x_1, x_2, \dots, x_{N+2}) \geq \delta_0 t^\sigma g(t, x_1, x_2, \dots, x_{N+2})$, for all $t \in [0, 1]$ and $x_i \in [0, \infty)$ ($1 \leq i \leq N + 2$), then the problem (1.2) has a unique solution $u^* \in P_k$, where $k(t) = t^{\alpha - \mu_N - 1}$ for all $t \in [0, 1]$. Moreover, the sequence

$$\begin{aligned} u_{n+1}(t) &= \int_0^1 \left[G(t, s) \right. \\ &\quad + \frac{t^{\alpha - \mu_N - 1} \sum_{j=1}^{m-2} a_j}{(\alpha - \mu_N - 1)(\alpha - \mu_N - 2)(\alpha - \mu_N - 3)(1 - \sum_{j=1}^{m-2} a_j \xi_j^{\alpha - \mu_N - 4})} \\ &\quad \left. \times H(\xi_j, s) \right] \times \left[\tilde{f}(s, u_n(s)) + \tilde{g}(s, u_n(s)) \right] ds \end{aligned}$$

converges to u^* for each initial value $u_0 \in P_k$.

Proof. Define the operators $A, B : P \rightarrow X$ by

$$\begin{aligned} Au(t) &= \int_0^1 \left[G(t, s) + \frac{t^{\alpha - \mu_N - 1} \sum_{j=1}^{m-2} a_j}{(\alpha - \mu_N - 1)(\alpha - \mu_N - 2)(\alpha - \mu_N - 3)(1 - \sum_{j=1}^{m-2} a_j \xi_j^{\alpha - \mu_N - 4})} \right. \\ &\quad \left. \times H(\xi_j, s) \right] \tilde{f}(s, u(s)) ds \end{aligned}$$

and

$$\begin{aligned} Bu(t) &= \int_0^1 \left[G(t, s) + \frac{t^{\alpha - \mu_N - 1} \sum_{j=1}^{m-2} a_j}{(\alpha - \mu_N - 1)(\alpha - \mu_N - 2)(\alpha - \mu_N - 3)(1 - \sum_{j=1}^{m-2} a_j \xi_j^{\alpha - \mu_N - 4})} \right. \\ &\quad \left. \times H(\xi_j, s) \right] \tilde{g}(s, u(s)) ds \end{aligned}$$

for all $t \in [0, 1]$. It is easy to check that u is a solution for the problem (1.2) if and only if $u = Au + Bu$. From the assumptions and Theorem 1.1, we know that the operators A and

B maps P into P . We show that A and B are increasing operators. Let $u \geq v$. Then, we have

$$\begin{aligned}
Au(t) &= \int_0^1 \left[G(t, s) \right. \\
&\quad \left. + \frac{t^{\alpha-\mu_N-1} \sum_{j=1}^{m-2} a_j}{(\alpha-\mu_N-1)(\alpha-\mu_N-2)(\alpha-\mu_N-3)(1-\sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\mu_N-4})} H(\xi_j, s) \right] \\
&\quad \times \tilde{f}(s, u(s)) ds \\
&= \int_0^1 \left[G(t, s) + \frac{t^{\alpha-\mu_N-1} \sum_{j=1}^{m-2} a_j}{(\alpha-\mu_N-1)(\alpha-\mu_N-2)(\alpha-\mu_N-3)(1-\sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\mu_N-4})} H(\xi_j, s) \right] \\
&\quad \times s^{-\sigma} s^\sigma \tilde{f}(s, u(s)) ds \\
&\geq \int_0^1 \left[G(t, s) + \frac{t^{\alpha-\mu_N-1} \sum_{j=1}^{m-2} a_j}{(\alpha-\mu_N-1)(\alpha-\mu_N-2)(\alpha-\mu_N-3)(1-\sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\mu_N-4})} H(\xi_j, s) \right] \\
&\quad \times s^{-\sigma} s^\sigma \tilde{f}(s, v(s)) ds \\
&= \int_0^1 \left[G(t, s) + \frac{t^{\alpha-\mu_N-1} \sum_{j=1}^{m-2} a_j}{(\alpha-\mu_N-1)(\alpha-\mu_N-2)(\alpha-\mu_N-3)(1-\sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\mu_N-4})} H(\xi_j, s) \right] \\
&\quad \times \tilde{f}(s, v(s)) ds = Av(t)
\end{aligned}$$

