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A contribution to the analysis of time fractional stochastic functional partial differential equations and applications

(Una contribución al análisis de las ecuaciones en derivadas parciales estocásticas funcionales con derivadas fraccionarias en tiempo y aplicaciones)

Memoria escrita por

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Dedicated to Prof. Tomás Caraballo on occasion of his 60^{th} birthday

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Contents

Notation

$\mathbb{R}^d$	the set of d-dimensions real numbers
$\mathbb{R}^+$	the set of positive real numbers
$\mathbb{C}$	the set of complex numbers
$\mathbb{Z}$	the set of positive integer
$\mathbb{N}$	the set of natural numbers
$\emptyset$	the empty set
Ω	the space in $\mathbb{R}$ or sample space
$C^m(\Omega)$	the space of functions f which are m times continuously differentiable on Ω
$L^p(a,b)$	the set of Lebesgue measurable function f on $[a,b]$, $p \in [1,\infty]$
$L^2(\Omega;\mathbb{H})$	the set of strongly-measurable, square-integrable $\mathbb{H}$-valued random variables
L^2_σ	the set of the subspace of the divergence-free vector fields in L^2
$W^{1,2}$	the subspace of L^2 consisting of functions that the weak derivative $\frac{\partial u}{\partial t}$ belongs to L^2
$AC[a,b]$	the space of functions f which are absolutely continuous on $[a,b]$
$E_{\alpha,\beta}$	Mittag-Leffler function with parameters α and β
$\mathbb{E}$	the expectation
$(\Omega,\mathcal{F},\mathbb{P})$	complete probability space
$\mathcal{L}(X;X)$	bounded linear operator from Banach space X to itself
$\Vert\cdot\Vert$	norm in $L^2(\Omega)$
$(\cdot,\cdot)$	inner product in $L^2(\Omega)$
$a \wedge b$	the smaller one between a and b, i.e., min{a,b}
C	an arbitrary positive constant, which may be different from line to line and even in the same line

Introduction

The classical heat equation $\partial_t u = \Delta u$ describes heat propagation in a homogeneous medium. The time fractional diffusion equation $\partial_t^\alpha u = \Delta u$ with $0 < \alpha < 1$ has been widely used to model anomalous diffusion exhibiting subdiffusive behavior, e.g., due to particle sticking and trapping phenomena. While in normal diffusions (described by the heat equations or more general parabolic equations), the mean square displacement of a diffusive particle behaves like const$\cdot t$ for $t \to \infty$, the time-fractional diffusion equation exhibits a behavior like const$\cdot t^\alpha$ for $t \to \infty$. This is the reason why time fractional equations with $0 < \alpha < 1$ are called subdiffusion equations in the literature and for the case $1 < \alpha < 2$ are called superdiffusion equations. Hence, in recent decades, scientists have developed many new models that naturally involve fractional differential equations, which demonstrate the anomalous diffusion phenomena appearing in the real world successfully, see e.g., [1, 21, 24, 26, 47, 56, 59, 45] and references therein.

Fractional Calculus has a long history, and its origins can be tracked back to the end of 17th century. The first main steps of the theory date back to the first half of the 19th century, although this subject has become very active only over the last few decades. Derivatives and integrals of non-integer order are very suitable for the description of properties of various real materials, e.g., in mechanics (theory of viscoelasticity and viscoplasticity), biochemistry (modeling of polymers and proteins), electrical engineering (transmission of ultrasound waves), medicine (modeling of human tissue under mechanical loads) etc. For more applications and references we refer the reader to [20, 21, 31, 45, 47, 52, 56, 59, 78] and references therein.

There are several kinds definitions of fractional calculus, here we only introduce Riemann-Liouville and Caputo time fractional derivatives which are frequently used in current literature. The classical form of fractional calculus is given by Riemann-Liouville integral, which is essentially described below,

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau,$$

where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) := \int_0^\infty t^{\alpha-1} e^{-t} dt$. Similar to definition of Riemann-Liouville integral, the Riemann-Liouville fractional derivative is defined by

$$({}^L D_t^\alpha f)(t) = \frac{d^n}{dt^n} I_t^{n-\alpha} f(t) = \frac{1}{n-\alpha} \left(\frac{d}{dt}\right)^n \int_0^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau, \quad n \leq \alpha < n+1.$$

Another option for computing fractional derivative is the Caputo fractional one. It was introduced by Michelo Caputo in 1967, Caputo's definition is illustrated as follows:

$$({}^C D_t^\alpha f)(t) := \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, \quad n-1 < \alpha < n.$$

In contrast to Riemann-Liouville fractional derivative, the advantages of Caputo factional derivative are: (a) when solving differential equations using Caputo's definition, it is not necessary to define the fractional order initial conditions; (b) $f^{(n)}(t)$ is zero when $f(t)$ is constant and its Laplace transform is expressed by means of the initial values of the function and its derivative.

From the mathematical point of view, the deterministic partial differential equations have been well studied. A. Friedman and B. Hu in [36] analyzed the bifurcation from stability to instability for a free boundary problem modeling tumor growth by Stokes equation. T. Caraballo and J. Real in [17] studied 2D Navier-Stokes equations with delays. R. N. Wang and D. H. Chen and T. J. Xiao in [78] investigated the abstract fractional Cauchy problems with almost sectorial operators etc. However, when we consider a physical system in the real world, we have to consider some influence of internal, external, or environmental noises. Besides, the whole background of physical system may be difficult to describe deterministically.

Therefore, in recent years, there has been growing interest in stochastic partial differential equations (see, e.g.,[12, 13, 14, 15, 29, 39, 40, 56, 58, 59, 79, 81]). In order to have a much better description of real models, we are able to consider some randomness which can be described by some kinds of white or colored noise or some other types of stochastic terms. In this project, we are mainly interested in applying two kinds of noises: *Brownian motion/Wiener Process* and *Fractional Brownian motion.*

- *Brownian motion/Wiener process*

In 1828 the Scottish botanish Robert Brown observed that pollen grains suspended in liquid performed an irregular motion. The motion was later explained by the random collisions with the molecules of the liquid. To describe the motion mathematically it is natural to use the concept of a stochastic process $B_t(\omega)$, interpreted as the position at time t of the pollen grain ω. In mathematics, Brownian motion is described by the Wiener process, a continuous time stochastic process. It is one of the best known Lévy processes and occurs frequently in pure and applied mathematics, economics and physics, etc.

The Brownian motion $B_t(\omega)$ is characterized by four facts:

(1) $B_0(\omega) = 0$;

(2) $B_t(\omega)$ is almost surely continuous;

(3) $B_t(\omega)$ has independent increments, which means that if $0 \le s_1 < t_1 \le s_2 < t_2$, then $B_{t_1}(\omega) - B_{s_1}(\omega)$ and $B_{t_2}(\omega) - B_{s_2}(\omega)$ are independent random variables;

(4) $B_t(\omega) - B_s(\omega) \sim \mathcal{N}(0, t-s)$ (for $0 \le s \le t$).

$\mathcal{N}(\mu, \sigma^2)$ denotes the normal distribution with expected value μ and variance σ^2.

Based on these properties of Brownian motion, it is possible to represent B_t as a generalized stochastic process called white noise process. Suppose that H is a right-continuous, adapted and locally bounded process, if $\{\pi_n\}$ is a sequence of partitions of $[0, t]$ with mesh going to zero, then the Itô integral of H with respect to B up to time t is a random variable

$$\int_0^t HdB = \lim_{n\to\infty} \sum_{[t_{i-}, t_i]\in\pi_n} H_{t_{i-1}}(B_{t_i} - B_{t_{i-1}}) \qquad \text{in} \quad L^2(\Omega).$$

Indeed, Itô integrals are martingales, this gives Itô integral an important computational advantage, such as, the Itô isometry (Lemma 1.11) which will be used frequently in our analysis.

- *Fractional Brownian motion*

A. N. Kolmogorov was the first to consider continuous Gaussian processes with stationary increments and with self-similarity properties, it means that for any $a > 0$, there exists $b > 0$ such that

$$Law(X(at); at \geq 0) = Law(bX(t); t \geq 0).$$

It turns out that such processes with zero mean have a special correlation function:

$$E\left(X(t)X(s)\right) = \frac{1}{2}(|s|^{2H} + |t|^{2H} - |t-s|^{2H}),$$

where $0 < H < 1$, which is called Hurst index. Notice that fractional Brownian motion is neither a semimartingale (except the case $H = 1/2$ when it is a Brownian motion) nor a Markov process. The former prevents the use of a well-established integration theory, the latter means that there is no direct connection between fractional Brownian motion and differential operators. However, it is closely connected with fractional calculus and can be represented as a " fractional integral" (with the help of a comparatively complicated hypergeometric kernel) via the Wiener process not only on infinite, but also on finite intervals. Such a representation, together with the Gaussian property of fractional Brownian motion and the Hölder property of its trajectories (fractional Brownian motion with Hurst index H is Hölder up to order H) permits us to create an interesting and specific stochastic calculus for fractional Brownian motion. But, the technique has one main point: it is harder to obtain a proper notion of integration as Hurst index H is smaller; the more irregular paths of the stochastic process are , the harder it is to integrate against them. It is worth mentioning that in [13], the authors established a technical lemma (Lemma 1.13) which is crucial to the stochastic integral with respect to fractional Brownian motion when considering the Hurst parameter $H \in (1/2, 1)$, this point of view is adopted in our thesis.

We also want to mention that, obviously, the future evolution of a system does not only depend on its current state, its past history does determine its future behavior too. Therefore, the retarded differential equations have been becoming an important area of applied mathematics, such as high velocity fields in wind tunnel experiments, or other memory processes, or biological motivations like species growth or incubating time in disease models among many others, for example, [12, 13, 16, 17, 18, 25, 57, 58, 62, 80, 81].

Motivated by above considerations, we will study in this work time fractional stochastic *lattice system, impulsive system and 2D Stokes system* to convince readers that such kind of differential equations is an important subject nowadays.

This PhD project is split into four chapters. In Chapter 1, we recall some basic definitions, properties and lemmas about fractional calculus and stochastic process which will be used throughout the project. Chapter 2 is devoted to studying the existence and uniqueness of solutions as well as the asymptotic behavior of stochastic lattice systems with a Caputo time fractional derivative. Next, in Chapter 3, we are going to investigate the well-posedness and dynamics of a kind of stochastic fractional impulsive differential equations with infinite delay in phase space $\mathcal{PC}$ (piecewise continuously). At beginning, a much more complicated model is considered (problem (6)), only well-posedness and asymptotic behavior can be analyzed in Sections 3.2 and 3.3. In order to go further step, later, a quite general model is studied in Section 3.4 (problem (7)), because of higher regularity of this original model, we are not only able to obtain well-posedness to this problem, but also global attracting sets (general case/ singleton case). At last, the main goal of Chapter 4 is to construct the well-posedness of stochastic time fractional 2D-Stokes equations with delay in the phase space $C((-h,0];L^2(\Omega;L^2_\sigma))$, where $h \leq \infty$.

Below we will describe in more details the content of each chapter.

- ***Preliminaries***

Chapter 1 is divided into three sections. In Section 1.1, we mainly recall basic concepts/properties to Riemann-Liouville time fractional derivative. Next, the Caputo fractional derivative is defined via Riemann-Liouville fractional derivative. As pointed out by Lemma 1.6, D^α is a left inverse of J^α, not a right inverse, when we want to obtain the mild solutions to time fractional differential equations, the Laplace transform of the Caputo fractional derivative provides us an alternative to obtain it.

In Section 1.2, some basic results concerning the theory of stochastic processes are presented. Indeed, Lemma 1.11 and inequality (1.6) guarantee us to deal with stochastic integral with respect to finite/infinite Brownian motion, respectively. Moreover, we are able to handle stochastic integrals with respect to fractional Brownian motion by virtue of Lemma 1.13.

Additionally, some examples of time fractional stochastic functional differential equations from applications are listed in Section 1.3 to end this chapter.

- ***Stochastic lattice systems with a Caputo time fractional derivative***

Lattice systems have attracted much attention in the literature, which arise naturally in a wide variety of applications where the spatial structure possesses a discrete character as well as in the spatial discretization of continuous problems. Therefore, in Chapter 2, we will study a kind of fractional stochastic lattice system.

Let X be a separable Hilbert space with norm $\|\cdot\|$ and inner product $(\cdot,\cdot)$. Then $L^2(\Omega;X)$ is a Hilbert space of X-valued random variable with norm $(\mathbb{E}\|\cdot\|^2)^{\frac{1}{2}}$ and inner product $\mathbb{E}(\cdot,\cdot)$.

In this chapter, when no confusion is possible, D^α and D_s^α denote the Caputo fractional derivative, and Caputo fractional substantial derivative (see Definition 1.4) of order α with $\sigma > 0$, respectively.

We first consider the existence of solutions of the fractional order SDEs,

$$\begin{cases} D^\alpha x(t) = f(t,x(t)) + g(t)\frac{dB(t)}{dt}, & t \geq 0, \dfrac{1}{2} < \alpha < 1, \\ x(0) = x_0 \in L^2(\Omega;X) \end{cases} \tag{1}$$

and

$$\begin{cases} D_s^\alpha x(t) = f(t,x(t)) + g(t)\frac{dB(t)}{dt}, & t \geq 0, \dfrac{1}{2} < \alpha < 1, \\ x(0) = x_0 \in L^2(\Omega;X), \end{cases} \tag{2}$$

in the Hilbert space $L^2(\Omega;X)$.

Instead of using a fixed point theorem to prove the existence and uniqueness of solutions to problems of this chapter, we will follow the approach of Lakshmikantham and Vatsala [52], who proved a Peano local existence result for fractional ordinary differential equations.

Let $f : [0,\infty) \times L^2(\Omega;X) \to L^2(\Omega;X)$ be sequentially weakly continuous in bounded sets, and $g : [0,\infty) \to X$ measurable.

(R_1) Suppose f and g are bounded maps, i.e.,

$$\mathbb{E}\|f(t,x)\|^2 \leq M^2, \qquad \|g(t)\|^2 \leq M^2, \qquad \text{for all } (t,x) \in R_0,$$

where $R_0 = \{(t,x) : 0 \leq t \leq T \text{ and } \mathbb{E}\|x - x_0\|^2 \leq b^2\}$. We are able to prove the initial value problem (1) possesses at least one solution $x(\cdot)$ defined globally in time (Theorems 2.5 and 2.6).

(R_2) In addition to conditions of (R_1), also let $b^2 \geq 12\mathbb{E}\|x_0\|^2$, the initial value problem (2) possesses at least one solution $x(\cdot)$ defined globally in time (Theorems 2.7 and 2.8).

Furthermore, imposing Lipschitz conditions on terms f and g, we prove the existence and uniqueness of solutions of the fractional order SDEs,

$$\begin{cases} D^\alpha x(t) = f(t,x(t)) + g(t,x(t))\frac{dB(t)}{dt}, & t \geq 0, \dfrac{1}{2} < \alpha < 1, \\ x(0) = x_0 \in L^2(\Omega;X) \end{cases} \tag{3}$$

and

$$\begin{cases} D_s^\alpha x(t) = f(t,x(t)) + g(t,x(t))\frac{dB(t)}{dt}, & t \geq 0, \dfrac{1}{2} < \alpha < 1, \\ x(0) = x_0 \in L^2(\Omega;X), \end{cases} \tag{4}$$

in $L^2(\Omega;X)$. Let $f,\ g:[0,\infty)\times L^2(\Omega;X)\to L^2(\Omega;X)$ be measurable functions satisfying for all $x,\ y\in L^2(\Omega;X)$ and $t\in[0,T]$,

$$\mathbb{E}\|f(t,x)-f(t,y)\|^2+\mathbb{E}\|g(t,x)-g(t,y)\|^2\le L\mathbb{E}\|x-y\|^2$$

for some constants L. In addition, let f and g be bounded maps, i.e.,

$$\mathbb{E}\|f(t,x)\|^2\le M^2,\qquad \mathbb{E}\|g(t,x)\|^2\le M^2,\qquad \text{for all}\ \ (t,x)\in R_0,$$

where $R_0=\{(t,x):0\le t\le T \text{ and } \mathbb{E}\|x-x_0\|^2\le b^2\}$. We obtain

(R_3) for every $x_0\in L^2(\Omega;X)$, there exists a unique solution to problems (3), (4), respectively (Theorem 2.10).

The details how to prove these results can be found in Theorem 2.5-Theorem 2.10 in Section 2.2.

Section 2.3 is devoted to investigating the following stochastic lattice system with Caputo fractional substantial time derivative of the form

$$\begin{cases} D_s^{\alpha}x_i(t)+(-1)^p\Delta^p x_i(t)+\lambda x_i(t)=f_i(x_i(t))+g_i(t)\frac{dB(t)}{dt}, & t\ge 0,\\ x_i(0)=x_0, & i\in\mathbb{Z}. \end{cases} \tag{5}$$

where, $1/2<\alpha<1$, $B(t)$ is a standard scalar Brownian motion on an underlying complete filtered probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\in\mathbb{R}},\mathbb{P})$, and $\lambda\in\mathbb{R}$, p is any positive integer, $\Delta^p=\Delta\circ\cdots\circ\Delta$, p times. Δ denotes the discrete one-dimensional Laplace operator, which is defined by $\Delta x_i=x_{i+1}+x_{i-1}-2x_i$. We also write $\partial^+x_i=x_{i+1}-x_i$, $\partial^-x_i=x_i-x_{i-1}$ and define

$$\mathbb{D}^p:=\begin{cases}\Delta^{\frac{p}{2}}, & p \text{ even},\\ \partial^+\Delta^{\frac{p-1}{2}}, & p \text{ odd}.\end{cases}$$

The natural phase space for such an infinite dimensional system of differential equations, fractional or not, is the Hilbert space

$$\ell^2=\left\{x=(x_i)_{i\in\mathbb{Z}},x_i\in\mathbb{R}:\sum_{i\in\mathbb{Z}}x_i^2<+\infty\right\},$$

with the inner product and norm

$$(x,y)=\sum_{i\in\mathbb{Z}}x_iy_i,\qquad \|x\|^2=\sum_{i\in\mathbb{Z}}x_i^2,\qquad \forall x=(x_i)_{i\in\mathbb{Z}},\ y=(y_i)_{i\in\mathbb{Z}}\in\ell^2.$$

Let $L^2(\Omega;\ell^2)$ denote the Hilbert space of all strongly measurable, square-integrable ℓ^2-valued random variables with the inner product and norm

$$\mathbb{E}(x,y)=\int_\Omega\sum_{i\in\mathbb{Z}}x_iy_id\mathbb{P},\qquad \|x\|_{L^2(\Omega;\ell^2)}=(\mathbb{E}\|x\|^2)^{\frac{1}{2}},\qquad \forall x,y\in L^2(\Omega;\ell^2).$$

In Section 2.3.1, suppose that

(H_1) The operators $f : L^2(\Omega;\ell^2) \to L^2(\Omega;\ell^2)$ and $g : [0,+\infty) \to \ell^2$, given componentwise by $(f(x))_i = f_i(x_i)$ and $(g(t))_i = g_i(t)$, $i \in \mathbb{Z}$, are well defined and bounded, and g is measurable.

(H_2) The $f_i : L^2(\Omega;\mathbb{R}) \to L^2(\Omega;\mathbb{R})$ are sequentially weakly continuous in bounded sets.

(H_3) The $f_i : L^2(\Omega;\mathbb{R}) \to L^2(\Omega;\mathbb{R})$ satisfy $\mathbb{E}|f_i(x)|^2 \leq |k_{i,1}|^2 + k_2^2\mathbb{E}|x|^2$ for all $x \in L^2(\Omega;\mathbb{R})$, where $k_1 = (k_{1,i})_{i\in\mathbb{Z}} \in \ell^2$ and $k_2 > 0$.

(H_4) There exists a positive constant M' such that for all $t \geq 0$,

$$\int_0^t (t-\tau)^{2\alpha-2} e^{-2\sigma(t-\tau)} \|g(\tau)\|^2 d\tau \leq M'.$$

(H_5) $\mathbb{E}|f_i(x) - f_i(y)|^2 \leq L'\mathbb{E}|x-y|^2$ for any x, $y \in L^2(\Omega;\mathbb{R})$, where $L' > 0$.

We obtain the following results:

(R_4) The initial value problem (5) has at least one solution if (H_1)-(H_2) hold (Theorem 2.11).

(R_5) Let σ (the coefficient of Caputo fractional substantial derivative) be large enough and (H_2)-(H_4) hold, then the solutions of initial value problem (5) define globally in time, also the estimation of absorbing set of problem (5) is derived (Corollary 2.13).

(R_6) Let σ be sufficiently large and conditions (H_1), (H_3)-(H_5) hold, then there exists a unique global solution to problem (5) (Theorem 2.14).