for all $t \in [0, 1]$. Hence, $Au \geq Av$. Similarly, we can show that $Bu \geq Bv$. Now, we show that A is γ -concave and B is a sub-homogeneous operator. Let $\lambda \in (0, 1)$ and $u \in P$. Then,

$$\begin{aligned}
A(\lambda u)(t) &= \int_0^1 \left[G(t, s) + \frac{t^{\alpha-\mu_N-1} \sum_{j=1}^{m-2} a_j}{(\alpha-\mu_N-1)(\alpha-\mu_N-2)(\alpha-\mu_N-3)(1-\sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\mu_N-4})} \right. \\
&\quad \left. \times H(\xi_j, s) \right] \times \tilde{f}(s, \lambda u(s)) ds \\
&= \int_0^1 \left[G(t, s) + \frac{t^{\alpha-\mu_N-1} \sum_{j=1}^{m-2} a_j}{(\alpha-\mu_N-1)(\alpha-\mu_N-2)(\alpha-\mu_N-3)(1-\sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\mu_N-4})} \right. \\
&\quad \left. \times H(\xi_j, s) \right] \times s^{-\sigma} s^\sigma \tilde{f}(s, \lambda u(s)) ds \\
&\geq \lambda^\gamma \left(\int_0^1 \left[G(t, s) + \frac{t^{\alpha-\mu_N-1} \sum_{j=1}^{m-2} a_j}{(\alpha-\mu_N-1)(\alpha-\mu_N-2)(\alpha-\mu_N-3)(1-\sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\mu_N-4})} \right. \right. \\
&\quad \left. \left. \times H(\xi_j, s) \right] \times s^{-\sigma} s^\sigma \tilde{f}(s, u(s)) ds \right) \\
&= \lambda^\gamma \left(\int_0^1 \left[G(t, s) + \frac{t^{\alpha-\mu_N-1} \sum_{j=1}^{m-2} a_j}{(\alpha-\mu_N-1)(\alpha-\mu_N-2)(\alpha-\mu_N-3)(1-\sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\mu_N-4})} \right. \right. \\
&\quad \left. \left. \times H(\xi_j, s) \right] \times \tilde{f}(s, u(s)) ds \right) = \lambda^\gamma Au(t)
\end{aligned}$$

for all $t \in [0, 1]$. Hence, $A(\lambda u) \geq \lambda^\gamma Au$ for all $\lambda \in (0, 1)$ and $u \in P$ and so the A is a γ -concave operator. By using similar calculations, we can show that the operator B is sub-homogeneous. Now, we show that $Ak, Bk \in P_k$. Note that,

$$Ak(t) = \int_0^1 \left[G(t, s) + \frac{t^{\alpha-\mu_N-1} \sum_{j=1}^{m-2} a_j}{(\alpha-\mu_N-1)(\alpha-\mu_N-2)(\alpha-\mu_N-3)(1-\sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\mu_N-4})} \right]$$

$$\begin{aligned}
 & \times H(\xi_j, s) \Big] \times \tilde{f}(s, k(s)) ds \\
 = & \int_0^1 \left[G(t, s) + \frac{t^{\alpha-\mu_N-1} \sum_{j=1}^{m-2} a_j}{(\alpha - \mu_N - 1)(\alpha - \mu_N - 2)(\alpha - \mu_N - 3)(1 - \sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\mu_N-4})} \right. \\
 & \left. \times H(\xi_j, s) \right] \times s^{-\sigma} s^\sigma \tilde{f}(s, k(s)) ds \\
 \leq & \frac{1}{\Gamma(\alpha - \mu_N)} k(t) \int_0^1 (1-s)^{\alpha-\mu_N-4} s^{-\sigma} s^\sigma \tilde{f}(s, 1) ds \\
 & + \frac{k(t) \sum_{j=1}^{m-2} a_j}{(\alpha - \mu_N - 1)(\alpha - \mu_N - 2)(\alpha - \mu_N - 3)(1 - \sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\mu_N-4})} \\
 & \times \int_0^1 H(\xi_j, s) s^{-\sigma} s^\sigma \tilde{f}(s, 1) ds := l_2 k(t)
 \end{aligned}$$

and

$$\begin{aligned}
 Ak(t) &= \int_0^1 \left[G(t, s) + \frac{t^{\alpha-\mu_N-1} \sum_{j=1}^{m-2} a_j}{(\alpha - \mu_N - 1)(\alpha - \mu_N - 2)(\alpha - \mu_N - 3)(1 - \sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\mu_N-4})} \right. \\
 & \left. \times H(\xi_j, s) \right] \times \tilde{f}(s, k(s)) ds \\
 &= \int_0^1 \left[G(t, s) + \frac{t^{\alpha-\mu_N-1} \sum_{j=1}^{m-2} a_j}{(\alpha - \mu_N - 1)(\alpha - \mu_N - 2)(\alpha - \mu_N - 3)(1 - \sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\mu_N-4})} \right. \\
 & \left. \times H(\xi_j, s) \right] \times s^{-\sigma} s^\sigma \tilde{f}(s, k(s)) ds \\
 &\geq \frac{1}{\Gamma(\alpha - \mu_N)} k(t) \int_0^1 s(s^2 - 3s + 3)(1-s)^{\alpha-\mu_N-4} s^{-\sigma} s^\sigma \tilde{f}(s, 0) ds \\
 &+ \frac{k(t) \sum_{j=1}^{m-2} a_j}{(\alpha - \mu_N - 1)(\alpha - \mu_N - 2)(\alpha - \mu_N - 3)(1 - \sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\mu_N-4})} \\
 & \times \int_0^1 H(\xi_j, s) s^{-\sigma} s^\sigma \tilde{f}(s, 0) ds := l_1 k(t)
 \end{aligned}$$

for all $t \in [0, 1]$. By using the assumptions, we get

$$s^\sigma \tilde{f}(s, 1) \geq s^\sigma \tilde{f}(s, 0) \geq s^\sigma f(s, 0, 0, \dots, 0) \geq \delta_0 s^\sigma g(s, 0, 0, \dots, 0) \geq 0.$$

Since $s^\sigma g(s, 0, 0, \dots, 0) \not\equiv 0$,

$$\int_0^1 s^\sigma \tilde{f}(s, 1) ds \geq \int_0^1 s^\sigma \tilde{f}(s, 0) ds \geq \delta_0 \int_0^1 s^\sigma g(s, 0, 0, \dots, 0) ds > 0$$

and so $l_1 > 0$ and $l_2 > 0$. Thus, $l_1 k(t) \leq Ak(t) \leq l_2 k(t)$ for all $t \in [0, 1]$ and so $Ak \in P_k$. Similarly, we can show that $Bk \in P_k$. Now, let $u \in P$. Then, we have

$$\begin{aligned}
 Au(t) &= \int_0^1 \left[G(t, s) + \frac{t^{\alpha-\mu_N-1} \sum_{j=1}^{m-2} a_j}{(\alpha - \mu_N - 1)(\alpha - \mu_N - 2)(\alpha - \mu_N - 3)(1 - \sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\mu_N-4})} \right. \\
 & \left. \times H(\xi_j, s) \right] \times \tilde{f}(s, u(s)) ds \\
 &= \int_0^1 \left[G(t, s) + \frac{t^{\alpha-\mu_N-1} \sum_{j=1}^{m-2} a_j}{(\alpha - \mu_N - 1)(\alpha - \mu_N - 2)(\alpha - \mu_N - 3)(1 - \sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\mu_N-4})} \right.
 \end{aligned}$$