Once we derive the estimation of absorbing set (Lemma 2.12), immediately, it is possible to study the existence of absorbing set to problem (5) (Theorem 2.15). Moreover, in the strong mean-square topology sense, the uniformly asymptotic stable of the solutions to problem (5) is shown (Theorem 2.16) to end Chapter 2.

- ***Fractional stochastic impulsive differential equations***

The theory of impulsive differential equations has become an active area due to its wide applications in communications, mechanics, electrical engineering, medicine, biology, etc.

So, the aim of Chapter 3 is to address the issues of well-posedness and dynamics to the following problems

$$\begin{cases} D_t^\alpha x(t) = Ax(t) + f(t,x_t) + g(t,x_t)\frac{dB(t)}{dt} + h(t)\frac{dB_Q^H(t)}{dt}, \quad t \geq 0, \\ \qquad t \neq t_k, \quad \frac{1}{2} < \alpha < 1, \\ \Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \quad k = 1,2,\cdots, \\ x(t) = \phi(t), \quad t \in (-\infty, 0], \end{cases} \tag{6}$$

and

$$\begin{cases} D_t^\alpha x(t) = Ax(t) + I_t^{1-\alpha} f(t, x_t) + [I_t^{1-\alpha} g(t, x_t)] \frac{dB(t)}{dt} + [I_t^{1-\alpha} h(t)] \frac{dB_Q^H(t)}{dt}, \\ \qquad t \geq 0, \quad t \neq t_k, \quad 0 < \alpha < 1, \\ \Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \quad t = t_k, \quad k = 1, 2, \cdots, \\ x(t) = \phi(t), \quad t \in (-\infty, 0], \end{cases} \tag{7}$$

where D_t^α is the Caputo fractional derivative of order $0 < \alpha < 1$, $I_t^{1-\alpha}$ is the $(1-\alpha)$-order fractional integral operator. For both models (6) and (7), $x(\cdot)$ takes values in the separable Hilbert space $\mathbb{H}$. $A : D(A) \subseteq \mathbb{H} \to \mathbb{H}$ is the infinitesimal generator of an α-order fractional compact and analytic operator $T_\alpha(t)(t \geq 0)$. $B(t)$ and $B_Q^H(t)$ denote, respectively, a $\mathbb{K}$-valued Q-cylindrical Brownian motion and fractional Brownian motion. The fixed time t_k, where the impulses take place, satisfy $0 = t_0 < t_1 < \cdots < t_k \to +\infty$ as $k \to \infty$.

To achieve our goal, in Section 3.1, we first present the abstract phase space $\mathcal{PC}$ in which we will establish our results properly. Let $L^2(\Omega; \mathbb{H})$ denote the Banach space of all strongly-measurable, square-integrable $\mathbb{H}$-valued random variables equipped with the norm $\|u(\cdot)\|_{L^2}^2 = \mathbb{E}\|u(\cdot)\|^2$. The abstract phase space $\mathcal{PC}$ is defined by

$$\mathcal{PC} = \Big\{ \xi : (-\infty, 0] \to L^2(\Omega; \mathbb{H}) \text{ is } \mathcal{F}_0\text{-adapted and continuous except in at most a countable number of points } \{\theta_k\}, \text{ at which there exist } \xi(\theta_k^+) \text{ and } \xi(\theta_k^-) \text{ with } \xi(\theta_k) = \xi(\theta_k^-), \text{ and } \sup_{\theta \in (-\infty, 0]} e^{\gamma\theta} \mathbb{E}\|\xi(\theta)\|^2 < \infty \Big\},$$

for some fixed parameter $\gamma > 0$. The norm of this Banach space $\mathcal{PC}$ is endowed with

$$\|\xi\|_{\mathcal{PC}} = \left(\sup_{\theta \in (-\infty, 0]} e^{\gamma\theta} \mathbb{E}\|\xi(\theta)\|^2 \right)^{\frac{1}{2}}, \qquad \xi \in \mathcal{PC}.$$

With the help of Laplace transform of Caputo fractional derivative, the mild solutions to problems (6) and (7) are stated in Definitions 3.8 and 3.9, respectively.

Secondly, in Section 3.2, the existence of mild solution to problem (6) is analyzed. To do this, we use the Picard method and do estimations on each impulsive interval, together with the assumptions that α-order fractional solution operator $T_\alpha(t)$ $(t > 0)$ and the α-resolvent family $S_\alpha(t)$ $(t > 0)$ are compact (Theorem 3.11). Analogously, the existence of mild solution to problem (7) is also proved (Theorem 3.12). Next, general results on the continuous dependence of mild solutions to problems (6) and (7) on initial value are proved (Theorems 3.14 and 3.15). When dealing with fractional impulsive differential equations, we need to do estimation on each impulse interval first, then combine them by induction to obtain final results.

Thirdly, we are interested in analyzing asymptotic behavior to our models. Proving the existence of global mild solution in time is first step to go further. To do so, we focus on dynamics of problem (6) in Section 3.3. Impose assumptions that

α-order fractional solution operator $T_\alpha(t)$ and an α-resolvent family $S_\alpha(t)$ are controlled by exponential decay functions,

$$\|T_\alpha(t)\| \le Me^{-\mu t}, \quad \|S_\alpha(t)\| \le Me^{-\mu t}(1+t^{\alpha-1}), \qquad \forall t>0,\ M\ge 1, \tag{8}$$

together with Lipschitz conditions on nonlinear terms f and g, it is proved mild solution to problem (6) is unique and defined globally in time (Theorem 3.3). We end up this section with showing the exponential asymptotic behavior to our model, although the idea to prove long time behavior is standard, infinite impulses pose big challenge for us. To overcome this difficulty, taking advantages of the definition of $\mathcal{PC}$ norm and properties of exponential calculus ($\gamma > 2\mu$), by doing estimation on each impulse interval, the expected result is proved (Theorem 3.17).

To conclude this chapter, Section 3.4 is devoted to analyzing the asymptotic behavior of problem (7). We have studied well-posedness and exponential decay behavior to problem (6) in Section 3.3, however, the lack of compactness of the α-order resolvent operator $S_\alpha(t)$ does not allow us to establish the existence and structure of attracting sets, which are a key concept for understanding the dynamical properties of the model. Heuristically, in our fractional situation, the α-order fractional solution operator $T_\alpha(t)$ is compact which has been proved in [78], therefore, the analysis of well-posedness and dynamics to problem (7) is presented in Section 3.4 to make our work complete. First, the globally existence and uniqueness of mild solution are proved due to assumption (8) on fractional solution operator $T_\alpha(t)$ (Theorem 3.19). Next, making use of the relationship between $\mathcal{PC}$ norm and exponential decay parameter of fractional solution operator $T_\alpha(t)$ ($\gamma > 2\mu$), we are capable of showing mild solutions to problem (7) are bounded uniformly with respect to bounded sets of initial conditions (Theorem 3.20). We emphasize such a priori estimation is crucial for our work in Sections 3.4.1 and 3.4.2.

In Section 3.4.1, a general result considering the existence of a minimal compact set which is globally attracting is presented in phase space $\mathcal{PC}$. Thanks to the compactness of fractional solution operator $T_\alpha(t)$, together with the definition of $\mathcal{PC}$ norm, by Arzelà-Ascoli theorem, the desired goal is obtained (Lemma 3.22). Furthermore, by a standard way, the properties of the omega limit set and the compactness of minimum attracting sets are proved (Theorems 3.23 and 3.24). Beyond these general results, if we want to obtain more details of the geometrical structure of this set, we need to impose stronger conditions. Therefore, in Section 3.4.2, Lipschitz condition ensures the uniqueness of solution to problem (7), moreover, a priori estimation about uniform boundness of solutions is exponential decay, which implies the attracting set is a singleton (Theorems 3.25, 3.27).

- ***Fractional stochastic 2D-Stokes differential equations***

The well-posedness of flow problems in a viscous fluid is crucial for many areas of science and engineering, for example, the automotive and aerospace industries, as well as nanotechnology. In the latter case of microfluidic structures, we often encounter flow problems at moderate viscosities, from the mathematical point of view, the Stokes equations provide a first approximation of the more general Navier-Stokes equations in situations where the flow is nearly steady and slow, and has small velocity gradients, so the inertial effects can be ignored.

For this reason, in Chapter 4, we will analyze the following time fractional stochastic delay incompressible flow problem, i.e., the non-stationary 2D-Stokes equations,

$$\begin{cases} D_t^\alpha u - \kappa \Delta u + \nabla p = f(t, u_t) + g(t, u_t)\frac{dW(t)}{dt} & \text{in } \mathbb{R}^2,\ t \geq 0, \\ \nabla \cdot u = 0 & \text{in } \mathbb{R}^2,\ t \geq 0, \\ u(t,x) = \varphi(t,x) & \text{in } \mathbb{R}^2,\ t \in [-h, 0], \end{cases} \tag{9}$$

where f and g are external forcing terms containing some hereditary or delay characteristics, and φ is the initial data in the interval of time $t \in [-h, 0]$, where h is a fixed positive number, and $W(t)$ is a standard scalar Brownian motion/ Wiener process on an underlying complete filtered probability space $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\}$.

Chapter 4 is divided into 3 sections. In Section 4.1, the background why we study stochastic fractional 2D-Stokes equations with delay is stated, also we introduce some basic lemmas with respect to Mittag-Leffler families $\mathbf{E}_\alpha(-t^\alpha \mathcal{A})$ and $\mathbf{E}_{\alpha,\alpha}(-t^\alpha \mathcal{A})$ that will be used throughout this chapter. In Section 4.2, we analyze the well-posedness results with bounded delay in a proper phase space $\mathcal{X}_2 := \{u : [-h, T] \times \Omega \to L^2(\Omega; L^2_\sigma)\}$. Using fixed point theory, we prove the existence and uniqueness of mild solution to problem (9) with bounded delay when external terms f and g are Lipschitz, T is small enough (Theorem 4.8). Next, by extension, it is possible to prove the solution of problem (9) is globally defined in time (Theorem 4.10), meanwhile, the continuity with respect to initial value of mild solution to our model is proved by the same method (Proposition 4.9). In Section 4.3, similarly, we prove the same results as in Section 4.2 to problem (9) with unbounded delay but in different phase space $\mathcal{C}_X(\mathbb{H}) := \{\varphi \in C((-\infty, 0]; \mathbb{H}) : \lim_{\theta \to -\infty} \varphi(\theta) \text{ exists in } \mathbb{H}\}$. In addition, the advantages of the phase space $\mathcal{C}_X$ that we adopt in Section 4.3 are illustrated comparing with another alternative phase space, $C^\gamma(\mathbb{H}) := \{\varphi \in C((-\infty, 0]; \mathbb{H}) : \sup_{\theta \in (-\infty, 0]} : e^{\gamma\theta} \|\varphi(\theta)\|_{\mathbb{H}} < \infty\}$ (Remark 4.11).

- ***Future work***

We conclude this PhD project with showing our future work: time fractional stochastic delay 2D-Navier Stokes equations with multiplicative noise, that is a further work of Chapter 4. P.M. Carvalho-Neto and G. Planas analyzed in [22] the following Navier-Stokes model with Caputo fractional time derivative,

$$\begin{cases} D_t^\alpha u - \kappa \Delta u + u \cdot \nabla u + \nabla p = f & \text{in } \mathbb{R}^N,\ t \geq 0, \\ \nabla \cdot u = 0 & \text{in } \mathbb{R}^N,\ t \geq 0, \\ u(0,x) = u_0 & \text{in } \mathbb{R}^N, \end{cases} \tag{10}$$

well-posedness of problem (10), the existence and eventual uniqueness of mild solutions as well as their regularity in time are analyzed completely. Combing our previous concerns, our model can be more realistic if we introduce stochastic and delay,

$$
\begin{cases}
D_t^\alpha u - \kappa \Delta u + u \cdot \nabla u + \nabla p = f(t, u_t) + g(t, u_t) \frac{dW(t)}{dt} & \text{in } \mathbb{R}^N, \ t \geq 0, \\
\nabla \cdot u = 0 & \text{in } \mathbb{R}^N, \ t \geq 0, \\
u(t, x) = \varphi(t, x) & \text{in } \mathbb{R}^N, \ t \in [-h, 0].
\end{cases} \tag{11}
$$

We are strongly interested in problem (11), it is well known that when we deal with the integer time stochastic Navier-Stokes equations in the phase space $L^2(\Omega; C([0, T]; X))$, with the help of Itô's isometry and Burkholder-Davis-Gundy's inequality, a priori estimation can be handled smoothly. However, for time fractional stochastic Navier-Stokes equations, if the same phase space were adopted, we would face essential troubles: (a) Itô's isometry only holds true for the integer time derivative rather than time fractional derivative; (b) Burkholder-Davis-Gundy's inequality cannot be used since the integral is not a martingale (the main reason is the singular kernel appearing in the stochastic integral). This inspires us to find a new technique to obtain some results to problem (11).

Resumen en español–Spanish Summary

La ecuación del calor clásica $\partial_t u = \Delta u$ describe la propagación del calor en un medio generalmente homogéneo. No obstante, la versión fraccionaria de dicha ecuación, $\partial_t^\alpha u = \Delta u$ with $0 < \alpha < 1$, ha sido usada para modelar difusiones anómalas que exhiben un comportamiento subdifusivo debido, por ejemplo, a fenómenos de captura y uniones de partículas. Mientras que en las difusiones normales (descritas por la ecuación del calor o ecuaciones en derivadas parciales parabólicas más generales) el desplazamiento en media cuadrática de una partícula difusiva es del orden de t cuando $t \to \infty$, en el caso de derivadas fraccionarias con respecto al tiempo dicho desplazamiento es del orden de t^α cuando $t \to \infty$. Por esta razón, las ecuaciones fraccionarias de orden $0 < \alpha < 1$ son conocidas como ecuaciones subdifusivas (o de subdifusión) en la literatura, mientras que en el caso $1 < \alpha < 2$ se llaman superdifusivas (o de superdifusión). Así, durante las últimas décadas, los científicos han desarrollado nuevos modelos que involucran de una manera natural ecuaciones diferenciales fraccionarias, mostrando de una forma exitosa los fenómenos de difusión anómala que aparecen en el mundo real (véanse, e.g., [1, 21, 24, 26, 47, 56, 59, 45] y las referencias mencionadas en estos trabajos).

El cálculo fraccionario posee una larga historia, y sus orígenes se remontan a los finales del siglo XXVII. Los primeros pasos de la teoría datan de la primera parte del siglo XIX, aunque ha sido durante las últimas décadas cuando esta teoría se ha mostrado más activa. Derivadas e integrales de órdenes no entero son muy adecuadas para describir propiedades de diversos materiales del mundo real, por ejemplo, en Mecánica (teoría de viscoelasticidad y viscoplasticidad), Bioquímica (modelado de polímeros y proteínas), Ingeniería eléctrica (transmisión de ondas de ultrasonidos), Medicina (modelado de tejidos humanos bajo el efecto de cargas mecánicas) etc. Para más aplicaciones y referencias sobre el tema se pueden consultar los trabajos [20, 21, 31, 45, 47, 52, 56, 59, 78] y las referencias allí mencionadas.

Aunque existen varias definiciones y conceptos para el cálculo fraccionario, en esta memoria sólamente introduciremos los conceptos de derivadas fraccionarias en el sentido de Riemann-Liouville y de Caputo ya que son las más usadas frecuentemente en la literatura actual. La forma clásica del cálculo fraccionario viene dada por la integral de Riemann-Liouville, que esencialmente está descrita por

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau,$$

donde $\Gamma(\alpha)$ es la función gamma de Euler dada por $\Gamma(\alpha) := \int_0^\infty t^{\alpha-1}e^{-t}dt$. De forma similar a la definición de la integral de Riemann-Liouville, la definición de la derivada fraccionaria en el sentido de Riemann-Liouville está dada por

$$({}^{L}D_t^\alpha f)(t) = \frac{d^n}{dt^n} I_t^{n-\alpha} f(t) = \frac{1}{n-\alpha}\left(\frac{d}{dt}\right)^n \int_0^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau, \quad n \leq \alpha < n+1.$$

Otra opción para computar las derivadas fraccionarias es utilizar la definición introducida en el año 1967 por Michelo Caputo, y que viene descrita como

$$({}^{C}D_t^\alpha f)(t) := \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, \quad n-1 < \alpha < n.$$

En contraste con la derivada fraccionaria en el sentido de Riemann-Liouville, la definición de Caputo tiene varias ventajas: (a) para resolver ecuaciones diferenciales con la definición de Caputo no es necesario definir las condiciones iniciales fraccionarias; (b) $f^{(n)}(t)$ se anula cuando $f(t)$ es constante y su transformada de Laplace se expresa en términos de los valores iniciales de la función y sus derivadas.

Desde un punto de vista matemático, las ecuaciones diferenciales han sido bien y extensamente estudiadas hoy en día. Por mencionar sólo algunos ejemplos, Friedman and Hu [36] han analizado problemas de bifurcación de estabilidad a inestabilidad para un modelo con frontera libre relacionado con los crecimientos de tumores usando la ecuación de Stokes; Caraballo and Real [17] han estudiado las ecuaciones bidimensionales de Navier-Stokes con términos de retardo; Wang, Chen y Xiao [78] han investigado los problemas de Cauchy fraccionarios abstractos para operadores casi sectoriales, etc. Sin embargo, cuando consideramos un problema físico del mundo real, debemos considerar aspectos adicionales para que el modelo sea más realista, en concreto, al menos se deben tener en cuenta algunas influencias de carácter interno, externo o medioambiental y que suelen llamarse "ruidos", lo que hace que el problema sea difícil de describir de forma determinista y pase a ser estocástico o aleatorio.

Por tanto, durante los últimos años ha habido un creciente interés por el estudio de las ecuaciones en derivadas parciales estocásticas en todas las variantes posibles, es decir, con retardos, no autónomas, con impulsos, multivaluadas, etc. (véanse, e.g.,[12, 13, 14, 15, 29, 39, 40, 56, 58, 59, 79, 81]). Para conseguir una mejor descripción de los modelos reales, podemos considerar algún tipo de aleatoriedad en las ecuaciones, que pueden venir descritas por alguna clase de ruido, bien de tipo ruido blanco o coloreado, o de algún otro tipo de términos estocásticos. En esta memoria, vamos a considerar principalmente dos clases de ruidos: *movimientos Brownianos/ procesos de Wiener* y *movimiento Browniano fraccionario.*

• *El movimiento Browniano/proceso de Wiener*

En 1828, el botánico escocés Robert Brown observó que los granos de polen suspendidos en un líquido experimentaban un movimiento irregular. Dicho movimiento fue explicado posteriormente por las colisiones aleatorias con las moléculas del líquido. Para describir matemáticamente este movimiento es natural usar el concepto de proceso estocástico $B_t(\omega)$, interpretado como la posición del grano de polen ω en el instante t. En matemáticas, el movimiento Browniano es descrito por el llamado proceso de Wiener, que es un proceso estocástico continuo, y es uno de los más estudiados y mejor conocidos procesos de tipo Lévy, y que ocurre frecuentemente en los problemas de matemáticas puras y aplicadas, en economía, física, etc.

El movimiento Browniano $B_t(\omega)$ se caracteriza por cuatro propiedades:

(1) $B_0(\omega) = 0$;

(2) $t \mapsto B_t(\omega)$ es continua casi seguramente;

(3) $B_t(\omega)$ posee incrementos independientes, es decir, si $0 \leq s_1 < t_1 \leq s_2 < t_2$, entonces $B_{t_1}(\omega) - B_{s_1}(\omega)$ y $B_{t_2}(\omega) - B_{s_2}(\omega)$ son variables aleatorias independientes;

(4) $B_t(\omega) - B_s(\omega) \sim \mathcal{N}(0, t-s)$ (para $0 \leq s \leq t$),

donde $\mathcal{N}(\mu, \sigma^2)$ denota la distribución normal con media μ y varianza σ^2.

Gracias a estas propiedades del movimiento Browniano, es posible representar B_t como un proceso estocástico generalizado llamado ruido blanco. Supongamos que H es un proceso localmente acotado, adaptado y continuo por la derecha, si $\{\pi_n\}$ es una sucesión de particiones del intervalo $[0, t]$ con diámetro decreciente a cero, entonces la integral de Itô de H con respecto a B es un proceso estocástico dado por el siguiente límite en media cuadrática:

$$\int_0^t H dB = \lim_{n\to\infty} \sum_{[t_{i-}, t_i] \in \pi_n} H_{t_{i-1}} (B_{t_i} - B_{t_{i-1}}) \qquad \text{en} \quad L^2(\Omega).$$

Más aún, los procesos dados por integrales de Itô suelen ser martingalas, lo que proporciona interesantes ventajas computacionales, como por ejemplo la isometría de Itô (Lema 1.11) que será utilizada frecuentemente en nuestro análisis.