$$\begin{aligned}
& \times H(\xi_j, s) \Big] \times s^{-\sigma} s^\sigma \tilde{f}(s, u(s)) ds \\
\geq & \delta_0 \int_0^1 \left[G(t, s) + \frac{t^{\alpha-\mu_N-1} \sum_{j=1}^{m-2} a_j}{(\alpha-\mu_N-1)(\alpha-\mu_N-2)(\alpha-\mu_N-3)(1-\sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\mu_N-4})} \right. \\
& \left. \times H(\xi_j, s) \right] \times s^{-\sigma} s^\sigma \tilde{g}(s, u(s)) ds \\
= & \delta_0 \int_0^1 \left[G(t, s) + \frac{t^{\alpha-\mu_N-1} \sum_{j=1}^{m-2} a_j}{(\alpha-\mu_N-1)(\alpha-\mu_N-2)(\alpha-\mu_N-3)(1-\sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\mu_N-4})} \right. \\
& \left. \times H(\xi_j, s) \right] \times \tilde{g}(s, u(s)) ds = \delta_0 Bu(t).
\end{aligned}$$

Hence, $Au \geq \delta_0 Bu$ for all $u \in P$. Now by using Theorem 1.3, the operator equation $Au + Bu = u$ has a unique solution $u^* \in P_k$. Moreover, the sequence $u_n = Au_{n-1} + Bu_{n-1}$ for $n \geq 1$ with initial value $u_0 \in P_k$ converges to u^* . This implies that the problem (1.2) has the unique positive solution $u^* \in P_k$. Thus, $I^{\mu_N} u^*$ is a unique positive solution for the problem (1.1). \square

Here, We list another similar result by using different conditions and Theorem 1.2 which we omit its proof.

Theorem 2.2. *Suppose that $g = 0$, $\beta_0 = 0$, $0 < \sigma < 1$, $f : (0, 1] \times \mathbb{R}^{N+2} \rightarrow [0, \infty)$ is continuous, $\lim_{t \rightarrow 0^+} f(t, \cdot, \cdot, \dots, \cdot) = +\infty$ and the map $t^\sigma f(t, y_1, y_2, \dots, y_{N+2})$ is continuous on $[0, 1] \times \mathbb{R}^{N+2}$. Assume there exist positive constants $\vartheta_1, \dots, \vartheta_L$ such that*

$$0 \leq h(t, s, x_0, x_1, \dots, x_L) - h(t, s, y_0, y_1, \dots, y_L) \leq \sum_{j=0}^L \vartheta_j (x_j - y_j)$$

for all $t, s \in [0, 1]$ and $x_j, y_j \in [0, \infty)$ with $x_j \geq y_j$ ($0 \leq j \leq L$). Then the problem (1.1) has a unique positive solution if there exist $p_1, \dots, p_{N+2} > 0$ such that $K \left(\sum_{i=1}^{N+2} p_i \right) \leq 1$ and there exist $0 < \theta_2 \leq \left(\gamma_0 \sum_{j=0}^L \vartheta_j \frac{1}{\Gamma(\mu_N - \beta_j + 1)} \right)^{-1}$, $0 < \theta_1 \leq \Gamma(\mu_N + 1)$, $0 < \theta_{i+2} \leq \Gamma(\mu_N - \mu_i + 1)$ for $1 \leq i \leq N-1$ and $0 < \theta_{N+2} \leq 1$ such that $0 \leq t^\sigma \left(f(t, u_1, u_2, \dots, u_{N+2}) - f(t, v_1, v_2, \dots, v_{N+2}) \right) \leq \sum_{i=1}^{N+2} p_i \phi(\theta_i (u_i - v_i))$ for all $t \in [0, 1]$ and $u_i, v_i \in [0, \infty)$ with $u_i \geq v_i$ ($1 \leq i \leq N+2$), where $\phi : [0, \infty) \rightarrow [0, \infty)$ is a nondecreasing continuous map such that $\psi : [0, \infty) \rightarrow [0, \infty)$ is nondecreasing, $\psi(0) = 0$ and ψ is positive on $(0, \infty)$. Here, $\psi(t) = t - \phi(t)$.