- *Movimiento Browniano fraccionario*

A. N. Kolmogorov fue el primero en considerar procesos Gaussianos continuos con incrementos estacionarios y propiedades de auto similitud, lo que significa que para cualquier $a > 0$, existe $b > 0$ tal que

$$Law(X(at); at \geq 0) = Law(bX(t); t \geq 0).$$

Ocurre que tales procesos con media cero poseen una especial función de correlación:

$$E\left(X(t)X(s)\right) = \frac{1}{2}(|s|^{2H} + |t|^{2H} - |t-s|^{2H}),$$

donde $0 < H < 1$ es llamado el índice de Hurst. Es importante observar que el movimiento Browniano fraccionario no es ni una semimartingala (excepto en el caso $H = 1/2$ cuando es un movimiento Browniano) ni un proceso de Markov. El primer hecho previene de la posibilidad de usar una bien establecida teoría de integración, mientras que lo segundo implica que no hay una conexión directa entre el movimiento Browniano fraccionario y los operadores diferenciales. Sin embargo, está muy conectada con el cálculo fraccionario y puede representarse como una “integral fraccionaria” (con la ayuda de un comparativamente complicado núcleo hipergeométrico) vía el proceso de Wiener no sólo en intervalos infinitos sino también finitos. Tal representación, junto con la propiedad Gaussiana del movimiento Browniano fraccionario y la propiedad de continuidad Hölder para sus trayectorias (el movimiento Browniano fraccionario con índice de Hurst H es Hölder continuo hasta el orden H) nos permite utilizar un interesante y específico cálculo estocástico para el movimiento Browniano fraccionario. Pero la técnica posee una dificultad importante: es más difícil de obtener una noción de integración apropiada cuando el índice de Hurst H es más pequeño; cuanto más irregular son las trayectorias del proceso estocástico, más difícil es la intregración respecto de las mismas. Es reseñable mencionar que en [13] los autores establecieron un lema técnico (Lemma 1.13) que es crucial para la integración estocástica con respecto al movimiento Browniano fraccionario con parámetro de Hurst $H \in (1/2, 1)$. Este es el punto de vista adoptado en la presente Tesis.

Merece la pena mencionar que, obviamente, la evolución futura de un sistema no depende sólo de su estado presente sino que la historia del fenómeno tiene también su importancia y determina su comportamiento futuro. Por esta razón, la teoría de ecuaciones diferenciales con retardos se ha convertido en un área importante de las matemáticas aplicadas, como por ejemplo, en los experimentos de campos de alta velocidad de viento en túneles, en materiales con memoria, o con motivaciones biológicas como en el crecimiento de especies, o en el tiempo de incubación en modelos de enfermedades entre muchos otros (véanse, e.g. [12, 13, 16, 17, 18, 25, 57, 58, 62, 80, 81]).

Motivado por las consideraciones expuestas anteriormente, en este trabajo vamos a estudiar modelos como *retículos de ecuaciones diferenciales ordinarias (lattice systems), sistemas estocásticos impulsivos con retardo, y sistemas estocásticos de Stokes bidimensionales* para convencer al lector de que estos modelos son de importancia hoy en día.

La presente memoria está estructurada en cuatro capítulos. En el Capítulo 1 hemos incluido algunos definiciones básicas, algunas propiedades y lemas sobre el cálculo fraccionario y sobre procesos estocásticos que serán utilizados a lo largo del trabajo. El Capítulo 2 está dedicado al estudio de la existencia y unicidad de soluciones, así como al análisis del comportamiento asintótico, de un retículo estocástico con derivada fraccionaria sustancial en el sentido de Caputo. En el tercer capítulo investigamos el buen planteamiento y la dinámica de una clase de ecuaciones diferenciales estocásticas con impulsos y retardos no acotados en el espacio de fases $\mathcal{PC}$ (de funciones continuas a trozos). Al principio, consideramos un modelo más complicado (problema (17)) para el que sólamente podemos demostrar el carácter bien planteado y analizamos el comportamiento asintótico en las secciones 3.2 y 3.3. Sin embargo, para ir un poco más allá, es necesario considerar otro modelo en la Sección 3.4, (el problema (18)) que contiene coeficientes más regulares, y para el que se puede demostrar además la existencia de conjuntos globalmente atrayentes para las soluciones del problema en el caso general de no unicidad de soluciones y en el de unicidad, en el que el conjunto atrayente es unitario. Finalmente, el principal objetivo del Capítulo 4 es demostrar el carácter bien planteado de un modelo estocástico bidimensional de Stokes con derivada fraccionaria y retardo en el espacio de fases $C((-h,0];L^2(\Omega;L^2_\sigma))$, donde $h \leq \infty$.

A continuación vamos a describir con algo más de detalle el contenido de cada capítulo de esta tesis.

- *Preliminares*

El Capítulo 1 está estructurado en tres secciones. En la Sección 1.1 recordamos algunas propiedades y conceptos básicos relacionados con la derivada fraccionaria de Riemann-Liouville. A continuación, definimos la derivada fraccionaria de Caputo via la de Riemann-Liouville. Como se señala en el Lema 1.6, D^α es el inverso por la izquierda de J^α, no por la derecha. Cuando queremos obtener las soluciones generalizadas (mild) de las ecuaciones diferenciales fraccionarias, la transformada de Laplace de la derivada fraccionaria de Caputo proporciona una forma alternativa de conseguirlo.

En la Sección 1.2, presentamos algunos resultados básicos relacionados con los procesos estocásticoss. En efecto, el Lema 1.11 y la desigualdad (1.6) nos garantizan que podamos tratar con la integral estocástica con respecto a procesos Brownianos finito e infinito dimensionales. Más aún, podremos tratar integrales estocásticas con respecto a movimientos Brownianos fraccionarios gracias al Lema 1.13.

Adicionalmente, mostraremos, en la Sección 1.3, algunos ejemplos de ecuaciones diferenciales estocásticas funcionales con derivadas fraccionarias en tiempo y que son de interés para las aplicaciones.

- *Retículos estocásticos con derivada fraccionaria de Caputo*

Los retículos diferenciales han recibido mucha atención recientemente en la literatura matemática ya que aparecen de una forma muy natural en una gran cantidad de situaciones en las que la estructura espacial del problema posee un carácter discreto, así como en problemas de discretización espacial de problemas continuos (por ejemplo en la discretización de problemas para ecuaciones en derivadas parciales). En consecuencia, en el Capítulo 2 estudiaremos una clase de retículos estocásticos con derivada fraccionaria de Caputo.

Sea X un espacio de Hilbert separable con norma $\|\cdot\|$ y producto escalar $(\cdot,\cdot)$. Entonces $L^2(\Omega;X)$ es un espacio de Hilbert formado por las variables aleatorias que toman valores en X con norma $(\mathbb{E}\|\cdot\|^2)^{\frac{1}{2}}$ y producto escalar $\mathbb{E}(\cdot,\cdot)$.

En este capítulo, cuando no haya lugar a confusión, D^α y D^α_s denotarán la derivada fraccionaria de Caputo, y la derivada fraccionaria sustancial de Caputo (véase la Definición 1.4) de órden α con $\sigma > 0$, respectivamente.

Primero analizamos la existencia de soluciones de los sistemas diferenciales estocásticos con derivada fraccionaria,

$$\begin{cases} D^\alpha x(t) = f(t,x(t)) + g(t)\frac{dB(t)}{dt}, & t \geq 0, \dfrac{1}{2} < \alpha < 1, \\ x(0) = x_0 \in L^2(\Omega;X) \end{cases} \tag{12}$$

y

$$\begin{cases} D_s^\alpha x(t) = f(t,x(t)) + g(t)\frac{dB(t)}{dt}, & t \geq 0, \dfrac{1}{2} < \alpha < 1, \\ x(0) = x_0 \in L^2(\Omega; X), \end{cases} \tag{13}$$

en el espacio de Hilbert $L^2(\Omega; X)$.

En lugar de utilizar un teorema de tipo punto fijo para demostrar la existencia y unicidad de soluciones para los problemas planteados en este capítulo, vamos a utilizar el punto de vista adoptado por Lakshmikantham y Vatsala [52], que usaron un resultado de existencia local de Peano para ecuaciones diferenciales ordinarias fraccionarias.

Sea $f : [0,\infty) \times L^2(\Omega; X) \to L^2(\Omega; X)$ secuencialmente débil continua en conjuntos acotados, y $g : [0,\infty) \to X$ medible.

(R_1) Suppongamos que f y g son aplicaciones acotadas, i.e.,

$$\mathbb{E}\|f(t,x)\|^2 \leq M^2, \qquad \|g(t)\|^2 \leq M^2, \qquad \text{para todo } (t,x) \in R_0,$$

donde $R_0 = \{(t,x) : 0 \leq t \leq T \text{ y } \mathbb{E}\|x - x_0\|^2 \leq b^2\}$. Vamos a ser capaces de demostrar que el problema de valores iniciales (12) posee al menos una solución $x(\cdot)$ definida globalmente en tiempo (teoremas 2.5 y 2.6).

(R_2) Además de las condiciones de (R_1), supongamos $b^2 \geq 12\mathbb{E}\|x_0\|^2$, entonces el probema de valores iniciales (13) posee al menos una solución $x(\cdot)$ definida globalmente en tiempo (teoremas 2.7 and 2.8).

Más aún, si imponemos condiciones de tipo Lipschitz en los términos f y g, demostramos la existencia y unicidad de soluciones de los sistemas diferenciales estocásticos con derivadas fraccionarias,

$$\begin{cases} D^\alpha x(t) = f(t,x(t)) + g(t,x(t))\frac{dB(t)}{dt}, & t \geq 0, \dfrac{1}{2} < \alpha < 1, \\ x(0) = x_0 \in L^2(\Omega; X) \end{cases} \tag{14}$$

y

$$\begin{cases} D_s^\alpha x(t) = f(t,x(t)) + g(t,x(t))\frac{dB(t)}{dt}, & t \geq 0, \dfrac{1}{2} < \alpha < 1, \\ x(0) = x_0 \in L^2(\Omega; X), \end{cases} \tag{15}$$

en $L^2(\Omega; X)$. Sean f, $g : [0,\infty) \times L^2(\Omega; X) \to L^2(\Omega; X)$ funciones medibles que satisfacen para todo x, $y \in L^2(\Omega; X)$ y $t \in [0,T]$,

$$\mathbb{E}\|f(t,x) - f(t,y)\|^2 + \mathbb{E}\|g(t,x) - g(t,y)\|^2 \leq L\mathbb{E}\|x - y\|^2$$

para alguna constante L. Además, supongamos que f y g son funciones acotadas, i.e.,

$$\mathbb{E}\|f(t,x)\|^2 \leq M^2, \qquad \mathbb{E}\|g(t,x)\|^2 \leq M^2, \qquad \text{para todo } \ (t,x) \in R_0,$$

donde $R_0 = \{(t,x) : 0 \leq t \leq T \text{ y } \ \mathbb{E}\|x - x_0\|^2 \leq b^2\}$. Obtenemos entonces que

(R_3) para cada $x_0 \in L^2(\Omega; X)$, existe una única solución de los problemas (14), (15), respectivamente (Teorema 2.10).

Los detalles sobre cómo se pueden demostrar estos resultados se pueden encontrar en los teoremas 2.5-2.10 en la Sección 2.2.

La Sección 2.3 está dedicada a investigar el siguiente retículo estocástico con derivada fraccionaria sustancial de Caputo:

$$\begin{cases} D_s^\alpha x_i(t) + (-1)^p \Delta^p x_i(t) + \lambda x_i(t) = f_i(x_i(t)) + g_i(t)\frac{dB(t)}{dt}, & t \geq 0, \\ x_i(0) = x_0, & i \in \mathbb{Z}, \end{cases} \tag{16}$$

donde $1/2 < \alpha < 1$, $B(t)$ es un movimiento Browniano estándar sobre un espacio de probabilidad completo y filtrado $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$, y $\lambda \in \mathbb{R}$, p es un número entero positivo, $\Delta^p = \Delta \circ \cdots \circ \Delta$, p veces. Δ denota el operador de Laplace unidimensional discreto, que viene definido como $\Delta x_i = x_{i+1} + x_{i-1} - 2x_i$. Escribiremos también $\partial^+ x_i = x_{i+1} - x_i$, $\partial^- x_i = x_i - x_{i-1}$ y definimos

$$\mathbb{D}^p := \begin{cases} \Delta^{\frac{p}{2}}, & p \text{ even}, \\ \partial^+ \Delta^{\frac{p-1}{2}}, & p \text{ odd}. \end{cases}$$

El espacio de fases natural para tal sistema infinito de ecuaciones diferenciales, ya sea fraccionario o no, es el espacio de Hilbert

$$\ell^2 = \left\{ x = (x_i)_{i\in\mathbb{Z}}, x_i \in \mathbb{R} : \sum_{i\in\mathbb{Z}} x_i^2 < +\infty \right\},$$

con el producto escalar y norma correspondiente

$$(x, y) = \sum_{i\in\mathbb{Z}} x_i y_i, \qquad \|x\|^2 = \sum_{i\in\mathbb{Z}} x_i^2, \qquad \forall x = (x_i)_{i\in\mathbb{Z}},\ y = (y_i)_{i\in\mathbb{Z}} \in \ell^2.$$

Sea $L^2(\Omega; \ell^2)$ el espacio de Hilbert de las variables aleatorias con valores en ℓ^2 que son fuertemente medibles y de cuadrado integrable, con el producto escalar y norma

$$\mathbb{E}(x, y) = \int_\Omega \sum_{i\in\mathbb{Z}} x_i y_i d\mathbb{P}, \qquad \|x\|_{L^2(\Omega;\ell^2)} = (\mathbb{E}\|x\|^2)^{\frac{1}{2}}, \qquad \forall x, y \in L^2(\Omega; \ell^2).$$

En la Sección 2.3.1 suponemos que

(H_1) Los operadores $f : L^2(\Omega; \ell^2) \to L^2(\Omega; \ell^2)$ y $g : [0, +\infty) \to \ell^2$, dados componente a componente por $(f(x))_i = f_i(x_i)$ y $(g(t))_i = g_i(t)$, $i \in \mathbb{Z}$, están bien definidos y son acotados, siendo además g medible.

(H_2) Las componentes $f_i : L^2(\Omega; \mathbb{R}) \to L^2(\Omega; \mathbb{R})$ son secuencialmente débil continuas en conjuntos acotados.

(H_3) Las componentes $f_i : L^2(\Omega; \mathbb{R}) \to L^2(\Omega; \mathbb{R})$ satisfacen $\mathbb{E}|f_i(x)|^2 \leq |k_{i,1}|^2 + k_2^2 \mathbb{E}|x|^2$ para todo $x \in L^2(\Omega; \mathbb{R})$, donde $k_1 = (k_{1,i})_{i\in\mathbb{Z}} \in \ell^2$ y $k_2 > 0$.

(H_4) Existe una constante positiva M' tal que para todo $t \geq 0$,

$$\int_0^t (t-\tau)^{2\alpha-2} e^{-2\sigma(t-\tau)} \|g(\tau)\|^2 d\tau \leq M'.$$

(H_5) $\mathbb{E}|f_i(x) - f_i(y)|^2 \leq L'\mathbb{E}|x-y|^2$ para cualesquiera x, $y \in L^2(\Omega; \mathbb{R})$, donde $L' > 0$.

En estas condiciones obtenemos los siguientes resultados:

(R_4) El problema de valores iniciales (16) posee al menos una solución si se verifican (H_1)-(H_2) (Teorema 2.11).

(R_5) Sea σ (el coeficiente de la derivada sustancial fraccionaria de Caputo) suficientemente grande y supongamos que se verifican (H_2)-(H_4), entonces las soluciones del problema de valores iniciales (16) están globalmente definidas en tiempo, y además se deduce una estimación pare el conjunto absorbente del problema (16) (véase el Corolario 2.13).

(R_6) Sea σ suficientemente grande y supongamos que se verifican las condiciones (H_1), (H_3)-(H_5), entonces existe una única solución global del problema (16) (véase el Teorema 2.14).

Una vez que obtenemos estimaciones adecuadas sobre la norma de las soluciones (Lema 2.12), es posible inmediatamente estudiar la existencia de conjuntos absorbentes para el problema (16) (Teorema 2.15). Más aún, usando la topología fuerte en media cuadrática, demostramos la estabilidad asintótica uniforme de las soluciones del problema (16) (Teorema 2.16) para finalizar el Capítulo 2.

- *Ecuaciones diferenciales estocásticas impulsivas y fraccionarias*

La teoría de ecuaciones diferenciales impulsivas se ha convertido en un área de investigación muy activa debido a las amplias aplicaciones que posee en comunicaciones, mecánica, ingeniería eléctrica, medicina, biología, etc.

Así, el objetivo del Capítulo 3 es investigar el carácter bien planteado y la dinámica de los siguientes problemas:

$$\begin{cases} D_t^\alpha x(t) = Ax(t) + f(t, x_t) + g(t, x_t)\frac{dB(t)}{dt} + h(t)\frac{dB_Q^H(t)}{dt}, \quad t \geq 0, \\ \qquad t \neq t_k, \quad \frac{1}{2} < \alpha < 1, \\ \Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \quad k = 1, 2, \cdots, \\ x(t) = \phi(t), \quad t \in (-\infty, 0], \end{cases} \tag{17}$$

y

$$\begin{cases} D_t^\alpha x(t) = Ax(t) + I_t^{1-\alpha} f(t, x_t) + [I_t^{1-\alpha} g(t, x_t)]\frac{dB(t)}{dt} + [I_t^{1-\alpha} h(t)]\frac{dB_Q^H(t)}{dt}, \\ \qquad t \geq 0, \quad t \neq t_k, \quad 0 < \alpha < 1, \\ \Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \quad t = t_k, \quad k = 1, 2, \cdots, \\ x(t) = \phi(t), \quad t \in (-\infty, 0], \end{cases} \tag{18}$$

donde D_t^α denota la derivada fraccionaria de Caputo de orden $0 < \alpha < 1$, e $I_t^{1-\alpha}$ es el operador integral fraccionario de orden $(1-\alpha)$. Para ambos modelos (17) y (18), $x(\cdot)$ toma valores en el espacio de Hilbert separable $\mathbb{H}$. $A : D(A) \subseteq \mathbb{H} \to \mathbb{H}$ es el generador infinitesimal de un operador $T_\alpha(t)(t \geq 0)$ analítico y compacto de order fraccionario α. $B(t)$ y $B_Q^H(t)$ denotan, respectivamente, un movimiento Browniano con valores en $\mathbb{K}$ y un movimiento Browniano fraccionario de tipo Q-cilíndrico respectivamente. Los instantes de tiempo t_k, donde los impulsos tienen lugar, satisfacen $0 = t_0 < t_1 < \cdots < t_k \to +\infty$ cuando $k \to \infty$.

Para alcanzar nuestro objetivo, en la Sección 3.1, presentamos en primer lugar el espacio de fases abstracto $\mathcal{PC}$ en el que estableceremos los resultados de una manera apropiada. Sea $L^2(\Omega; \mathbb{H})$ el espacio de Banach formado por las variables aleatorias fuertemente medibles, de cuadrado integrable con valores en H, equipado con la norma $\|u(\cdot)\|_{L^2}^2 = \mathbb{E}\|u(\cdot)\|^2$. El espacio de fases abstracto $\mathcal{PC}$ está definido como

$$\mathcal{PC} = \Big\{ \xi : (-\infty, 0] \to L^2(\Omega; \mathbb{H}) \text{ is } \mathcal{F}_0\text{-adaptado y continuo excepto en}$$
$$\text{a lo más un conjunto numerable de puntos } \{\theta_k\}, \text{ en los que existe } \xi(\theta_k^+)$$
$$\text{y } \xi(\theta_k^-) \text{ con } \xi(\theta_k) = \xi(\theta_k^-), \text{ y } \sup_{\theta \in (-\infty, 0]} e^{\gamma\theta} \mathbb{E}\|\xi(\theta)\|^2 < \infty \Big\},$$

para algún parámetro fijo $\gamma > 0$. La norma de este espacio de Banach $\mathcal{PC}$ está definida como

$$\|\xi\|_{\mathcal{PC}} = \left(\sup_{\theta \in (-\infty, 0]} e^{\gamma\theta} \mathbb{E}\|\xi(\theta)\|^2 \right)^{\frac{1}{2}}, \qquad \xi \in \mathcal{PC}.$$

Con la ayuda de la transformada de Laplace de la derivada fraccionaria de Caputo, se puede establecer la definición de la solución generalizada (mild) de los problemas (17) y (18) en las correspondientes definiciones 3.8 y 3.9.

En segundo lugar, en la Sección 3.2 analizamos la existencia de solución generalizada del problema (17). Para ello, usamos el método de Picard y realizamos estimaciones en cada intervalo impulsivo, teniendo en cuenta las hipótesis de compacidad de los operadores $T_\alpha(t)$ $(t > 0)$ y $S_\alpha(t)$ $(t > 0)$ (Teorema 3.11). Análogamente demostramos también la existencia de solución generalizada del problema (18) (Teorema 3.12). Posteriormente, demostramos resultados de dependencia continua de las soluciones respecto de los datos iniciales de los problemas (17) y (18) (teoremas 3.14 y 3.15). Cuando trabajamos con ecuaciones diferenciales fraccionarias impulsivas, necesitamos realizar las estimaciones en cada intervalo impulsivo y luego proceder por inducción para conseguir el resultado final.

En tercer lugar, estamos interesados en analizar el comportamiento asintótico de nuestros modelos. Demostrar que las soluciones generalizadas existen globalmente en tiempo es lo primero que tenemos que asegurar. A continuación nos centramos en estudiar la dinámica del problema (17) en la Sección 3.3. Para ello necesitamos imponer algunas hipótesis adicionales a los operadores $T_\alpha(t)$ y $S_\alpha(t)$, en concreto que están controlados por funciones con decrecimiento exponencial:

$$\|T_\alpha(t)\| \leq Me^{-\mu t}, \quad \|S_\alpha(t)\| \leq Me^{-\mu t}(1 + t^{\alpha-1}), \qquad \forall t > 0, \; M \geq 1. \tag{19}$$

Estas condiciones, junto con el carácter Lipschitz de los términos no lineales f y g, nos permiten demostrar que la solución generalizada del problema (17) es única y está definida globalmente en tiempo (Teorema 3.3). Finalizamos esta sección demostrando el decaimiento asintótico exponencial de las soluciones de nuestro modelo. Es interesante remarcar que, si bien la idea para analizar el comportamiento asintótico es estándar, el hecho de considerar infinitos impulsos impone un gran desafío para nosotros. Para solventar esta dificultad, gracias a la definición del espacio de fases $\mathcal{PC}$, su norma y propiedades del cálculo exponencial (escogiendo $\gamma > 2\mu$), podemos realizar estimaciones en cada intervalo impulsivo y conseguimos demostar el resultado deseado (Teorema 3.17).

Para concluir este capítulo, la Sección 3.4 está dedicada al análisis del comportamiento asintótico del problema (18). Estudiamos el carácer bien planteado y el decaimiento exponencial de las soluciones del problema (17) en la Sección 3.3, pero, la falta de compacidad del operador $S_\alpha(t)$ no nos permitió establecer la existencia de conjuntos atrayentes, que es un concepto clave para la comprensión de las propiedades dinámicas del modelo. Heurísticamente, el operador fraccionario $T_\alpha(t)$ es compacto (como así fue demostrado en [78]). De esta forma, en la Sección 3.4 analizamos el carácter bien planteado y la dinámica del problema (18). Primero, gracias a la hipótesis (19) sobre el operador solución fraccionario $T_\alpha(t)$, demostramos la existencia y unicidad global de la solución generalizada (Teorema 3.19). Luego, a la vista de la relación existente entre la norma de $\mathcal{PC}$ y el parámetro de decaimiento exponencial de $T_\alpha(t)$ (con $\gamma > 2\mu$), somos capaces de demostrar que las soluciones generalizadas del problema (18) son acotadas uniformemente con respecto a conjuntos acotados de condiciones iniciales (Teorema 3.20). Enfatizamos que estas estimaciones a priori obtenidas aquí serán de una importancia crucial en el trabajo realizado en las secciones 3.4.1 y 3.4.2.

En la Sección 3.4.1 presentamos un resultado que garantiza la existencia de un conjunto compacto minimal que es globalmente atrayente en el espacio $\mathcal{PC}$. Gracias a la compacidad del operador $T_\alpha(t)$, junto con la definición de la norma de $\mathcal{PC}$, y el teorema de Arzelà-Ascoli, conseguimos demostrar el resultado deseado (Lema 3.22). Más aún, realizando un análisis estándar, demostramos las propiedades de los conjuntos omega límites y la compacidad del conjunto atrayente minimal (teoremas 3.23 y 3.24). Más allá de estos resultados generales, si queremos obtener más detalles sobre la estructura geométrica de este conjunto, necesitamos imponer condiciones más restrictivas. Por eso, en la Sección 3.4.2, la propiedad de Lipschitz asegura la unicidad de soluciones del problema (18), y además, unas estimaciones a priori sobre las soluciones y su decaimiento exponencial, permiten demostrar que el conjunto atrayente está formado por un único punto (teoremas 3.25 y 3.27).

- *Ecuaciones bidimensionales de Stokes estocásticas y fraccionarias*

El carácter bien planteado de problemas de flujos en un fluido viscoso es crucial para muchas áreas de la ciencia e ingeniería. Por ejemplo, para las industrias aeroespaciales y la nanotecnología. En este último caso de estructuras microfluidas, a menudo encontramos problemas de flujos a velocidades moderadas. Desde un punto de vista matemático, las ecuaciones de Stokes proporcionan una primera aproximación de las más generales ecuaciones de Navier-Stokes en situaciones donde el flujo es casi estacionario y lento, y posee pequeños gradientes de velocidad, de manera que los efectos inerciales pueden ser ignorados.

Por estas razones, en el Capítulo 4 analizamos la siguiente ecuación incompresible estocástica fraccionaria con retardo, i.e., las ecuaciones bidimensionales de Stokes no estacionarias:

$$\begin{cases} D_t^\alpha u - \kappa\Delta u + \nabla p = f(t,u_t) + g(t,u_t)\frac{dW(t)}{dt} & \text{en } \mathbb{R}^2,\ t \geq 0, \\ \nabla \cdot u = 0 & \text{en } \mathbb{R}^2,\ t \geq 0, \\ u(t,x) = \varphi(t,x) & \text{en } \mathbb{R}^2,\ t \in [-h,0], \end{cases} \tag{20}$$

donde f y g son fuerzas externas conteniendo algunas propiedades hereditarias y retardos, y φ es el dato inicial en el intervalo de tiempo $t \in [-h,0]$, donde h es un número positivo, y $W(t)$ es un proceso de Wiener estándar sobre un espacio de probabilidad completo y filtrado $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\}$.

El Capítulo 4 está estructurado en tres secciones. En la Sección 4.1 recordamos el marco en el que vamos a estudiar nuestras ecuaciones estocásticas fraccionarias de Stokes bidimensionales con retardos, y también introducimos algunos lemas relacionados con las familias de operadores de Mittag-Leffler denotados como $\mathbf{E}_\alpha(-t^\alpha\mathcal{A})$ y $\mathbf{E}_{\alpha,\alpha}(-t^\alpha\mathcal{A})$, y que serán usados a lo largo de todo el capítulo. En la Sección 4.2 analizamos el carácter bien planteado en el caso de retardos acotados en un adecuado espacio de fases $\mathcal{X}_2 := \{u : [-h,T] \times \Omega \to L^2(\Omega; L^2_\sigma)\}$. Haciendo uso de la teoría del punto fijo demostramos la existencia y unicidad de solución generalizada del problema (20) con retardo acotado cuando los términos de fuerza externa f y g son Lipschitz y T es suficientemente pequeño (Teorema 4.8). Posteriormente, demostramos que la solución se puede extender de manera que sea globalmente definida en tiempo (Teorema 4.10), y del mismo modo demostramos también la continuidad con respecto a los datos iniciales (Proposición 4.9). En la Sección 4.3, demostramos los mismos resultados que en la Sección 4.2 para el problema (20) con retardo no acotado pero en un espacio de fases diferente, en concreto en $\mathcal{C}_X(\mathbb{H}) := \{\varphi \in C((-\infty,0];\mathbb{H}) : \lim_{\theta\to-\infty}\varphi(\theta) \text{ existe en } \mathbb{H}\}$. Además, las ventajas de este espacio de fases $\mathcal{C}_X$ son ilustradas comparando con otro posible espacio como es $C^\gamma(\mathbb{H}) := \{\varphi \in C((-\infty,0];\mathbb{H}) : \sup_{\theta\in(-\infty,0]} : e^{\gamma\theta}\|\varphi(\theta)\|_{\mathbb{H}} < \infty\}$ (Nota 4.11).

- *Cuestiones para trabajar en el futuro*

Concluimos esta memoria mostrando algunos puntos en los que pretendemos trabajar en el futuro inmediato: Las ecuaciones de Navier-Stokes bidimensionales estocásticas fraccionarias con retardo y ruido multiplicativo que son la continuación natural del contenido del Capítulo 4. P.M. Carvalho-Neto y G. Planas analizaron en [22] el siguiente modelo de Navier-Stokes con derivadas fraccionarias de tipo Caputo,

$$\begin{cases} D_t^\alpha u - \kappa\Delta u + u\cdot\nabla u + \nabla p = f & \text{en } \mathbb{R}^N,\ t \geq 0, \\ \nabla \cdot u = 0 & \text{en } \mathbb{R}^N,\ t \geq 0, \\ u(0,x) = u_0 & \text{en } \mathbb{R}^N. \end{cases} \tag{21}$$

Establecieron el buen planteamiento del problema (21), la existencia y eventual unicidad de solución generalizada, así como la regularidad en tiempo de las soluciones. A la luz de las cuestiones anteriores, nuestro modelo podría ser más realista si introducimos retardos y ruidos en dicho modelo:

$$\begin{cases} D_t^\alpha u - \kappa\Delta u + u\cdot\nabla u + \nabla p = f(t,u_t) + g(t,u_t)\frac{dW(t)}{dt} & \text{en } \mathbb{R}^N,\ t\geq 0, \\ \nabla\cdot u = 0 & \text{en } \mathbb{R}^N,\ t\geq 0, \\ u(t,x) = \varphi(t,x) & \text{en } \mathbb{R}^N,\ t\in[-h,0]. \end{cases} \tag{22}$$

Estamos muy interesados en el problema (22). Es bien conocido que cuando trabajamos con las ecuaciones de Navier-Stokes estocásticas con derivadas enteras en el espacio de fases $L^2(\Omega; C([0,T];X))$, gracias a la isometría de Itô y la desigualdad de Burkhölder-Davis-Gundy, podemos obtener estimaciones a priori de una forma adecuada. Sin embargo, en el caso de derivadas fraccionarias, si considerásemos el mismo espacio de fases, nos enfrentaríamos con problemas esenciales: (a) la isometría de Itô no ha sido demostrada en el caso fraccionario; (b) la desigualdad de Burkhölder-Davis-Gundy no puede ser usada ya que la integral estocástica no es una martingala (siendo la principal razón que el núcleo que aparece en la integral estocástica tiene una singularidad). Esto nos inspira y motiva para tener que buscar y diseñar una nueva técnica que nos permita abordar el problem (22) y obtener resultados interesantes sobre el mismo.

Chapter 1

Abstract results on the theory of time fractional stochastic ordinary/partial differential equations

This chapter contains the definitions and some properties of fractional integrals/ derivatives and stochastic processes.

The fractional calculus is a name for the theory of integrals and derivatives of arbitrary order, which unifies and generalizes the notions of integer-order differentiation and n-fold integration. There are several different kinds of definitions to fractional derivatives and integrals, such as, Grünwald-Letnikov fractional derivative, Riemann-Liouville fractional derivative and Caputo fractional derivative. However, in the models of this thesis we consider Caputo time fractional derivative, whose advantage is, comparing with Riemann-Liouville derivative [45] (Lemma1.3), Caputo derivatives remove singularities at the origin and share many similarities with the classical derivative so that they are suitable for initial value problems (Lemma 1.5).

1.1 Basic concepts/properties to time fractional calculus

1.1.1 Riemann-Liouville fractional derivative

Let $I = [a, b]$ $(-\infty < a < b < \infty)$ be a finite interval on the real axis $\mathbb{R}$. The *Riemann-Liouville fractional integral* $I^\alpha_{a+}f$ and $I^\alpha_{b-}f$ of order $\alpha \in \mathbb{R}^+$ are defined by

$$(I^\alpha_{a+}f)(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \qquad (x > a) \tag{1.1}$$

and

$$(I^\alpha_{b-}f)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)}{(t-x)^{1-\alpha}} dt, \qquad (x < b) \tag{1.2}$$

respectively. Here $\Gamma(\alpha)$ is the Gamma function. The integrals are called the *left-sided* and the *right-sided* fractional integrals.

Accordingly, the *Riemann-Liouville fractional derivatives* $D^\alpha_{a+}y$ and $D^\alpha_{b-}y$ of order $\alpha \in \mathbb{R}^+$ are defined by

$$\begin{aligned}(D^\alpha_{a+}y)(x) :&= \left(\frac{d}{dx}\right)^n (I^{n-\alpha}_{a+}y)(x)\\ &= \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \int_a^x \frac{y(t)}{(x-t)^{\alpha-n+1}}dt, \qquad (n=[\alpha]+1, x>a)\end{aligned} \tag{1.3}$$

and

$$\begin{aligned}(D^\alpha_{b-}y)(x) :&= \left(-\frac{d}{dx}\right)^n (I^{n-\alpha}_{b-}y)(x)\\ &= \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dx}\right)^n \int_x^b \frac{y(t)}{(t-x)^{\alpha-n+1}}dt, \qquad (n=[\alpha]+1, x<b)\end{aligned} \tag{1.4}$$

separately, where $[\alpha]$ means the integral part of α.

If $0<\alpha<1$, then

$$(D^\alpha_{a+}y)(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x \frac{y(t)}{(x-t)^\alpha}dt, \qquad (x>a),$$

$$(D^\alpha_{b-}y)(x) = -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_x^b \frac{y(t)}{(t-x)^\alpha}dt, \qquad (x<b).$$

The semigroup property of the fractional integration operators I^α_{a+} and I^α_{b-} are given by the following result.

Lemma 1.1. *If $\alpha>0$ and $\beta>0$, then the equations*

$$(I^\alpha_{a+}I^\beta_{a+}f)(x) = (I^{\alpha+\beta}_{a+}f)(x) \quad \text{and} \quad (I^\alpha_{b-}I^\beta_{b-}f)(x) = (I^{\alpha+\beta}_{b-}f)(x)$$

are satisfied at almost every point $x \in [a,b]$ for $f \in L^p[a,b]$ $(1 \le p \le \infty)$.

Lemma 1.2. *If $\alpha>0$ and $f \in L^p(a,b)$ $(1 \le p \le \infty)$, then the following equalities*

$$(D^\alpha_{a+}I^\alpha_{a+}f)(x) = f(x) \quad \text{and} \quad (D^\alpha_{b-}D^\alpha_{b-}f)(x) = f(x) \tag{1.5}$$

hold almost everywhere on $[a,b]$.

Lemma 1.3. *Let $\alpha>0$, $n=[\alpha]+1$ and let $f_{n-\alpha} = (I^{n-\alpha}_{a+}f)$ be the fractional integral (1.1) of order $n-\alpha$. If $f \in L^1(a,b)$ and $f_{n-\alpha} \in AC^n[a,b]$, then the equality*

$$(I^\alpha_{a+}D^\alpha_{a+}f)(x) = f(x) - \sum_{j=1}^n \frac{f^{n-j}_{n-\alpha}(a)}{\Gamma(n-j+1)}(x-a)^{\alpha-j},$$

holds almost everywhere on $[a, b]$. *We denote by* $AC^n[a, b]$ *the space of real-valued functions* f *which have continuous derivatives up to order* $n-1$ *on* $[a, b]$ *such that* $f^{(n-1)} \in AC[a, b]$*:*

$$AC^n[a, b] = \left\{ f : [a, b] \to \mathbb{R} \;\; and \;\; [D^{n-1} f] \in AC[a, b], \;\; D = \frac{d}{dx} \right\}.$$

In particular, $AC^1[a, b] = AC[a, b]$.

1.1.2 Caputo fractional derivative

Next we present the definitions and some properties of the *Caputo fractional derivatives.* Let $D^\alpha_{a+}[y(t)](x) \equiv (D^\alpha_{a+} y)(x)$ and $D^\alpha_{b-}[y(t)](x) \equiv (D^\alpha_{b-} y)(x)$ be the Riemann-Liouville fractional derivatives of order $\alpha \in \mathbb{R}^+$ defined by (1.3) and (1.4), respectively. The fractional derivatives $({}^C D^\alpha_{a+} \, y)(x)$ and $({}^C D^\alpha_{b-} \, y)(x)$ of order $\alpha \in \mathbb{R}^+$ on $[a, b]$ are defined via the above Riemann-Liouville fractional derivatives by

$$({}^C D^\alpha_{a+} \, y)(x) := \left(D^\alpha_{a+} \left[y(t) - \sum_{k=0}^{n-1} \frac{y^{(k)}(a)}{k!} (t-a)^k \right] \right) (x)$$

and

$$({}^C D^\alpha_{b-} \, y)(x) := \left(D^\alpha_{b-} \left[y(t) - \sum_{k=0}^{n-1} \frac{y^{(k)}(a)}{k!} (b-t)^k \right] \right) (x)$$

separately, where $n = [\alpha] + 1$. These derivatives are called *left-sided* and *right-sided Caputo fractional derivatives* of order α.

Definition 1.4. *Let* $\alpha > 0$ *and let* $n = [\alpha] + 1$*. If* $y \in AC^n[a, b]$*, then the Caputo fractional derivatives* $({}^C D^\alpha_{a+} \, y)(x)$ *and* $({}^C D^\alpha_{b-} \, y)(x)$ *exist almost everywhere on* $[a, b]$*, they are represented by*

$$({}^C D^\alpha_{a+} \, y)(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{y^{(n)}(t)}{(x-t)^{\alpha-n+1}} dt =: (I^{n-\alpha}_{a+} y)(x)$$

and

$$({}^C D^\alpha_{b-} \, y)(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b \frac{y^{(n)}(t)}{(t-x)^{\alpha-n+1}} dt =: (-1)^n (I^{n-\alpha}_{b-} y)(x),$$

respectively, where $D = d/dx$.

Lemma 1.5. *Let* $\alpha > 0$ *and* $n = [\alpha] + 1$*. If* $y \in AC^n[a, b]$ *or* $y \in C^n[a, b]$*, then*

$$(I^\alpha_{a+} \, {}^C D^\alpha_{a+} \, y)(x) = y(x) - \sum_{k=0}^{n-1} \frac{y^{(k)}(a)}{k!} (x-a)^k$$

and

$$(I^\alpha_{b-} \, {}^C D^\alpha_{b-} \, y)(x) = y(x) - \sum_{k=0}^{n-1} \frac{(-1)^k y^{(k)}(b)}{k!} (b-x)^k.$$

Lemma 1.2 (equality (1.5)) shows Riemann-Liouville/Caputo fractional derivative D^α_{a+} (D^α_{b-}) is a left inverse of J^α_{a+} (J^α_{b-}), but in general, not a right inverse (see Lemmas 1.3 and 1.5). Hence when we want to obtain the mild solutions to time fractional differential equations, we need other methods, that is the following assertion, which yields the Laplace transform of the Caputo fractional derivative.

Lemma 1.6. *Let $\alpha > 0$, $n-1 < \alpha \le n$ ($n \in \mathbb{N}$) such that $y \in C^n(\mathbb{R}^+)$, $y^{(n)} \in L^1(0,b)$ for any $b > 0$, the estimates*

$$|y^{(n)}(x)| \le Ce^{q_0 x} \quad (x > b > 0) \quad \text{for constants} \quad B > 0 \quad \text{and} \quad q_0 > 0$$

hold for any $y^{(n)}$, the Laplace transforms $(\mathcal{L}y)(t)$ and $\mathcal{L}(D^n y(t))$ exist, and $\lim_{x\to+\infty}$ $(D^k y)(x) = 0$ for $k = 0, 1, \cdots, n-1$. Then the following relation holds:

$$(\mathcal{L}\, {}^C D^\alpha_{a+}\, y)(s) = s^\alpha (\mathcal{L}y)(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1}(D^k y)(0).$$

In particular, if $0 < \alpha \le 1$, then

$$(\mathcal{L}\, {}^C D^\alpha_{a+}\, y)(s) = s^\alpha (\mathcal{L}y)(s) - s^{\alpha-1} y(0).$$

At the end of this subsection, we present some properties of two special functions. Denote by $E_{\alpha,\beta}$ the generalized *Mittag-Leffler* special function defined by

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} = \frac{1}{2\pi i} \int_\gamma \frac{\lambda^{\alpha-\beta} e^\lambda}{\lambda^\alpha - z} d\lambda, \quad \alpha, \beta > 0, z \in \mathbb{C},$$

where γ is a contour which starts and ends at $-\infty$ and encircles the disc $|\lambda| \le |z|^{\frac{1}{\alpha}}$ counter-clockwise. If $0 < \alpha < 1$, $\beta > 0$, then the asymptotic expansion of $E_{\alpha,\beta}$ as $z \to \infty$ is given by

$$E_{\alpha,\beta}(z) = \begin{cases} \frac{1}{\alpha} z^{1-\beta/\alpha} \exp(z^{1/\alpha}) + \varepsilon_{\alpha,\beta}(z), & |\arg z| \le \frac{1}{2}\alpha\pi, \\ \varepsilon_{\alpha,\beta}(z), & |\arg(-z)| < (1 - \frac{1}{2}\alpha)\pi, \end{cases}$$

where

$$\varepsilon_{\alpha,\beta}(z) = -\sum_{n=1}^{N-1} \frac{z^{-n}}{\Gamma(\beta - \alpha n)} + O(|z|^{-N}), \quad \text{as} \quad z \to \infty.$$

For short, set

$$E_\alpha(z) := E_{\alpha,1}(z), \qquad e_\alpha(z) := E_{\alpha,\alpha}(z).$$

Remark 1.7. *Throughout this thesis, we only adopt left-sided Caputo fractional derivative on the interval $[0,t]$. Therefore, as a matter of convenience, we denote $D^\alpha_t y$ by ${}^C D^\alpha_{a+}\, y$.*

1.2 Basic concepts and properties to stochastic process

In this section, we present some basic results for the theory of stochastic processes. To start off, we first show the precise definitions of *random variable* and *stochastic process.*

Definition 1.8. *If Ω is a given set, then a σ-algebra $\mathcal{F}$ on Ω is a family $\mathcal{F}$ of subsets of Ω with the following properties:*

(i) $\emptyset \in \mathcal{F}$;

(ii) $F \in \mathcal{F} \Rightarrow F^C \in \mathcal{F}$, *where* $F^C = \Omega \setminus F$ *is the complement of* F *in* Ω;

(iii) $A_1, A_2, \cdots \in \mathcal{F} \Rightarrow A := \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

Definition 1.9. *Let T be an ordered set, $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space, and $(E, \mathcal{G})$ a measurable space. A stochastic process is a collection of random variables $X = \{X_t; t \in T\}$ such that for each fixed $t \in T$, X_t is a random variable from $(\Omega, \mathcal{F}, \mathbb{P})$ to $(E, \mathcal{G})$. The set Ω is known as the sample space, where E is the state space of the stochastic process X_t.*

The set T can be either discrete, for example the set of positive integers $\mathbb{Z}^+$, or continuous, $T = \mathbb{R}^+$. The state space E will usually be $\mathbb{R}^d$ equipped with the σ-algebra of Borel sets.