Now, we give the following example to illustrate our main result.

Example 2.1. *Consider the singular boundary value problem*

$$\begin{aligned}
D^{\frac{36}{5}} u(t) &= \frac{1}{\sqrt{t}} \left[|u(t)|^{\frac{1}{2}} + p(t) \left(\frac{|\varphi u(t) + u(t)|}{1 + |\varphi u(t) + u(t)|} \right)^{\frac{1}{3}} + q(t) \left(\arctan |D^{\frac{1}{4}} u(t)| \right)^{\frac{1}{4}} \right. \\
&+ \left(|D^{\frac{10}{9}} u(t)|^3 + |D^{\frac{31}{10}} u(t)| \right)^{\frac{1}{12}} + \arctan(|u(t) + D^{\frac{1}{4}} u(t)|) + r(t) \frac{|\varphi u(t)|}{1 + |\varphi u(t)|} + s(t) \\
&\left. \times \frac{|D^{\frac{1}{4}} u(t) + D^{\frac{10}{9}} u(t)|}{1 + |D^{\frac{1}{4}} u(t) + D^{\frac{10}{9}} u(t)|} + \ln \left(1 + \left(|D^{\frac{10}{9}} u(t)|^3 + |D^{\frac{31}{10}} u(t)| \right)^{\frac{1}{12}} \right) + a(t) + \frac{\pi}{2} + b \right] \quad (2.1)
\end{aligned}$$

with boundary conditions $D^{\frac{1}{4}}u(0) = D^{\frac{10}{9}}u(0) = D^{\frac{31}{10}}u(0) = D^{\frac{41}{10}}u(0) = D^{\frac{51}{10}}u(0) = D^{\frac{61}{10}}u(0) = D^{\frac{1}{7}}u(0) = D^{\frac{4}{3}}u(0) = D^{\frac{21}{10}}u(0) = 0$ and $D^{\frac{61}{10}}u(1) = \frac{\pi}{4}D^{\frac{61}{10}}u(\frac{1}{13}) + \frac{1}{17}D^{\frac{61}{10}}u(\frac{1}{7}) + \frac{e^{-3}}{2}D^{\frac{61}{10}}u(\frac{3}{4}) + \frac{3}{19}D^{\frac{61}{10}}u(\frac{5}{6}) + \frac{4}{21}D^{\frac{61}{10}}u(\frac{7}{8})$, where $b > 0$ is a constant, $a, p, q, r, s : [0, 1] \rightarrow [0, \infty)$ are continuous, $(\varphi u)(t) = \int_0^t \frac{(t-s)^5}{\sqrt{1+s^2}} \left[\frac{e^{-t^3} \sin^2 s}{\sqrt{1+t^4}} + \frac{(3 + \cos(s^3))}{1+s^2} \right] \times \left(\ln(1+|u(s)|) + |u(s)|^{\alpha_1} |D^{\frac{1}{7}}u(s)|^{\alpha_2} |D^{\frac{4}{3}}u(s)|^{\alpha_3} |D^{\frac{21}{10}}u(s)|^{\alpha_4} + \left(b_1|u(s)|^p + b_2|D^{\frac{1}{7}}u(s)|^p + b_3|D^{\frac{4}{3}}u(s)|^p + b_4|D^{\frac{21}{10}}u(s)|^p \right)^{\frac{1}{p}} \right) ds, \alpha_i,$