1.2.1 Finite dimensional Brownian motion/Wiener process

To describe the irregular motion mathematically it is natural to use the concept of a stochastic process $B_t(\omega)$ (Brownian motion/Wiener process), interpreted as the position at time t of the sample ω.

Definition 1.10. *A one-dimensional standard* Brownian motion $B(t) : \mathbb{R}^+ \to \mathbb{R}$ *is a real-valued stochastic process with almost surely (a.s.) continuous paths such that $B(0) = 0$, it has independent increments, and for every $t > s \geq 0$, the increment density of the random variable $B(t)$-$B(s)$ has a Gaussian distribution with mean* 0 *and variance $t - s$, i.e.,the density of the random variable $B(t)$-$B(s)$ is*

$$g(x; t, s) = (2\pi(t-s))^{-\frac{1}{2}} \exp\left(-\frac{x^2}{2(t-s)}\right).$$

A standard d-dimensional Brownian motion $B(t) : \mathbb{R}^+ \to \mathbb{R}^d$ is a vector of d independent one-dimensional Brownian motions:

$$B(t) = (B_1(t), \cdots, B_d(t)),$$

where $B_i(t)$, $i = 1, \cdots, d$ are independent one-dimensional Brownian motions. The density of the Gaussian random vector $B(t)$-$B(s)$ is thus

$$g(\mathbf{x}; t, s) = (2\pi(t-s))^{-d/2} \exp\left(-\frac{|\mathbf{x}|^2}{2(t-s)}\right),$$

where $|\cdot|$ is the usual norm of $\mathbb{R}^d$.

Although it is reasonable next to present the construction of stochastic integral, we will only recall Itô's isometry which will be used throughout the paper frequently. For more details about the construction of stochastic integral, see [66] and references therein.

Lemma 1.11. (**The Itô isometry**) *Let $\mathcal{V} = \mathcal{V}(S,T)$ be the class of functions*

$$f(\cdot,\cdot) : [0,\infty) \times \Omega \to \mathbb{R}$$

such that

(i) $(t,\omega) \to f(t,\omega)$ *is* $\mathbf{B} \times \mathcal{F}$*-measurable, where* $\mathbf{B}$ *denotes the Borel* σ*-algebra on* $[0,\infty)$,

(ii) $f(t,\cdot)$ *is* $\mathcal{F}_t$*-adapted,*

(iii) $\mathbb{E}\left[\int_S^T f(t,\omega)^2 dt\right] < \infty$.

Then

$$\mathbb{E}\left[\left(\int_S^T f(t,\omega) dB_t\right)^2\right] = \mathbb{E}\left[\int_S^T f^2(t,\omega) dt\right] \quad \textit{for all} \quad f \in \mathcal{V}(S,T).$$

1.2.2 Infinite dimensional Brownian motion/ fractional Brownian motion

In this subsection, we introduce the basic definitions of $\mathbb{K}$-valued Q-cylindrical fractional Brownian motion as well as Brownian motion. Let $\mathbb{H}$ and $\mathbb{K}$ be two separable Hilbert spaces and $\mathcal{L}(\mathbb{K},\mathbb{H})$ be the space of all bounded linear operators from $\mathbb{K}$ to $\mathbb{H}$, $\mathcal{L}_Q^0(\mathbb{K};\mathbb{H})$ denotes the space of all $\xi \in \mathcal{L}(\mathbb{K},\mathbb{H})$ such that $\xi Q^{1/2}$ is a Hilbert-Schmidt operator, separately. For convenience, we will use the same notation $\|\cdot\|$ to denote the norms in $\mathbb{H}$, $\mathbb{K}$ and $\mathcal{L}(\mathbb{K},\mathbb{H})$, and use $(\cdot,\cdot)$ to denote the inner product of $\mathbb{H}$ and $\mathbb{K}$ without any confusion. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$).

Let $B = (B(t))_{t\geq 0}$ and $B_Q^H = (B_Q^H(t))_{t\geq 0}$ be a $\mathbb{K}$-valued Q-cylindrical *Brownian motion* and *fractional Brownian motion* respectively, defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with $TrQ < \infty$, where Q is a symmetric nonnegative trace class operator from $\mathbb{K}$ into itself. We assume that there exists a complete orthonormal basis $\{e_k\}_{k\geq 1}$

in $\mathbb{K}$, a bounded sequence of nonnegative real numbers λ_k such that $Qe_k = \lambda_k e_k$, $k = 1, 2, \cdots$. Then for arbitrary $t \in [0, T]$, $B(\cdot)$, $B_Q^H(\cdot)$ have the expansions

$$B(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\beta_k(t)e_k, \qquad B_Q^H(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\beta_k^H(t)e_k, \quad t \geq 0,$$

where $\{\beta_k\}_{k\geq 1}$ and $\{\beta_k^H\}_{k\geq 1}$ are, respectively, a sequence of two-sided one-dimensional real valued standard Brownian motions and a sequence of fractional Brownian motions mutually independent on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$.

According to above construction, first, we consider some properties about Brownian motion. For $\varphi, \psi \in \mathcal{L}(\mathbb{K}, \mathbb{H})$, we define $(\varphi, \psi) = Tr[\varphi Q \psi^*]$, where ψ^* is the adjoint operator of ψ. Then, for any bounded operator $\psi \in \mathcal{L}(\mathbb{K}, \mathbb{H})$,

$$\|\psi\|_Q^2 = Tr[\psi Q \psi^*] = \sum_{k=1}^{\infty} \|\sqrt{\lambda_k}\psi e_k\|^2.$$

If $\|\psi\|_Q^2 < \infty$, then ψ is called a Q-Hilbert-Schmidt operator, we denote by $\mathcal{L}_Q^0(\mathbb{K}, \mathbb{H})$ the space of all $\xi \in \mathcal{L}(\mathbb{K}, \mathbb{H})$ such that $\xi Q^{\frac{1}{2}}$ is a Hilbert-Schmidt operator. By Proposition 2.8 in [26], if ψ is an $\mathcal{L}(\mathbb{K}, \mathbb{H})$-valued stochastic process on $T \times \Omega$ such that $\psi(t)$ is measurable relative to $\mathcal{F}_t$ for all $t \in [0, T]$, and satisfies

$$\int_0^T \mathbb{E}\|\psi(t)\|^2 dt < \infty,$$

then we have the following property,

$$\mathbb{E}\left\| \int_0^T \psi(s) dB(s) \right\|^2 \leq Tr(Q) \int_0^T \mathbb{E}\|\psi(s)\|^2 ds. \tag{1.6}$$

Next, we would love to recall some properties of fractional Brownian motion, introduced in [13] by Caraballo et al., which will be used in the thesis frequently. Let $\varphi : [0, T] \to L_Q^0(\mathbb{K}, \mathbb{H})$ such that

$$\sum_{k=1}^{\infty} \|K_H^*(\varphi Q^{1/2} e_k)\|_{L^2([0,T];\mathbb{H})} < \infty, \tag{1.7}$$

where K_H^* is a linear operator from the linear space of $\mathbb{R}$-valued step functions on $[0, T]$ to $L^2[0, T]$.

Definition 1.12. *Let $\varphi : [0, T] \to \mathcal{L}_Q^0(\mathbb{K}, \mathbb{H})$ satisfy (1.7). Then, its stochastic integral with respect to the fractional Brownian motion B_Q^H is defined for $t \geq 0$, as follows*

$$\int_0^t \varphi(s) dB_Q^H(s) := \sum_{k=1}^{\infty} \int_0^t \varphi(s) Q^{1/2} e_k d\beta_k^H = \sum_{k=1}^{\infty} \int_0^t (K_H^*(\varphi Q^{1/2} e_k))(s) dW(s).$$

Notice that if

$$\sum_{k=1}^{\infty} \|\varphi Q^{1/2} e_k\|_{L^{1/H}([0,T];\mathbb{H})} < \infty, \tag{1.8}$$

then in particular (1.7) holds.

Lemma 1.13. *For any $\varphi : [0,T] \to \mathcal{L}_Q^0(\mathbb{K}, \mathbb{H})$ such that (1.8) holds, and for any α, $\beta \in [0,T]$ with $\alpha > \beta$,*

$$\mathbb{E}\left|\int_\beta^\alpha \varphi(s)dB_Q^H(s)\right|_{\mathbb{H}}^2 \le cH(2H-1)(\alpha-\beta)^{2H-1}\sum_{n=1}^{\infty}\int_\beta^\alpha |\varphi(s)Q^{1/2}e_n|_{\mathbb{H}}^2 ds,$$

where $c = c(H)$.

If, in addition, $\sum_{n=1}^{\infty} |\varphi(t)Q^{1/2}e_n|_{\mathbb{H}}$ is uniformly convergent for $t \in [0,T]$, then

$$\mathbb{E}\left|\int_\beta^\alpha \varphi(s)dB_Q^H(s)\right|_{\mathbb{H}}^2 \le cH(2H-1)(\alpha-\beta)^{2H-1}\int_\beta^\alpha \|\varphi(s)\|_Q^2 ds. \tag{1.9}$$

For more details about fractional Brownian motions, the reader is referred to [13] and the references therein.

1.3 Some examples in applications

Fractional differential equations now play a central role in the modelling of anomalous diffusion processes [19]. They arise naturally in a wide variety applications such as physics, fluid mechanics, viscoelasticity, heat conduction in materials with memory, chemistry and engineering [3, 8, 47]. Fractional differential equations with the fractional substantial derivative also appear in the transport equation of describing the time evolution of the partial differential equation (PDE) of a Lévy walk, which is a model with the spatiotemporal coupled PDFs of waiting time and jump length [35, 74].

Example 1.14. *The authors in [56] investigated the finite element approximation for the following initial boundary value problem with $1 < \alpha < 2$ and $\frac{1}{2} < \beta \le 1$:*

$$\begin{cases} \partial_t^\alpha u(t,x) + (-\Delta)^\beta u(t,x) = \frac{\partial^2 W(t,x)}{\partial t \partial x}, & 0 < t < T,\ 0 < x < 1, \\ u(t,0) = u(t,1) = 0, \qquad 0 < t < T, \\ u(0,x) = v_1(x), \partial_t u(0,x) = v_2(x), \qquad 0 < x < 1, \end{cases}$$

where ∂_t^α denotes the left-sided Caputo fractional derivative of order α with respect to t; $(-\Delta)^\beta$ is the fractional Laplacian, and $W(t,x)$ represents a space-time infinite dimensional Brownian motion.

Example 1.15. *The authors in [22] studied generalized Navier-Stokes equations with time fractional differential operators:*

$$\begin{cases} D_t^\alpha u - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f, & \text{in } \mathbb{R}^N,\ t>0,\\ \nabla\cdot u = 0, & \text{in } \mathbb{R}^N,\ t>0,\\ u(x,0) = u_0, & \text{in } \mathbb{R}^N, \end{cases}$$

where $\alpha \in (0,1)$ is a fixed number and D_t^α is the Caputo fractional derivative.

Example 1.16. *The authors in [78] considered the linear and semilinear time fractional evolution equations involving the linear part. The existence and uniqueness of mild solution and classical solutions for the inhomogeneous linear abstract Cauchy problem*

$$\begin{cases} D_t^\alpha u(t) + Au(t) = f(t), & 0 < t \le T,\\ u(0) = u_0, \end{cases}$$

are firstly studied, where $A \in \Theta_\omega^\gamma(X)$ (see [78] Definition 1.1 for more details) with $-1 < \gamma < 0$ and $0 < \omega < \pi/2$, D_t^α $(0 < \alpha < 1)$ is the Caputo fractional derivative of order α, and u_0 is given belonging to a subset of Banach space X.

Next, the authors applied the theory of time fractional derivative and their properties to the nonlinear fractional abstract Cauchy problem

$$\begin{cases} D_t^\alpha u(t) + Au(t) = f(t,u(t)), & 0 < t \le T,\\ u(0) = u_0, \end{cases}$$

where A and D_t^α has the same meaning with the above linear abstract Cauchy problem.

Chapter 2

Stochastic lattice systems with Caputo time fractional derivative

Lattice systems have attracted much attention in recent decades. They arise naturally in a wide variety of applications where the spatial structure possesses a discrete character as well as in the spatial discretization of continuous problems. The asymptotic behavior of both deterministic and stochastic lattice systems has been investigated extensively in the literatures, see, e.g., deterministic [5, 16, 27, 86], stochastic [4, 14, 15, 39, 40, 41, 42]. The existence and uniqueness for a class of fractional stochastic delay and evolution differential equations were given in [33], while the existence of solutions for fractional stochastic differential equations (SDEs) with infinite delay was obtained in [25, 70] using fixed point theory. There has, however, been little mention of deterministic or stochastic lattice differential equations with time fractional derivative.

Hence, in this chapter, we will study a stochastic lattice system with Caputo fractional substantial time derivative, the asymptotic behavior of this kind of problem is investigated. In particular, the existence of a global forward attracting set in the weak mean-square topology is established. A general theorem on the existence of solutions for a fractional SDE in a Hilbert space under the assumption that the nonlinear term is weakly continuous in a given sense is established and applied to the lattice system. The existence and uniqueness of solutions for a more general fractional SDEs are also obtained under a Lipschitz condition. Instead of using fixed pointed theory, we will follow the approach of Lakshmikaantham and Vatsata [52], who proved a Peano local existence result for fractional ordinary differential equations (ODEs).

The results of this chapter can be found in [79].

2.1 Statement of the problem, some definitions and lemmas

The Caputo fractional substantial time derivative [45] is defined as

$$D_s^{\mu} f(s) = I_s^{\nu}[D_s^m f(t)], \qquad \nu = m - \mu,$$

when m is the smallest integer that exceeds μ. Here $I_s^\nu f(t)$ is the fractional substantial integral [20, 31] defined by

$$I_s^\nu f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-\tau)^{\nu-1} e^{-\sigma(t-\tau)} f(\tau) d\tau, \qquad \nu > 0,$$

where σ is a constant or a function, and

$$D_s^m = \left(\frac{\partial}{\partial t} + \sigma \right)^m = (D+\sigma)^m = (D+\sigma)(D+\sigma)\cdots(D+\sigma).$$

In fact, we only adopt Caputo fractional substantial time derivative and Caputo fractional time derivative, at the same time, in this chapter. Hence, to avoid confusion with notations, we use D_s^α and D^α to represent Caputo time substantial derivative and Caputo time derivative, respectively.

We investigate a stochastic lattice system with Caputo fractional substantial time derivative of the form

$$\begin{cases} D_s^\alpha x_i(t) + (-1)^p \Delta^p x_i(t) + \lambda x_i(t) = f_i(x_i(t)) + g_i(t)\frac{dB(t)}{dt}, & t \geq 0, \dfrac{1}{2} < \alpha < 1, \\ x_i(0) = x_0, & i \in \mathbb{Z}. \end{cases}$$

Here Δ is the discrete analogue of the one-dimensional Laplacian, which will be defined later. In addition, $B(t)$ is a standard scalar Brownian motion on an underlying complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$. In particular, $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$ is an increasing and right continuous family of σ-sub-algebras of $\mathcal{F}$, which contains of $\mathbb{P}$-null sets, and $B(t)$ is $\mathcal{F}_t$ measurable for each $t \in \mathbb{R}^+$. Essentially, $\mathcal{F}_t$ represents the information about the randomness until time t.

The natural phase space for such an infinite dimensional system of differential equations, fractional or not, is the Hilbert space

$$\ell^2 = \left\{ x = (x_i)_{i\in\mathbb{Z}}, x_i \in \mathbb{R} : \sum_{i\in\mathbb{Z}} x_i^2 < +\infty \right\},$$

with the inner product and norm

$$(x,y) = \sum_{i\in\mathbb{Z}} x_i y_i, \qquad \|x\|^2 = \sum_{i\in\mathbb{Z}} x_i^2, \qquad \forall x = (x_i)_{i\in\mathbb{Z}},\ y = (y_i)_{i\in\mathbb{Z}} \in \ell^2.$$

Moreover, let X be a separable Hilbert space with norm $\|\cdot\|$ and inner product $(\cdot,\cdot)$. Then $L^2(\Omega; X)$ is a Hilbert space of X-valued random variables with norm $(\mathbb{E}\|\cdot\|^2)^{\frac{1}{2}}$ and inner product $\mathbb{E}(\cdot,\cdot)$.

We first consider the existence of solutions of the fractional order SDEs

$$\begin{cases} D^\alpha x(t) = f(t, x(t)) + g(t)\frac{dB(t)}{dt}, & t \geq 0, \dfrac{1}{2} < \alpha < 1, \\ x(0) = x_0 \in L^2(\Omega; X) \end{cases} \tag{2.1}$$

and

$$\begin{cases} D_s^\alpha x(t) = f(t, x(t)) + g(t)\frac{dB(t)}{dt}, & t \geq 0, \dfrac{1}{2} < \alpha < 1, \\ x(0) = x_0 \in L^2(\Omega; X), \end{cases} \tag{2.2}$$

in the Hilbert space $L^2(\Omega; X)$, where $f : [0, \infty) \times L^2(\Omega; X) \to L^2(\Omega; X)$ and $g : [0, \infty) \to X$. Here $\frac{1}{2} < \alpha < 1$, and D_s^α is the Caputo fractional substantial derivative of order α with $\sigma > 0$, while D^α is the Caputo fractional time derivative of order α (see Definition 1.4, like D_s^α but with $\sigma = 0$).

We assume that the nonlinear term f is sequentially weakly continuous in bounded sets. This concept has been introduced by Caraballo et al. [16] in the context of delay differential equations in Banach spaces with a classical derivative. In addition, with Lipschitz conditions we prove the existence and uniqueness of solutions of the fractional order SDEs,

$$\begin{cases} D^\alpha x(t) = f(t, x(t)) + g(t, x(t))\frac{dB(t)}{dt}, & t \geq 0, \dfrac{1}{2} < \alpha < 1, \\ x(0) = x_0 \in L^2(\Omega; X) \end{cases} \tag{2.3}$$

and

$$\begin{cases} D_s^\alpha x(t) = f(t, x(t)) + g(t, x(t))\frac{dB(t)}{dt}, & t \geq 0, \dfrac{1}{2} < \alpha < 1, \\ x(0) = x_0 \in L^2(\Omega; X), \end{cases} \tag{2.4}$$

in $L^2(\Omega; X)$, where $g : [0, \infty) \times L^2(\Omega; X) \to L^2(\Omega; X)$.

Next we will introduce some notations which will be used throughout this chapter. We denote by $C(a, b; L^2(\Omega; X)) = C(a, b; L^2(\Omega; \mathcal{F}, \mathbb{P}, X))$ the Banach space of all continuous functions from $[a, b]$ into $L^2(\Omega; X)$ equipped with the supremum norm. Let $L_w^2(\Omega; X)$ be the space $L^2(\Omega; X)$ endowed with the weak topology.