$b_i \geq 0$ ($1 \leq i \leq 4$), $\sum_{i=1}^4 \alpha_i \leq 1$, $p > 0$, $\alpha = \frac{36}{5}$, $N = 3$, $\mu_1 = \frac{1}{4}$, $\mu_2 = \frac{10}{9}$, $\mu_3 = \frac{31}{10}$, $L = 3$, $\beta_1 = \frac{1}{7}$, $\beta_2 = \frac{4}{3}$, $\beta_3 = \frac{21}{10}$, $a_1 = \frac{\pi}{4}$, $a_2 = \frac{1}{17}$, $a_3 = \frac{e^{-3}}{2}$, $a_4 = \frac{3}{19}$, $a_5 = \frac{4}{21}$, $\xi_1 = \frac{1}{13}$, $\xi_2 = \frac{1}{7}$, $\xi_3 = \frac{3}{4}$, $\xi_4 = \frac{5}{6}$, $\xi_5 = \frac{7}{8}$. Put $f(t, x_1, x_2, x_3, x_4, x_5) = \frac{1}{\sqrt{t}} \left[|x_1|^{\frac{1}{2}} + p(t) \left(\frac{|x_2+x_1|}{1+|x_2+x_1|} \right)^{\frac{1}{3}} + q(t) (\arctan |x_3|)^{\frac{1}{4}} + (|x_4|^3 + |x_5|)^{\frac{1}{12}} + a(t) + \frac{\pi}{2} + c \right]$,

$$g(t, x_1, x_2, x_3, x_4, x_5) = \frac{1}{\sqrt{t}} \left[\arctan(|x_1 + x_3|) + r(t) \frac{|x_2|}{1 + |x_2|} + s(t) \frac{|x_3 + x_4|}{1 + |x_3 + x_4|} + \ln \left(1 + (|x_4|^3 + |x_5|)^{\frac{1}{12}} \right) + b - c \right]$$

and $h(t, s, y_1, y_2, y_3, y_4) = \frac{e^{-t^3} \sin^2 s}{\sqrt{1+t^4}} + \frac{(3+\cos(s^3))}{1+s^2} \left(\ln(1 + |y_1|) + |y_1|^{\alpha_1} |y_2|^{\alpha_2} |y_3|^{\alpha_3} |y_4|^{\alpha_4} + (b_1|y_1|^p + b_2|y_2|^p + b_3|y_3|^p + b_4|y_4|^p)^{\frac{1}{p}} \right)$ for all $x_i, y_j \in \mathbb{R}$ ($1 \leq i \leq 5, 1 \leq j \leq 4$) and $t, s \in [0, 1]$. Let $\sigma = \frac{1}{2}$, $\gamma = \frac{1}{2}$, $r_{\max} = \max\{r(t) : t \in [0, 1]\}$, $s_{\max} = \max\{s(t) : t \in [0, 1]\}$ and $0 < c \leq \frac{b}{2}$. It is easy to check that the maps $t^\sigma f(t, \dots, \dots)$, $t^\sigma g(t, \dots, \dots)$ and $h(t, s, \dots, \dots)$ are increasing with respect to their components on $[0, \infty)$ for all $t, s \in [0, 1]$ and $t^\sigma g(t, 0, 0, \dots, 0) = b - c > 0$. Also, we have

$$\begin{aligned} t^\sigma g(t, \lambda x_1, \lambda x_2, \lambda x_3, \lambda x_4, \lambda x_5) &= \arctan(\lambda(x_1 + x_3)) + r(t) \frac{\lambda x_2}{1 + \lambda x_2} + s(t) \frac{\lambda(x_3 + x_4)}{1 + \lambda(x_3 + x_4)} \\ &\quad + \ln \left(1 + (\lambda^3 x_4^3 + \lambda x_5)^{\frac{1}{12}} \right) + b - c \\ &\geq \lambda \left(\arctan(x_1 + x_3) + r(t) \frac{x_2}{1 + x_2} + s(t) \frac{x_3 + x_4}{1 + x_3 + x_4} + \ln \left(1 + (x_4^3 + x_5)^{\frac{1}{12}} \right) + b - c \right) \\ &= \lambda t^\sigma g(t, x_1, x_2, x_3, x_4, x_5), \\ t^\sigma f(t, \lambda x_1, \lambda x_2, \lambda x_3, \lambda x_4, \lambda x_5) &= \lambda^{\frac{1}{2}} x_1^{\frac{1}{2}} + p(t) \left(\frac{\lambda(x_2 + x_1)}{1 + \lambda(x_2 + x_1)} \right)^{\frac{1}{3}} + q(t) (\arctan(\lambda x_3))^{\frac{1}{4}} \\ &\quad + (\lambda^3 x_4^3 + \lambda x_5)^{\frac{1}{12}} + a(t) + \frac{\pi}{2} + c \\ &\geq \lambda^{\frac{1}{2}} \left(x_1^{\frac{1}{2}} + p(t) \left(\frac{x_2 + x_1}{1 + x_2 + x_1} \right)^{\frac{1}{3}} + q(t) (\arctan(x_3))^{\frac{1}{4}} + (x_4^3 + x_5)^{\frac{1}{12}} + a(t) + \frac{\pi}{2} + c \right) \\ &= \lambda^\gamma t^\sigma f(t, x_1, x_2, x_3, x_4, x_5) \text{ and} \end{aligned}$$

$$\begin{aligned} h(t, s, \lambda y_1, \lambda y_2, \lambda y_3, \lambda y_4) &= \frac{e^{-t^3} \sin^2 s}{\sqrt{1+t^4}} + \frac{(3 + \cos(s^3))}{1+s^2} \left(\ln(1 + \lambda y_1) \right. \\ &\quad \left. + \lambda (\sum_{i=1}^4 \alpha_i) y_1^{\alpha_1} y_2^{\alpha_2} y_3^{\alpha_3} y_4^{\alpha_4} + \lambda (b_1 y_1^p + b_2 y_2^p + b_3 y_3^p + b_4 y_4^p)^{\frac{1}{p}} \right) \geq \lambda \left(\frac{e^{-t^3} \sin^2 s}{\sqrt{1+t^4}} \right. \\ &\quad \left. + \frac{(3 + \cos(s^3))}{1+s^2} \left(\ln(1 + y_1) + y_1^{\alpha_1} y_2^{\alpha_2} y_3^{\alpha_3} y_4^{\alpha_4} + (b_1 y_1^p + b_2 y_2^p + b_3 y_3^p + b_4 y_4^p)^{\frac{1}{p}} \right) \right) \end{aligned}$$

$= \lambda h(t, s, y_1, y_2, y_3, y_4)$ for all $t, s \in [0, 1]$, $\lambda \in (0, 1)$ and $x_i, y_j \in [0, \infty)$ ($1 \leq i \leq 5, 1 \leq j \leq 4$). If δ_0 belongs to $(0, \frac{c}{r_{\max} + s_{\max} + b - c}]$, then we get

$$\begin{aligned} t^\sigma f(t, x_1, x_2, x_3, x_4, x_5) &= x_1^{\frac{1}{2}} + p(t) \left(\frac{x_2 + x_1}{1 + x_2 + x_1} \right)^{\frac{1}{3}} \\ &\quad + q(t) (\arctan(x_3))^{\frac{1}{4}} + (x_4^3 + x_5)^{\frac{1}{12}} + a(t) + \frac{\pi}{2} + c \geq \frac{\pi}{2} + (x_4^3 + x_5)^{\frac{1}{12}} + c \\ &\geq \arctan(x_1 + x_3) + \ln(1 + (x_4^3 + x_5)^{\frac{1}{12}}) + \frac{c}{r_{\max} + s_{\max} + b - c} \times (r_{\max} + s_{\max} + b - c) \\ &\geq \delta_0 \left(\arctan(x_1 + x_3) + r(t) \frac{x_2}{1 + x_2} + s(t) \frac{x_3 + x_4}{1 + x_3 + x_4} + \ln \left(1 + (x_4^3 + x_5)^{\frac{1}{12}} \right) + b - c \right) \\ &= \delta_0 t^\sigma g(t, x_1, x_2, x_3, x_4, x_5). \end{aligned}$$

Now by using Theorem 2.1, we get the problem (2.1) has a unique positive solution.