Also we recall two concepts who are introduced in [16]. We say that $x_n \to x \in C(0, T; L_\omega^2(\Omega; X))$ in $C(0, T; L_w^2(\Omega; X))$, if $x_n(s_n) \to x(s)$ in $L_\omega^2(\Omega; X)$ for all $s_n \to s \in [0, T]$. We will also say that the function f is sequentially weakly continuous in bounded sets, if $t_n \to t$, $x_n(t_n) \to x(t)$ in $L_\omega^2(\Omega; X)$ and $(\mathbb{E}\|x_n(t_n)\|^2)^{\frac{1}{2}} \leq M$ for all n imply that $f(t_n, x(t_n)) \to f(t, x(t))$ in $L_\omega^2(\Omega; X)$.

On the other hand, we will say that the function f is bounded if it maps bounded subsets of $[0, \infty) \times L^2(\Omega; X)$ onto bounded subsets of $L^2(\Omega; X)$.

In what follows, we define what we mean by solutions of the above initial problems.

Definition 2.1. *The map $x : [0, T] \to L^2(\Omega; X)$ is called a solution of initial value problem (2.3) if $x(0) = x_0$ and $x(\cdot)$ is continuous and satisfies for $t \in [0, T]$,*

$$\begin{aligned} x(t) = x(0) &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} f(\tau, x(\tau)) d\tau \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} g(\tau, x(\tau)) dB(\tau), \quad \mathbb{P}\text{-}a.s. \end{aligned} \tag{2.5}$$

This definition also applies for problem (2.2) with g depending only on the t variable.

Definition 2.2. *The map $x : [0,T] \to L^2(\Omega; X)$ is called a solution of initial value problem (2.4) if $x(0) = x_0$ and $x(\cdot)$ is continuous and satisfies for $t \in [0,T]$,*

$$\begin{aligned} x(t) = x(0)e^{-\sigma t} &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} e^{-\sigma(t-\tau)} f(\tau, x(\tau)) d\tau \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} e^{-\sigma(t-\tau)} g(\tau, x(\tau)) dB(\tau) \quad \mathbb{P}\text{-}a.s. \end{aligned} \tag{2.6}$$

The following generalization of Gronwall's lemma for singular kernels [44] will be used in the sequel.

Lemma 2.3. *Suppose $b \geq 0$, $\beta \geq 0$ and $a(t)$ is a nonnegative function locally integrable on $0 \leq t < T$ (some $T \leq +\infty$), and suppose that $u(t)$ is nonnegative and locally integrable on $0 \leq t < T$ with*

$$u(t) \leq a(t) + b \int_0^t (t-s)^{\beta-1} u(s) ds$$

on this interval. Then

$$u(t) \leq a(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(b\Gamma(\beta))^n}{\Gamma(n\beta)} (t-s)^{n\beta-1} a(s) \right] ds, \qquad 0 \leq t < T.$$

Corollary 2.4. *Suppose $\alpha > 0$, $M > 0$, $\sigma > (M\Gamma(\alpha)2^\alpha)^{\frac{1}{\alpha}}$, and a, b are nonnegative constants, and suppose $u(t)$ is nonnegative and locally integrable on $0 \leq t \leq T$ (some $T \leq \infty$) with*

$$u(t) \leq ae^{-\sigma t} + be^{\sigma t} + M \int_0^t (t-s)^{\alpha-1} u(s) ds, \quad 0 \leq t \leq T.$$

Then

$$u(t) \leq ae^{-\sigma t} + be^{\sigma t} + (ae^{\frac{\sigma}{2}t} + be^{\sigma t}) \sum_{n=1}^{\infty} \left(\frac{M\Gamma(\alpha)2^\alpha}{\sigma^\alpha} \right)^n.$$

Proof. By Lemma 2.3, we deduce that

$$\begin{aligned} u(t) &\leq ae^{-\sigma t} + be^{\sigma t} + \int_0^t \sum_{n=1}^{\infty} \frac{(M\Gamma(\alpha))^n}{\Gamma(n\alpha)} (t-s)^{n\alpha-1} (ae^{-\sigma s} + be^{\sigma s}) ds \\ &\leq ae^{-\sigma t} + be^{\sigma t} + a \sum_{n=1}^{\infty} \frac{(M\Gamma(\alpha))^n}{\Gamma(n\alpha)} \int_0^t (t-s)^{n\alpha-1} e^{-\sigma s} ds \\ &\quad + b \sum_{n=1}^{\infty} \frac{(M\Gamma(\alpha))^n}{\Gamma(n\alpha)} \int_0^t (t-s)^{n\sigma-1} e^{\sigma s} ds. \end{aligned} \tag{2.7}$$

Noticing that

$$
\begin{aligned}
\int_0^t (t-s)^{n\alpha-1}e^{-\sigma s}ds &= e^{\frac{\sigma}{2}t}\int_0^t (t-s)^{n\alpha-1}e^{-\frac{\sigma}{2}(t-s)}e^{-\frac{3}{2}\sigma s}ds \\
&\le e^{\frac{\sigma}{2}t}\left(\frac{\sigma}{2}\right)^{1-n\alpha}\int_0^t \left(\frac{\sigma}{2}(t-s)\right)^{n\alpha-1}e^{-\frac{\sigma}{2}(t-s)}ds \\
&\le e^{\frac{\sigma}{2}t}\left(\frac{\sigma}{2}\right)^{-n\alpha}\Gamma(n\alpha),
\end{aligned}
\tag{2.8}
$$

and

$$
\begin{aligned}
\int_0^t (t-s)^{n\alpha-1}e^{\sigma s}ds &= e^{\sigma t}\int_0^t (t-s)^{n\alpha-1}e^{-\sigma(t-s)}ds \\
&= e^{\sigma t}\sigma^{1-n\alpha}\int_0^t (\sigma(t-s))^{n\alpha-1}e^{-\sigma(t-s)}ds \\
&\le e^{\sigma t}\sigma^{-n\alpha}\Gamma(n\alpha).
\end{aligned}
\tag{2.9}
$$

It follows from (2.7)-(2.9) that

$$
u(t) \le ae^{-\sigma t} + be^{\sigma t} + (ae^{\frac{\sigma}{2}t} + be^{\sigma t})\sum_{n=1}^{\infty}\left(\frac{M\Gamma(\alpha)2^{\alpha}}{\sigma^{\alpha}}\right)^n. \tag{2.10}
$$

This completes the proof. □

2.2 Existence results

We consider the existence of solutions of the initial value problems (2.1) and (2.2) possibly without uniqueness.

Theorem 2.5. *Let $f : [0,+\infty) \times L^2(\Omega; X) \to L^2(\Omega; X)$ be sequentially weakly continuous in bounded sets, let $g : [0,+\infty) \to X$ be measurable, and let f and g be bounded maps, i.e.,*

$$
\mathbb{E}\|f(t,x)\|^2 \le M^2, \qquad \|g(t)\|^2 \le M^2 \qquad \text{for all } (t,x) \in R_0,
$$

where $R_0 = \{(t,x) : 0 \le t \le T \text{ and } \mathbb{E}\|x - x_0\|^2 \le b^2\}$. Then initial value problem (2.1) possesses at least one solution $x(\cdot)$ defined on $[0,T_b]$, where

$$
T_b = \min\left\{T, \left[\frac{b^2\Gamma^2(\alpha+1)}{4M^2}\right]^{\frac{1}{2\alpha}}, \ \left[\frac{b^2\Gamma^2(\alpha)(2\alpha-1)}{4M^2}\right]^{\frac{1}{2\alpha-1}}\right\}, \quad \frac{1}{2} < \alpha < 1.
$$

Proof. Let $x_0(t)$ be a mean-square continuous function on $[-\delta, 0]$, $0 < \delta < 1$, such that $x_0(0) = x_0$ and $\mathbb{E}\|x_0(t) - x_0\|^2 \le b^2$. For any $0 < \epsilon < \delta$, we define

$$
x_\epsilon(t) = \begin{cases} x_0(t), \quad t \in [-\delta, 0] \\ x_0 + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-\tau)^{\alpha-1}f(\tau, x_\epsilon(\tau-\epsilon))d\tau \\ +\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-\tau)^{\alpha-1}g(\tau)dB(\tau), \quad t \in [0, T_1], \end{cases} \tag{2.11}
$$

where $T_1 = \min\{T_b, \epsilon\}$. Then we find that

$$\begin{aligned}
\mathbb{E}\|x_\epsilon(t) - x_0\|^2 &\le \frac{2}{\Gamma^2(\alpha)}\mathbb{E}\left\|\int_0^t (t-\tau)^{\alpha-1} f(\tau, x_\epsilon(\tau-\epsilon))d\tau\right\|^2 \\
&\quad + \frac{2}{\Gamma^2(\alpha)}\mathbb{E}\left\|\int_0^t (t-\tau)^{\alpha-1} g(\tau)dB(\tau)\right\|^2 \\
&\le \frac{2}{\Gamma^2(\alpha)}\int_0^t (t-\tau)^{\alpha-1}d\tau \times \int_0^t (t-\tau)^{\alpha-1}\mathbb{E}\|f(\tau, x_\epsilon(\tau-\epsilon))\|^2 d\tau \\
&\quad + \frac{2}{\Gamma^2(\alpha)}\int_0^t (t-\tau)^{2\alpha-2}\|g(\tau)\|^2 d\tau \\
&\le \frac{2M^2 t^{2\alpha}}{\Gamma^2(\alpha+1)} + \frac{2M^2 t^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha-1)} \le b^2,
\end{aligned} \tag{2.12}$$

because of the choice of T_1. If $T_1 < T_b$, then we can use (2.11) to extend as a continuous function on $[-\delta, T_2]$ with $T_2 = \min\{T_b, 2\epsilon\}$, such that $\mathbb{E}\|x_\epsilon(t) - x_0\|^2 \le b^2$ holds.

Continuing this process, we can define $x_\epsilon(t)$ over $[-\delta, T_b]$ so that $\mathbb{E}\|x_\epsilon(t) - x_0\|^2 \le b^2$ is satisfied on $[-\delta, T_b]$.

Let $0 \le t_1 \le t_2 \le T_b$. We find that

$$\begin{aligned}
\mathbb{E}\|x_\epsilon(t_2) - x_\epsilon(t_1)\|^2 &\le \frac{2}{\Gamma^2(\alpha)}\mathbb{E}\bigg\| \int_0^{t_1} ((t_1-\tau)^{\alpha-1} - (t_2-\tau)^{\alpha-1}) f(\tau, x_\epsilon(\tau-\epsilon))d\tau \\
&\quad + \int_{t_1}^{t_2} (t_2-\tau)^{\alpha-1} f(\tau, x_\epsilon(\tau-\epsilon))d\tau\bigg\|^2 \\
&\quad + \frac{2}{\Gamma^2(\alpha)}\mathbb{E}\bigg\| \int_{t_1}^{t_2} (t_2-\tau)^{\alpha-1} g(\tau)dB(\tau) \\
&\quad + \int_0^{t_1} ((t_1-\tau)^{\alpha-1} - (t_2-\tau)^{\alpha-1}) g(\tau)dB(\tau)\bigg\|^2 \\
&:= P_1 + P_2.
\end{aligned} \tag{2.13}$$

Then, using Young's inequality and Hölder's inequality, we obtain

$$\begin{aligned}
P_1 &\le \frac{4}{\Gamma^2(\alpha)}\int_0^{t_1} ((t_1-\tau)^{\alpha-1} - (t_2-\tau)^{\alpha-1})d\tau \\
&\quad \times \int_0^{t_1} ((t_1-\tau)^{\alpha-1} - (t_2-\tau)^{\alpha-1})\mathbb{E}\|f(\tau, x_\epsilon(\tau-\epsilon))\|^2 d\tau \\
&\quad + \frac{4}{\Gamma^2(\alpha)}\int_{t_1}^{t_2} (t_2-\tau)^{\alpha-1}d\tau \times \int_{t_1}^{t_2} (t_2-\tau)^{\alpha-1}\mathbb{E}\|f(\tau, x_\epsilon(\tau-\epsilon))\|^2 d\tau \qquad (2.14) \\
&\le \frac{4M^2}{\Gamma^2(\alpha+1)}(t_1^\alpha - t_2^\alpha + (t_2-t_1)^\alpha)^2 + \frac{4M^2}{\Gamma^2(\alpha+1)}(t_2-t_1)^{\alpha-1} \\
&\le \frac{8M^2}{\Gamma^2(\alpha+1)}(t_2-t_1)^{2\alpha},
\end{aligned}$$

and

$$
\begin{aligned}
P_2 \le& \frac{4}{\Gamma^2(\alpha)}\mathbb{E}\left\|\int_0^{t_1}((t_1-\tau)^{\alpha-1}-(t_2-\tau)^{\alpha-1})g(\tau)dB(\tau)\right\|^2\\
&+\frac{4}{\Gamma^2(\alpha)}\mathbb{E}\left\|\int_{t_1}^{t_2}(t_2-\tau)^{\alpha-1}g(\tau)dB(\tau)\right\|^2\\
\le& \frac{4}{\Gamma^2(\alpha)}\int_0^{t_1}((t_1-\tau)^{\alpha-1}-(t_2-\tau)^{\alpha-1})^2\|g(\tau)\|^2d\tau\\
&+\frac{4}{\Gamma^2(\alpha)}\int_{t_1}^{t_2}(t_2-\tau)^{2\alpha-2}\|g(\tau)\|^2d\tau\\
\le& \frac{4M^2}{\Gamma^2(\alpha)}\left(\int_0^{t_1}(t_1-\tau)^{2\alpha-2}d\tau-\int_0^{t_1}(t_2-\tau)^{2\alpha-2}d\tau\right)\\
&+\frac{4M^2}{\Gamma^2(\alpha)}\int_{t_1}^{t_2}(t_2-\tau)^{2\alpha-2}d\tau \le \frac{8M^2}{\Gamma^2(\alpha)(2\alpha-1)}(t_2-t_1)^{2\alpha-1}.
\end{aligned}
\tag{2.15}
$$

Hence, (2.13)-(2.15) imply that

$$
\mathbb{E}\|x_\epsilon(t_2)-x_\epsilon(t_1)\|^2 \le \frac{8M^2}{\Gamma^2(\alpha+1)}(t_2-t_1)^{2\alpha}+\frac{8M^2}{\Gamma^2(\alpha)(2\alpha-1)}(t_2-t_1)^{2\alpha-1}. \tag{2.16}
$$

Since X is a Hilbert space, we deduce from (2.12) that for any $t \in [0,T_b]$, there exists a sequence $\{x_{\epsilon_n}(t)\}$ with $\epsilon_n \to 0$ as $n \to +\infty$, which is relatively compact in $L^2_w(\Omega;X)$. By the diagonal method and (2.16), arguing as in the proof of Theorem 4 in Caraballo [16], we obtain the existence of a mean-square continuous function $x(\cdot)$ and a subsequence of $\{x_{\epsilon_n}(\cdot)\}$ (denoted again x_{ϵ_n}) such that

$$
x_{\epsilon_n}(t) \to x(t) \quad \text{in } \ L^2_w(\Omega;X) \ \text{ for all } \ t\in[0,T_b]. \tag{2.17}
$$

It follows from (2.16) and (2.17) that

$$
x_{\epsilon_n}(t_n) \to x(t_0) \quad \text{in } \ L^2_w(\Omega;X) \ \text{ if } \ t_n \to t_0 \in [0,T_b]. \tag{2.18}
$$

Let $x(t)=x_0(t)$, $\forall t\in[-\delta,0]$. Then we have

$$
x_{\epsilon_n}(t_n) \to x(t_0) \quad \text{in } \ L^2_w(\Omega;X) \ \text{ if } \ t_n \to t_0 \in [-\delta,T_b]. \tag{2.19}
$$

Now we prove that the limit $x(\cdot)$ is a solution of (2.1). For this aim we will pass to the limit in the integral

$$
\begin{aligned}
x_{\epsilon_n}(t) = x_0 &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau,x_{\epsilon_n}(\tau-\epsilon_n))d\tau\\
&+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}g(\tau)dB(\tau), \quad t\in[0,T_b].
\end{aligned}
\tag{2.20}
$$

Since f is sequentially weakly continuous in bounded sets, by (2.19) we have for any $\tau\in[0,T_b]$ and any $v\in L^2(\Omega;X)$,

$$
f(\tau,x_{\epsilon_n}(\tau-\epsilon_n)) \to f(\tau,x(\tau)) \qquad \text{in } \ L^2_w(\Omega;X) \tag{2.21}
$$

as $n \to \infty$. Then, by (2.21), we deduce from Lebesgue's theorem that for any $v \in L^2(\Omega; X)$,

$$\mathbb{E}\left(\int_0^t (t-\tau)^{\alpha-1} f(\tau, x_{\epsilon_n}(\tau-\epsilon_n))d\tau, v\right) = \int_0^t (t-\tau)^{\alpha-1}\mathbb{E}(f(\tau, x_{\epsilon_n}(\tau-\epsilon_n)), v)d\tau$$
$$\longrightarrow \int_0^t (t-\tau)^{\alpha-1}\mathbb{E}(f(\tau, x(\tau)), v)d\tau = \mathbb{E}\left(\int_0^t (t-\tau)^{\alpha-1} f(\tau, x(\tau))d\tau, v\right). \tag{2.22}$$

From (2.22) and Hölder's inequality, we have

$$\mathbb{E}(x(t), v) = \mathbb{E}(x_0, v) + \mathbb{E}\left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau, x(\tau))d\tau, v\right)$$
$$+ \mathbb{E}\left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} g(\tau)dB(\tau), v\right).$$

As $v \in L^2(\Omega; X)$ is arbitrary, we have for all $t \in [0, T_b]$,

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau, x(\tau))d\tau + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} g(\tau)dB(\tau), \quad \mathbb{P}\text{-a.s.}$$

This implies that $x(t)$ satisfies (2.1) and completes the proof. □

Theorem 2.6. *Assume the conditions of Theorem 2.5. If a solution $x(\cdot)$ of equation (2.1) has a maximal interval of existence, $[0, T^*]$, and there exists $K > 0$ such that $\mathbb{E}\|x(t)\|^2 \leq K$ for all $t \in [0, T^*)$, then $T^* = +\infty$, i.e., $x(\cdot)$ is a globally defined solution.*

Proof. Since f and g are bounded, by the similar arguments as for (2.13)-(2.15), we see that x is uniformly continuous on $[0, T^*)$. Therefore, the limit $\lim_{t\to T^*-} x(t) = x^*$ exists. Then we consider

$$\begin{cases} D^\alpha x(t) = f(t, x(t)) + g(t)\frac{dB(t)}{dt}, & t \geq T^*, \frac{1}{2} < \alpha < 1, \\ x(0) = x^* \in L^2(\Omega; X), \end{cases} \tag{2.23}$$

and by Theorem 2.5, we obtain that the solution $x(\cdot)$ can be extended to the interval $[0, T^* + \delta)$, $\delta > 0$, which is contradiction. □

In the following results, we establish the existence of solutions to initial value problem (2.2).

Theorem 2.7. *Assume the conditions of Theorem 2.5. Let $b^2 \geq 12\mathbb{E}\|x_0\|^2$ and $\frac{1}{2} < \alpha < 1$. Then initial value problem (2.2) possesses at least one solution $x(t)$ defined on $[0, T']$, where*

$$T' = \min\left\{T,\ \left[\frac{\Gamma^2(\alpha+1)}{6M^2}(b^2 - 12\mathbb{E}\|x_0\|^2)\right]^{\frac{1}{2\alpha}},\right.$$
$$\left.\left[\frac{\Gamma^2(\alpha)(2\alpha-1)}{6M^2}(b^2 - 12\mathbb{E}\|x_0\|^2)\right]^{\frac{1}{2\alpha-1}}\right\}.$$

Proof. Let $x_0(t)$ be a mean-square continuous function on $[-\delta, 0]$, where $0 < \delta < 1$, such that $x_0(0) = x_0$ and $\mathbb{E}\|x_0(t) - x_0\|^2 \leq b^2$. For $0 < \epsilon \leq \delta$, we define

$$x_\epsilon(t) = \begin{cases} x_0(t), & t \in [-\delta, 0], \\ x_0 e^{-\sigma t} + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-\tau)^{\alpha-1} e^{-\sigma(t-\tau)} f(\tau, x_\epsilon(\tau - \epsilon)) d\tau \\ + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-\tau)^{\alpha-1} e^{-\sigma(t-\tau)} g(\tau) dB(\tau), & t \in [0, T_1'], \end{cases} \tag{2.24}$$

where $T_1' = \min\{T', \epsilon\}$. Using (2.12), we obtain

$$\begin{aligned} \mathbb{E}\|x_\epsilon(t) - x_0\|^2 &\leq 3\mathbb{E}\|x_0 - x_0 e^{-\sigma t}\|^2 \\ &\quad + \frac{3}{\Gamma^2(\alpha)} \mathbb{E} \left\| \int_0^t (t-\tau)^{\alpha-1} f(\tau, x_\epsilon(\tau - \epsilon)) d\tau \right\|^2 \\ &\quad + \frac{3}{\Gamma^2(\alpha)} \mathbb{E} \left\| \int_0^t (t-\tau)^{\alpha-1} g(\tau) dB(\tau) \right\|^2 \\ &\leq 12\mathbb{E}\|x_0\|^2 + \frac{3M^2 t^{2\alpha}}{\Gamma^2(\alpha+1)} + \frac{3M^2 t^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha-1)} \leq b^2, \end{aligned} \tag{2.25}$$

because of the choice of T_1'. If $T_1' < T'$, we can use (2.24) to extend as a continuous function on $[-\delta, T_2']$, where $T_2' = \min\{T', 2\epsilon\}$ such that $\mathbb{E}\|x_\epsilon(t) - x_0\|^2 \leq b^2$ holds.

Continuing this process, we can define $x_\epsilon(t)$ over $[-\delta, T']$ so that $\mathbb{E}\|x_\epsilon(t) - x_0\|^2 \leq b^2$ is satisfied on $[-\delta, T']$. Let $0 \leq t_1 \leq t_2 \leq T'$, then we obtain

$$\begin{aligned} &\mathbb{E}\|x_\epsilon(t_2) - x_\epsilon(t_1)\|^2 \leq C\mathbb{E}\|x_0\|^2 (e^{-\sigma t_1} - e^{-\sigma t_2})^2 \\ &+ \frac{C}{\Gamma^2(\alpha)} \mathbb{E} \left\| \int_0^{t_1} \left((t_1-\tau)^{\alpha-1} e^{-\sigma(t_1-\tau)} - (t_2-\tau)^{\alpha-1} e^{-\sigma(t_2-\tau)} \right) f(\tau, x_\epsilon(\tau-\epsilon)) d\tau \right\|^2 \\ &+ \frac{C}{\Gamma^2(\alpha)} \mathbb{E} \left\| \int_{t_1}^{t_2} (t_2-\tau)^{\alpha-1} e^{-\sigma(t_2-\tau)} f(\tau, x_\epsilon(\tau-\epsilon)) d\tau \right\|^2 \\ &+ \frac{C}{\Gamma^2(\alpha)} \mathbb{E} \left\| \int_0^{t_1} \left((t_1-\tau)^{\alpha-1} e^{-\sigma(t_1-\tau)} - (t_2-\tau)^{\alpha-1} e^{-\sigma(t_2-\tau)} \right) g(\tau) dB(\tau) \right\|^2 \\ &+ \frac{C}{\Gamma^2(\alpha)} \mathbb{E} \left\| \int_{t_1}^{t_2} (t_2-\tau)^{\alpha-1} e^{-\sigma(t_2-\tau)} g(\tau) dB(\tau) \right\|^2 \\ &:= C\mathbb{E}\|x_0\|^2 (e^{-\sigma t_1} - e^{-\sigma t_2})^2 + P_3 + P_4 + P_5 + P_6. \end{aligned} \tag{2.26}$$

Thanks to the arguments for (2.14)-(2.15), we have

$$\begin{aligned}
P_3 \le& \frac{C}{\Gamma^2(\alpha)}\int_0^{t_1}\left((t_1-\tau)^{\alpha-1}e^{-\sigma(t_1-\tau)}-(t_2-\tau)^{\alpha-1}e^{-\sigma(t_2-\tau)}\right)d\tau\\
&\times\int_0^{t_1}\left((t_1-\tau)^{\alpha-1}e^{-\sigma(t_1-\tau)}-(t_2-\tau)^{\alpha-1}e^{-\sigma(t_2-\tau)}\right)\mathbb{E}\|f(\tau,x_\epsilon(\tau-\epsilon))\|^2d\tau\\
\le& \frac{CM^2}{\Gamma^2(\alpha)}\int_0^{t_1}\left((t_1-\tau)^{2\alpha-2}e^{-2\sigma(t_1-\tau)}-(t_2-\tau)^{2\alpha-2}e^{-2\sigma(t_2-\tau)}\right)d\tau\\
\le& \frac{CM^2}{\Gamma^2(\alpha)}\int_0^{t_1}\left((t_1-\tau)^{2\alpha-2}e^{-2\sigma(t_1-\tau)}-(t_2-\tau)^{2\alpha-2}e^{-2\sigma(t_1-\tau)}\right)d\tau\\
&+\frac{CM^2}{\Gamma^2(\alpha)}\int_0^{t_1}\left((t_2-\tau)^{2\alpha-2}e^{-2\sigma(t_1-\tau)}-(t_2-\tau)^{2\alpha-2}e^{-2\sigma(t_2-\tau)}\right)d\tau\\
\le& \frac{CM^2(t_2-t_1)^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha-1)}+\frac{CM^2(t_2-t_1)^{2\alpha-2}}{\Gamma^2(\alpha)}\int_0^{t_1}(e^{-2\sigma(t_1-\tau)}-e^{-2\sigma(t_2-\tau)})d\tau\\
\le& \frac{CM^2(t_2-t_1)^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha-1)}+\frac{CM^2(t_2-t_1)^{2\alpha-1}}{\Gamma^2(\alpha)}+\frac{CM^2(t_2-t_1)^{2\alpha-1}}{\Gamma^2(\alpha)}\frac{o(t_2-t_1)}{t_2-t_1},
\end{aligned}\tag{2.27}$$

where $\lim_{t_2-t_1\to 0}\left(\frac{o(t_2-t_1)}{t_2-t_1}\right)=0$;

$$\begin{aligned}
P_4 \le& \frac{C}{\Gamma^2(\alpha)}\int_{t_1}^{t_2}(t_2-\tau)^{\alpha-1}d\tau\int_{t_1}^{t_2}(t_2-\tau)^{\alpha-1}\mathbb{E}\|f(\tau,x_\epsilon(\tau-\epsilon))\|^2d\tau\\
\le& \frac{CM^2(t_2-t_1)^{2\alpha}}{\Gamma^2(\alpha+1)},
\end{aligned}\tag{2.28}$$

$$\begin{aligned}
P_5 \le& \frac{C}{\Gamma^2(\alpha)}\int_0^{t_1}\left((t_1-\tau)^{\alpha-1}e^{-\sigma(t_1-\tau)}-(t_2-\tau)^{\alpha-1}e^{-\sigma(t-\tau)}\right)^2\|g(\tau)\|^2d\tau\\
\le& \frac{CM^2}{\Gamma^2(\alpha)}\int_0^{t_1}\left((t_1-\tau)^{2\alpha-2}e^{-2\sigma(t_1-\tau)}-(t_2-\tau)^{2\alpha-2}e^{-2\sigma(t_2-\tau)}\right)d\tau\\
\le& \frac{CM^2(t_2-t_1)^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha-1)}+\frac{CM^2(t_2-t_1)^{2\alpha-1}}{\Gamma^2(\alpha)}+\frac{CM^2(t_2-t_1)^{2\alpha-1}}{\Gamma^2(\alpha)}\frac{o(t_2-t_1)}{t_2-t_1}
\end{aligned}\tag{2.29}$$

and

$$P_6 \le \frac{C}{\Gamma^2(\alpha)}\int_{t_1}^{t_2}(t_2-\tau)^{2\alpha-2}e^{-2\sigma(t_2-\tau)}\|g(\tau)\|^2d\tau \le \frac{CM^2(t_2-t_1)^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha-1)}.\tag{2.30}$$

Then (2.26)-(2.30) imply that

$$\begin{aligned}
\mathbb{E}\|x_\epsilon(t_2)-x_\epsilon(t_1)\|^2 \le& C\mathbb{E}\|x_0\|^2(e^{-\sigma t_1}-e^{-\sigma t_2})^2+\frac{CM^2(t_2-t_1)^{2\alpha}}{\Gamma^2(\alpha+1)}\\
&+\frac{CM^2(t_2-t_1)^{2\alpha-1}}{\Gamma^2(\alpha)}+\frac{CM^2(t_2-t_1)^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha-1)}\\
&+\frac{CM^2(t_2-t_1)^{2\alpha-1}}{\Gamma^2(\alpha)}\frac{o(t_2-t_1)}{t_2-t_1}.
\end{aligned}\tag{2.31}$$

This finishes the proof. □

Theorem 2.8. *Assume the conditions of Theorem 2.5. If a solution $x(\cdot)$ of equation (2.2) has a maximal interval of existence $[0, T'')$, and there exists $K > 0$ such that $\mathbb{E}\|x(t)\|^2 \leq K$ for all $t \in [0, T'')$, then $T'' = +\infty$, i.e., $x(\cdot)$ is a globally defined solution.*

Proof. The proof of this theorem is similar to that of Theorem 2.5, so will be omitted.

Next we consider the existence and uniqueness of solutions to initial value problems (2.3) and (2.4).

Theorem 2.9. *Let $T > 0$ and let $f(\cdot,\cdot)$, $g(\cdot,\cdot) : [0,T] \times L^2(\Omega; X) \to L^2(\Omega; X)$ be measurable functions satisfying for all x, $y \in L^2(\Omega; X)$ and $t \in [0,T]$,*

$$\mathbb{E}\|f(t,x) - f(t,y)\|^2 + \mathbb{E}\|g(t,x) - g(t,y)\|^2 \leq L\mathbb{E}\|x - y\|^2 \tag{2.32}$$

for some constants L. In addition, let f and g be bounded maps, i.e.,

$$\mathbb{E}\|f(t,x)\|^2 \leq M^2, \qquad \mathbb{E}\|g(t,x)\|^2 \leq M^2, \qquad \text{for all } \ (t,x) \in R_0,$$

where $R_0 = \{(t,x) : 0 \leq t \leq T$ and $\mathbb{E}\|x - x_0\|^2 \leq b^2\}$. Then, for every $x_0 \in L^2(\Omega; X)$, there exists a unique solution to (2.3).

Proof. We start the proof by checking the uniqueness of solutions. Assume x, $y \in L^2(\Omega; X)$ are two solutions of (2.3). Then by Young's inequality, Hölder's inequality, Itô's isometry and the Lipschitz condition (2.32), we deduce that

$$\begin{aligned}
\mathbb{E}\|x(t) - y(t)\|^2 &\leq \frac{2}{\Gamma^2(\alpha)}\mathbb{E}\left\|\int_0^t (t-\tau)^{\alpha-1}(f(\tau,x(\tau)) - f(\tau,y(\tau)))d\tau\right\|^2 \\
&\quad + \frac{2}{\Gamma^2(\alpha)}\mathbb{E}\left\|\int_0^t (t-\tau)^{\alpha-1}(g(\tau,x(\tau)) - g(\tau,y(\tau)))dB(\tau)\right\|^2 \\
&\leq \frac{2t}{\Gamma^2(\alpha)}\int_0^t (t-\tau)^{2\alpha-2}\mathbb{E}\|f(\tau,x(\tau)) - f(\tau,y(\tau))\|^2 d\tau \\
&\quad + \frac{2}{\Gamma^2(\alpha)}\int_0^t (t-\tau)^{2\alpha-2}\mathbb{E}\|g(\tau,x(\tau)) - g(\tau,y(\tau))\|^2 d\tau \\
&\leq \frac{(2T+2)L}{\Gamma^2(\alpha)}\int_0^t (t-\tau)^{2\alpha-2}\mathbb{E}\|x(\tau) - y(\tau)\|^2 d\tau.
\end{aligned} \tag{2.33}$$

By Lemma 2.3, it follows that $\mathbb{E}\|x(t) - y(t)\|^2 = 0$ for all $t \in [0,T]$ and the uniqueness is thereby proved.

Now we prove the existence of solutions to problem (2.3). First of all, let x_0 be a mean-square continuous function on $[-\delta, 0]$, where $0 < \delta < 1$, such that $x_0(0) = x_0$ and $\mathbb{E}\|x_0(t) - x_0\|^2 \leq b^2$. For $0 < \epsilon \leq \delta$, we define

$$x_\epsilon(t) = \begin{cases} x_0(t), & t \in [-\delta, 0] \\ x_0 + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-\tau)^{\alpha-1} f(\tau, x_\epsilon(\tau - \epsilon))d\tau & \\ +\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-\tau)^{\alpha-1} g(\tau, x_\epsilon(\tau - \epsilon))dB(\tau). & \end{cases} \tag{2.34}$$

Arguing as in the proof of Theorem 2.5, we can define $x_\epsilon(t)$ over $[-\delta, T_b]$ so that

$$\mathbb{E}\|x_\epsilon(t) - x_0\|^2 \le b^2 \tag{2.35}$$

is satisfied on $[-\delta, T_b]$, there exists a continuous function $x(\cdot)$ and a subsequence of $\{x_{\epsilon_n}(\cdot)\}$ (denoted again x_{ϵ_n}) such that

$$x_{\epsilon_n}(t) \to x(t) \qquad \text{in } L^2_w(\Omega; X) \qquad \text{for all } \ t \in [0, T_b], \tag{2.36}$$

and for any t_1, $t_2 \in [0, T_b]$ with $t_1 \le t_2$,

$$\mathbb{E}\|x_\epsilon(t_2) - x_\epsilon(t_1)\|^2 \le \frac{8M^2}{\Gamma^2(\alpha+1)}(t_2 - t_1)^{2\alpha} + \frac{8M^2}{\Gamma^2(\alpha)(2\alpha-1)}(t_2 - t_1)^{2\alpha-1}.$$

Combing this with the definition of $x_\epsilon(t)$, we obtain that $x_\epsilon(t)$ is equi-continuous on $[-\delta, T_b]$, i.e., for any $\eta > 0$ there exits $\delta_1 > 0$ such that for t_1, $t_2 \in [-\delta, T_b]$ with $|t_1 - t_2| < \delta_1$, we have

$$\mathbb{E}\|x_\epsilon(t_2) - x_\epsilon(t_1)\|^2 \le \eta \tag{2.37}$$

for all $\epsilon \in [0, \delta]$.

Then we show that the sequence $\{x_{\epsilon_n}\}$ given in the proof of Theorem 2.5 is a Cauchy sequence in $C(0, T_b; L^2(\Omega; X))$. Similar to the argument for (2.33), in view of (2.37), we find that for all n, m sufficiently large,

$$\begin{aligned}
\mathbb{E}\|x_{\epsilon_n}(t) - x_{\epsilon_m}(t)\|^2 &\le \frac{(2T+2)L}{\Gamma^2(\alpha)}\int_0^t (t-\tau)^{2\alpha-2}\mathbb{E}\|x_{\epsilon_n}(\tau - \epsilon_n) - x_{\epsilon_m}(\tau - \epsilon_m)\|^2 d\tau \\
&\le \frac{(2T+2)L}{\Gamma^2(\alpha)}\int_0^t (t-\tau)^{2\alpha-2}\Big(3\mathbb{E}\|x_{\epsilon_n}(\tau-\epsilon_n) - x_{\epsilon_n}(\tau)\|^2 \\
&\quad + 3\mathbb{E}\|x_{\epsilon_n}(\tau) - x_{\epsilon_m}(\tau)\|^2 + 3\mathbb{E}\|x_{\epsilon_m}(\tau) - x_{\epsilon_m}(\tau-\epsilon_m)\|^2\Big) d\tau \\
&\le \frac{6(2T+2)LT^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha-1)} + \frac{3(2T+2)L}{\Gamma^2(\alpha)} \\
&\quad \times \int_0^t (t-\tau)^{2\alpha-2}\mathbb{E}\|x_{\epsilon_n}(\tau) - x_{\epsilon_m}(\tau)\|^2 d\tau.
\end{aligned} \tag{2.38}$$

Applying Lemma 2.3, we have for n, m large enough

$$\begin{aligned}
&\mathbb{E}\|x_{\epsilon_n}(t) - x_{\epsilon_m}(t)\|^2 \\
&\le \frac{6(2T+2)LT^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha-1)}\eta\left[1 + \sum_{n=1}^{\infty}\frac{(\frac{3(2T+2)L}{\Gamma^2(\alpha)}\Gamma(2\alpha-1))^n}{\Gamma(n(2\alpha-1))}\int_0^t (t-\tau)^{n(2\alpha-1)-1}d\tau\right] \\
&\le \frac{6(2T+2)LT^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha-1)}\eta\left[1 + \sum_{n=1}^{\infty}\frac{\left(\frac{3(2T+2)L}{\Gamma^2(\alpha)}\Gamma(2\alpha-1)T^{2\alpha-1}\right)^n}{\Gamma(n(2\alpha-1)+1)}\right] \\
&\le \frac{6(2T+2)LT^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha-1)}\eta\left[1 + E_{2\alpha-1,1}\left(\frac{6(T+1)L}{\Gamma^2(\alpha)}\Gamma(2\alpha-1)T^{2\alpha-1}\right)\right].
\end{aligned} \tag{2.39}$$

Consequently,

$$\begin{aligned}&\sup_{0\le t\le a}\mathbb{E}\|x_{\epsilon_n}(t)-x_{\epsilon_m}(t)\|^2\\&\le\frac{6(2T+2)LT^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha-1)}\eta\left[1+E_{2\alpha-1,1}\left(\frac{6(T+1)L}{\Gamma^2(\alpha)}\Gamma(2\alpha-1)T^{2\alpha-1}\right)\right],\end{aligned}\tag{2.40}$$

so $\{x_{\epsilon_n}\}$ is a Cauchy sequence in $C(0,T_b;L^2(\Omega;X))$.

Finally, we check that the limit x of the sequence $\{x_{\epsilon_n}\}$ is a solution of (2.3). For this aim we will pass to the limit in the integral

$$\begin{aligned}x_{\epsilon_n}(t)&=x(0)+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}f(\tau,x_{\epsilon_n}(\tau-\epsilon_n))d\tau\\&\quad+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}g(\tau,x_{\epsilon_n}(\tau-\epsilon_n))dB(\tau),\qquad t\in[0,a].\end{aligned}\tag{2.41}$$

By Hölder's inequality, (2.37) and (2.39), we obtain for all n large enough

$$\begin{aligned}&\mathbb{E}\left\|\int_0^t(t-\tau)^{\alpha-1}f(\tau,x_{\epsilon_n}(\tau-\epsilon_n))d\tau-\int_0^t(t-\tau)^{\alpha-1}f(\tau,x(\tau))d\tau\right\|^2\\&\le TL\int_0^t(t-\tau)^{2\alpha-2}\mathbb{E}\|x_{\epsilon_n}(\tau-\epsilon_n)-x(\tau)\|^2d\tau\\&\le TL\int_0^t(t-\tau)^{2\alpha-2}(2\mathbb{E}\|x_{\epsilon_n}(\tau-\epsilon_n)-x_{\epsilon_n}(\tau)\|^2\\&\quad+2\mathbb{E}\|x_{\epsilon_n}(\tau)-x(\tau)\|^2)d\tau\le C\eta,\end{aligned}\tag{2.42}$$

and by Itô's isometry, it follows that for all n sufficiently large,

$$\begin{aligned}&\mathbb{E}\left\|\int_0^t(t-\tau)^{\alpha-1}g(\tau,x_{\epsilon_n}(\tau-\epsilon_n))dB(\tau)-\int_0^t(t-\tau)^{\alpha-1}g(\tau,x(\tau))dB(\tau))\right\|^2\\&=\int_0^t(t-\tau)^{2\alpha-2}\mathbb{E}\|g(\tau,x_{\epsilon_n}(\tau-\epsilon_n))-g(\tau,x(\tau))\|^2d\tau\\&\le L\int_0^t(t-\tau)^{2\alpha-2}\mathbb{E}\|x_{\epsilon_n}(\tau-\epsilon_n)-x(\tau)\|^2d\tau\le C\eta.\end{aligned}\tag{2.43}$$

Therefore, (2.40) and (2.42)-(2.43) imply that x is the solution of (2.3). □

Theorem 2.10. *Assume that conditions of Theorem 2.9. Then for every $x_0\in L^2(\Omega;X)$ there exists a unique solution to (2.4).*

Proof. The proof of this theorem is similar to that of Theorem 2.9, so is omitted here.

2.3 Fractional stochastic lattice systems

Stochastic lattice system involving classical Itô SDEs have been investigated extensively, e.g.,[4, 14, 15, 39, 41, 42], but seemingly not for those with a fractional time derivative.