REFERENCES

- [1] R. P. Agarwal, D. O'regan, S. Stanek, *Positive solutions for Dirichlet problem of singular nonlinear fractional differential equations*, J. Math. Anal. Appl. 371 (2010) 57–68.
- [2] Z. Bai, W. Sun, *Existence and multiplicity of positive solutions for singular fractional boundary value problems*, Comput. Math. with Appl. 63 (2012) 1369–1381.
- [3] A. Cabada, G. Wang, *Positive solution of nonlinear fractional differential equations with integral boundary value conditions*, J. Math. Anal. Appl. 389 (2012) 403–411.
- [4] I. J. Cabrera, J. Harjani, K. B. Sadarangani, *Positive and nondecreasing solutions to m-point boundary value problem for nonlinear fractional differential equation*, Abstr. Appl. Anal. (2012) Article ID 826580, 15 pages.
- [5] D. Guo, V. Lakshmikantham, *Nonlinear problems in abstract cones*, Academic Press (1988).
- [6] J. Harjani, K. Sadarangani, *Fixed point theorems for weakly contractive mappings in partially ordered sets*, Nonlinear Anal. 71 (2009) 3403–3410.
- [7] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, Vol. 204, Elsevier Science, Amsterdam (2006).
- [8] S. Liang, J. Zhang, *Existence of multiple positive solutions for m-point fractional boundary value problems on an infinite interval*, Math. Comput. Model. 54 (2011) 1334–1346.
- [9] S. Liang, J. Zhang, *Existence and uniqueness of positive solutions to m-point boundary value problem for nonlinear fractional differential equation*, J. Appl. Math. Comput. 38 (2012) 225–241.
- [10] Y. Liu, P. J. Y. Wong, *Global existence of solutions for a system of singular fractional differential equations with impulse effects*, J. Appl. Math. Inform. 33 (2015) No. 3-4, 327–342.
- [11] J. Nieto, R. Rodriguez-Lopez, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order 22 (2005) 223–239.
- [12] N. Nyamoradi, T. Bashiri, S. M. Vaezpour, D. Baleanu, *Uniqueness and existence of positive solutions for singular fractional differential equations*, Electron. J. Diff. Eq. (2014) No. 130, 13 pages.
- [13] I. Podlubny, *Fractional differential equations*, Academic Press (1999).
- [14] Sh. Rezapour, M. Shabibi, *A singular fractional differential equation with Riemann-Liouville integral boundary condition*, J. Adv. Math. Stud. 8 (2015) No. 1, 80–88.
- [15] J. Sabatier, O. P. Agrawal, J. A. T. Machado, *Advances in fractional calculus: theoretical developments and applications in physics and engineering*, Springer (2007).
- [16] H. A. H. Salem, *On the fractional order m-point boundary value problem in reflexive Banach spaces and weak topologies*, J. Comput. Math. Appl. 224 (2009) 565–572.
- [17] Y. Tian, Y. Zhou, *Positive solution for multi-point boundary value problem of fractional differential equations*, J. Appl. Comput. 38 (2012) 417–427.
- [18] S. W. Vong, *Positive solutions of singular fractional differential equations with integral boundary conditions*, Math. Computer Model. 57 (2013) 1053–1059.
- [19] Y. Wang, L. Liu, *Necessary and sufficient condition for the existence of positive solution to singular fractional differential equations*, Adv. Diff. Eq. (2015) 2014:207.
- [20] C. Zhai, D. R. Anderson, *A sum operator equation and applications to nonlinear elastic beam equations and Lane-Emden-Fowler equations*, J. Math. Anal. Appl. 375 (2011) 388–400.
- [21] C. Zhai, W. Yan, C. Yang, *A sum operator method for the existence and uniqueness of positive solutions to Riemann-Liouville fractional differential equation boundary value problems*, Commun Nonlinear Sci. Numer Simulat. 18 (2013) 858–866.