2.3.1 Statement of the problem, existence and uniqueness results

In this chapter, we consider the following stochastic lattice system with a Caputo fractional substantial time derivative,

$$\begin{cases} D_s^\alpha x_i(t) + (-1)^p \Delta^p x_i(t) + \lambda x_i(t) = f_i(x_i(t)) + g_i(t)\frac{dB(t)}{dt}, & t \geq 0, \\ x_i(0) = x_{0,i}, & i \in \mathbb{Z}, \end{cases} \tag{2.44}$$

where $\frac{1}{2} < \alpha < 1$, $\lambda \in \mathbb{R}$, p is any positive integer, $\Delta^p = \Delta \circ \cdots \circ \Delta$, p times. Here Δ denotes the discrete one-dimensional Laplace operator, which is defined by $\Delta x_i = x_{i+1} + x_{i-1} - 2x_i$. We also write $\partial^+ x_i = x_{i+1} - x_i$, $\partial^- x_i = x_i - x_{i-1}$ and define

$$\mathbb{D}^p := \begin{cases} \Delta^{\frac{p}{2}}, & p \text{ even}, \\ \partial^+ \Delta^{\frac{p-1}{2}}, & p \text{ odd}. \end{cases}$$

Recall that ℓ^2 is a Hilbert space of square summable real-valued bi-infinite sequences with the inner product $(x, y) = \sum_{i\in\mathbb{Z}} x_i y_i$ and norm $\|x\|^2 = \sum_{i\in\mathbb{Z}} x_i^2$ for all $x = (x_i)_{i\in\mathbb{Z}}$, $y = (y_i)_{i\in\mathbb{Z}} \in \ell^2$. Let $L^2(\Omega; \ell^2)$ denote the Hilbert space of all strongly measurable, square-integrable ℓ^2-valued random variables with the inner product and norm

$$\mathbb{E}(x, y) = \int_\Omega \sum_{i\in\mathbb{Z}} x_i y_i d\mathbb{P}, \qquad \|x\|_{L^2(\Omega;\ell^2)} = (\mathbb{E}\|x\|^2)^{\frac{1}{2}}, \qquad \forall x, y \in L^2(\Omega; \ell^2).$$

Then it is easy to see by induction that

$$\sum_{i\in\mathbb{Z}} \mathbb{E}|\mathbb{D}^p x_i(\cdot)|^2 \leq 4^p \mathbb{E}\|x(\cdot)\|^2, \tag{2.45}$$

for all $x = (x_i)_{i\in\mathbb{Z}} \in L^2(\Omega; \ell^2)$. Let $L^2_w(\Omega; \ell^2)$ be the space $L^2(\Omega; \ell^2)$ endowed with the weak topology. We shall use the similar notations $L^2(\Omega; \mathbb{R})$ and $L^2_w(\Omega; \mathbb{R})$.

We consider the following conditions:

(H_1) The operators $f : L^2(\Omega; \ell^2) \to L^2(\Omega; \ell^2)$ and $g : [0, +\infty) \to \ell^2$, given componentwise by $(f(x))_i = f_i(x_i)$ and $(g(t))_i = g_i(t)$, $i \in \mathbb{Z}$, are well defined and bounded, and g is measurable.

(H_2) The $f_i : L^2(\Omega; \mathbb{R}) \to L^2(\Omega; \mathbb{R})$ are sequentially weakly continuous in bounded sets.

Theorem 2.11. *Let (H_1)-(H_2) hold. Then for every $x_0 \in L^2(\Omega; \ell^2)$, the initial value problem (2.44) has at least one local solution $x(t)$ defined on $[0, T)$ for some $T > 0$.*

Proof. We can rewrite equation (2.44) as

$$\begin{cases} D_s^\alpha x(t) = F(x(t)) + g(t)\frac{dB(t)}{dt}, & t \geq 0, \frac{1}{2} < \alpha < 1, \\ x(0) = x_0, \end{cases} \tag{2.46}$$

where

$$F(x(t)) = f(x(t)) - (-1)^p \Delta^p x(t) - \lambda x(t).$$

First, we show that $F : L^2(\Omega; \ell^2) \to L^2(\Omega; \ell^2)$ is sequentially weakly continuous in bounded sets.

Let $x^n(t) \to x(t)$ in $L^2_w(\Omega; \ell^2)$, $\mathbb{E}\|x^n(t)\|^2 \le C^*$ for all $n \in \mathbb{N}$ and let $v \in L^2(\Omega; \ell^2)$ be arbitrary. Then for each i,

$$x_i^n(t) \to x_i(t) \qquad \text{in } L^2_w(\Omega; \mathbb{R}), \tag{2.47}$$

and for any $\epsilon > 0$, there exists $M(\epsilon) > 0$ such that

$$\mathbb{E} \sum_{|i| \ge M} |v_i|^2 < \epsilon^2. \tag{2.48}$$

By (H_2) and (2.47), it follows that there exists $N(\epsilon, M) > 0$ such that

$$\mathbb{E} \sum_{i \le M} (f_i(x_i^n(t)) - f_i(x_i(t))) v_i \le \epsilon, \qquad \text{if } n \ge N, \tag{2.49}$$

and by (H_1) we obtain there exists $C' > 0$ such that for all $n \in \mathbb{N}$,

$$\mathbb{E}\|f(x^n(t))\|^2 \le C', \qquad \mathbb{E}\|f(x(t))\|^2 \le C'. \tag{2.50}$$

Hence, we find for all $n \ge \mathbb{N}$ that

$$\begin{aligned} |\mathbb{E}(f(x^n(t)) - f(x(t)), v)| &\le \mathbb{E} \sum_{i \le M} (f_i(x_i^n(t)) - f_i(x_i(t))) v_i \\ &+ \left(\sqrt{\mathbb{E}\|f(x^n(t))\|^2} + \sqrt{\mathbb{E}\|f(x(t))\|^2} \right) \sqrt{\mathbb{E} \sum_{i \ge M} |v_i|^2} \le C_\epsilon. \end{aligned} \tag{2.51}$$

The result for the operator Δ^p can be proved similarly.

Since f is bounded, in view of (2.45) and (H_1), F and g are bounded. Then the results follows from Theorem 2.5. □

In order to show that every solution is globally defined, we now need the following estimates of solutions.

Lemma 2.12. *Assume (H_2) holds and also that*

(H_3) *The $f_i : L^2(\Omega; \mathbb{R}) \to L^2(\Omega; \mathbb{R})$ satisfy $\mathbb{E}|f_i(x)|^2 \le |k_{i,1}|^2 + k_2^2 \mathbb{E}|x|^2$ for all $x \in L^2(\Omega; \mathbb{R})$, where $k_1 = (k_{1,i})_{i \in \mathbb{Z}} \in \ell^2$ and $k_2 > 0$.*

(H_4) *There exists a positive constant M' such that for all $t \ge 0$,*

$$\int_0^t (t - \tau)^{2\alpha - 2} e^{-2\sigma(t-\tau)} \|g(\tau)\|^2 d\tau \le M'.$$

Then, every solution $x(\cdot)$ with $x(0) = x_0 \in L^2(\Omega; \ell^2)$ satisfies

$$\mathbb{E}\|x(t)\|^2 \leq (C\mathbb{E}\|x_0\|^2 e^{-\frac{\sigma}{2}t} + C)\left(1 + \sum_{n=1}^{\infty}\left(\frac{C\Gamma(\alpha)}{\sigma^\alpha}\right)^n\right), \quad \forall t \in [0, T^*),$$

where T^* is the maximal time of existence and σ is large enough to ensure the convergence of the above series.

Proof. By (H_3) and Itô's isometry, we deduce that

$$\begin{aligned}
\mathbb{E}|x_i(t)|^2 \leq{}& C\mathbb{E}|x_i(0)|^2 e^{-2\sigma t} + \frac{C}{\Gamma^2(\alpha)}\int_0^t (t-\tau)^{\alpha-1}e^{-\sigma(t-\tau)}d\tau \\
&\times \int_0^t (t-\tau)^{\alpha-1}e^{-\sigma(t-\tau)}\Big(\mathbb{E}|(\Delta^p x_i)(\tau)|^2 + \lambda^2\mathbb{E}|x_i(\tau)|^2 \\
&+ |k_{1,i}|^2 + k_2^2\mathbb{E}|x_i(\tau)|^2\Big)d\tau \\
&+ \frac{C}{\Gamma^2(\alpha)}\int_0^t (t-\tau)^{2\alpha-2}e^{-2\sigma(t-\tau)}|g_i(\tau)|^2 d\tau.
\end{aligned} \tag{2.52}$$

Then (2.9),(2.45), (2.52) and (H_4) imply that

$$\mathbb{E}\|x(t)\|^2 \leq C\mathbb{E}\|x_0\|^2 e^{-2\sigma t} + C\int_0^t (t-\tau)^{\alpha-1}e^{-\sigma(t-\tau)}\mathbb{E}\|x(\tau)\|^2 d\tau + C. \tag{2.53}$$

Hence,

$$e^{\sigma t}\mathbb{E}\|x(t)\|^2 \leq C\mathbb{E}\|x_0\|^2 e^{-\sigma t} + C\int_0^t (t-\tau)^{\alpha-1}e^{\sigma t}\mathbb{E}\|x(\tau)\|^2 d\tau + Ce^{\sigma t}. \tag{2.54}$$

From Corollary 2.4, we obtain

$$\mathbb{E}\|x(t)\|^2 \leq (C\mathbb{E}\|x_0\|^2 e^{-\frac{\sigma}{2}t} + C)\left(1 + \sum_{n=1}^{\infty}\left(\frac{C\Gamma(\alpha)}{\sigma^\alpha}\right)^n\right). \tag{2.55}$$

This completes the proof of the lemma. □

Thanks to the Theorem 2.11 and the Gronwall Lemma 2.12, it follows from Theorem 2.8 that

Corollary 2.13. *Let (H_2)-(H_4) hold, and let σ be sufficiently large. Then every local solution of initial value problem (2.44) is defined globally.*

Furthermore, we can prove the uniqueness of solutions for the fractional stochastic lattice system (2.44) when the nonlinearity satisfies the following Lipschitz property.

(H_5) $\mathbb{E}|f_i(x) - f_i(y)|^2 \leq L'\mathbb{E}|x - y|^2$ for any x, $y \in L^2(\Omega; \mathbb{R})$, where $L' > 0$.

Theorem 2.14. *Let (H_1) and (H_3)-(H_5) hold, and let σ be sufficiently large. Then for every $x_0 \in L^2(\Omega; \ell^2)$, there exists a unique globally defined solution x to (2.44).*

Proof. Define $F(x(t)) := f(x(t)) - (-1)^p \Delta^p x(t) - \lambda x(t)$. We first prove that $F : L^2(\Omega; \ell^2) \to L^2(\Omega; \ell^2)$ is Lipschitz. By (2.45) and (H_5), we have

$$\begin{aligned}\mathbb{E}\|F(x(t)) - F(y(t))\|^2 &= \sum_{i\in\mathbb{Z}} \mathbb{E}|F_i(x_i(t)) - F_i(y_i(t))|^2 \\ &\le \sum_{i\in\mathbb{Z}} \Big(3\mathbb{E}|f_i(x_i(t)) - f_i(y_i(t))|^2 + 3\mathbb{E}|\Delta^p (x_i(t) - y_i(t))|^2 \\ &\quad + 3\lambda^2 \mathbb{E}|x_i(t) - y_i(t)|^2 \Big) \\ &\le \sum_{i\in\mathbb{Z}} C\mathbb{E}|x_i(t) - y_i(t)|^2 = C\mathbb{E}\|x(t) - y(t)\|^2.\end{aligned} \tag{2.56}$$

Thus, using (2.45) and (H_3), we deduce that

$$\begin{aligned}\mathbb{E}\|F(x(t))\|^2 &\le \sum_{i\in\mathbb{Z}} \left(3\mathbb{E}|f_i(x_i(t))|^2 + 3\mathbb{E}|\Delta^p x_i(t)|^2 + 3\lambda^2\mathbb{E}|x_i(t)|^2\right) \\ &\le C \sum_{i\in\mathbb{Z}} (|k_{i,1}|^2 + k_2^2 \mathbb{E}|x_i(t)|^2 + \mathbb{E}|x_i(t)|^2 + \lambda^2 \mathbb{E}|x_i(t)|^2) \\ &\le C + C\mathbb{E}\|x(t)\|^2,\end{aligned} \tag{2.57}$$

and thus $F : L^2(\Omega; \ell^2) \to L^2(\Omega; \ell^2)$ is well defined and bounded. It follows from Theorem 2.10 that for any $x_0 \in L^2(\Omega; \ell^2)$, there exists a unique local solution to (2.44). Finally, Theorem 2.8 and Lemma 2.12 imply that the local solution of (2.44) can be extended globally. □

2.3.2 Asymptotic behavior

Mean-square asymptotic properties of random systems are frequently used in physics and engineering. Recently, mean-square random dynamical systems and their mean-square attractors were introduced by Kloeden and Lorenz [49]. The existence of a mean-square attractor was established in [49] as well as in [32, 81] for random dynamical systems generated by a mean-field SDE and a stochastic delay equation. These mean-square attractors are defined in terms of pullback convergence, which uses information about the system in the distant past. They usually say nothing about the future asymptotic behavior of the system, in which case we need to talk about forward attractors or attracting sets [50].

These results do not, however, apply in the present context because the fractional SDEs do not generate a pathwise cocycle mapping or a mean-square two-parameter semi-group. Nevertheless, we can establish the existence of a forward global attracting set in the weak topology for the fractional stochastic lattice model possibly without uniqueness of solutions.

Theorem 2.15. *Let (H_2)-(H_4) hold, and let σ be sufficiently large. Then*

(1) *for any bounded subset B of $L^2(\Omega;\ell^2)$, any sequence $\{t_n\}$ with $t_n \to +\infty$ as $n \to +\infty$, $\{x_0^n\} \in B$, and any sequence of solutions $\{x^n(\cdot)\}$ of problem (2.44) with $x^n(0) = x_0^n \in B$, the sequence $\{x^n(t_n)\}$ is relatively compact in $L^2_w(\Omega;\ell^2)$;*

(2) *for any bounded subset B of $L^2(\Omega;\ell^2)$, the set*

$$\omega_w(B) = \Big\{x : \exists t_n \to +\infty, x_0^n \in B \text{ and a sequence of solutions } x^n(\cdot) \text{ of problem (2.44) with } x^n(0) = x_0^n \in B \text{ such that } x^n(t_n) \to x \text{ in } L^2_w(\Omega;\ell^2)\Big\}$$

is nonempty, compact and attracts B in the weak topology;

(3) *the set*

$$A_w = \overline{\bigcup\{\omega_w(B) : \ B \subset L^2(\Omega;\ell^2), B \text{ bounded}\}}^w$$

is bounded in $L^2(\Omega;\ell^2)$, compact in the topology of $L^2_w(\Omega;\ell^2)$, and, moreover, is the minimal weakly closed set that attracts all bounded subsets of $L^2(\Omega;\ell^2)$ in the weak topology.

Proof. Since $L^2(\Omega;\ell^2)$ is reflexive, conclusion (1) follows from Lemma 2.12, and consequently $\omega_w(B)$ is nonempty and weakly compact.

Now we show that $\omega_w(B)$ attracts B in the weak topology. Assume, otherwise, there exist $\varepsilon_0 > 0$ and sequences $\{t_n\}$ with $t_n \to +\infty$ as $n \to +\infty$, $\{x_0^n\}$ with $x_0^n \in B$ and solutions $\{x^n(\cdot)\}$ of (2.44) with $x^n(0) = x_0^n$ such that

$$\mathrm{dist}_w(x^n(t_n), \omega_w(B)) > \varepsilon_0, \qquad \forall n \in \mathbb{N}, \tag{2.58}$$

where $dist_w(\cdot,\cdot)$ is in the sense of weak topology. Using conclusion (1), we obtain that $x^n(t_n)$ is relatively compact in $L^2_w(\Omega;\ell^2)$ and possesses at least one cluster point w. By the definition of $\omega_w(B)$, it is clear that $w \in \omega_w(B)$, but this contradicts (2.58).

From Lemma 2.12 and the conclusion (2), we see that A_w is bounded in $L^2(\Omega;\ell^2)$ and attracts all bounded subsets of $L^2(\Omega;\ell^2)$ in the weak topology. It is clear that A_w is compact in the topology of $L^2_w(\Omega;\ell^2)$.

Finally, we prove that A_w is the minimal weakly closed set attracting any bounded set $B \subset L^2(\Omega;\ell^2)$ in the weak topology. Indeed, if there is a weakly closed set A' which attracts any bounded set $B \subset L^2(\Omega;\ell^2)$ in the weak topology, then by the definition of $\omega_w(B)$, we deduce that $\omega_w(B) \subset A'$, and thus $\bigcup\{\omega_w(B)\} : B \subset L^2(\Omega;\ell^2), B$ bounded$\}$ belongs to A'. Since A' is weakly closed, we have

$$A_w = \overline{\bigcup\{\omega_w(B) : \ B \subset L^2(\Omega;\ell^2), B \text{ bounded }\}}^w \subset A'.$$

This completes the proof of Theorem 2.15. □

Theorem 2.16. *Assume that (H_1) and (H_3)-(H_5) hold, and let σ be sufficiently large. Then the solutions of (2.44) are uniformly asymptotically stable in the strong* *mean-square topology, i.e., for any bounded subset B of $L^2(\Omega;\ell^2)$ and $\eta > 0$, there exists a $T_\eta > 0$ such that*

$$\mathbb{E}\|x(t) - y(t)\|^2 \le \eta, \quad \textit{for all} \;\; t \ge T_\eta, \tag{2.59}$$

for any two solutios $x(t)$ and $y(t)$ of problem (2.44) corresponding to initial values x_0 and y_0 belong to B.

Proof. Let $x(t)$ and $y(t)$ be two solutions of problem (2.44) corresponding to initial values x_0 and y_0. Then by (2.45), (2.9) and (H_5), using Hölder's inequality, we obtain

$$\begin{aligned}
\mathbb{E}|x_i(t) - y_i(t)|^2 \le{} & C\mathbb{E}|x_i(0) - y_i(0)|^2 e^{-2\sigma t} + \frac{C}{\Gamma^2(\alpha)} \int_0^t (t-\tau)^{\alpha-1} e^{-\sigma(t-\tau)} d\tau \\
& \times \int_0^t (t-\tau)^{\alpha-1} e^{-\sigma(t-\tau)} \Big(\mathbb{E}|f_i(x_i(\tau)) \\
& - f_i(y_i(\tau))|^2 + \mathbb{E}|x_i(\tau) - y_i(\tau)|^2 \Big) d\tau \\
\le{} & C\mathbb{E}|x_i(0) - y_i(0)|^2 e^{-2\sigma t} + \frac{C}{\Gamma^2(\alpha)} \\
& \times \int_0^t (t-\tau)^{\alpha-1} e^{-\sigma(t-\tau)} \mathbb{E}|x_i(\tau) - y_i(\tau)|^2 d\tau.
\end{aligned} \tag{2.60}$$

Therefore,

$$\begin{aligned}
\mathbb{E}\|x(t) - y(t)\|^2 \le{} & C\mathbb{E}\|x_0 - y_0\|^2 e^{-2\sigma t} \\
& + \frac{C}{\Gamma^2(\alpha)} \int_0^t (t-\tau)^{\alpha-1} e^{-\sigma(t-\tau)} \mathbb{E}\|x(\tau) - y(\tau)\|^2 d\tau.
\end{aligned}$$

Let $w(t) = e^{\sigma t}\mathbb{E}\|x(t) - y(t)\|^2$, then we have

$$w(t) \le C\mathbb{E}\|x_0 - y_0\|^2 e^{-\sigma t} + C \int_0^t (t-\tau)^{\alpha-1} w(\tau) d\tau.$$

Applying Lemma 2.3, we have

$$\begin{aligned}
w(t) \le{} & C\mathbb{E}\|x_0 - y_0\|^2 e^{-\sigma t} + \int_0^t \sum_{n=1}^{\infty} \frac{(C\Gamma(\alpha))^n}{\Gamma(n\alpha)} (t-\tau)^{n\alpha-1} C\mathbb{E}\|x_0 - y_0\|^2 e^{-\sigma\tau} d\tau \\
\le{} & C\mathbb{E}\|x_0 - y_0\|^2 e^{-\sigma t} + C\mathbb{E}\|x_0 - y_0\|^2 \sum_{n=1}^{\infty} \frac{(C\Gamma(\alpha))^n}{\Gamma(n\alpha)} \int_0^t (t-\tau)^{n\alpha-1} d\tau \\
\le{} & C\mathbb{E}\|x_0 - y_0\|^2 e^{-\sigma t} + C\mathbb{E}\|x_0 - y_0\|^2 \sum_{n=1}^{\infty} \frac{(C\Gamma(\alpha) t^\alpha)^n}{\Gamma(n\alpha+1)} \\
\le{} & C\mathbb{E}\|x_0 - y_0\|^2 e^{-\sigma t} + C\mathbb{E}\|x_0 - y_0\|^2 E_{\alpha,1}(C\Gamma(\alpha) t^\alpha),
\end{aligned}$$

where

$$|E_{\alpha,1}(C\Gamma(\alpha)t^\alpha)| \le A_1 \exp(C\Gamma(\alpha)t^\alpha)^{\frac{1}{\alpha}} + \frac{A_2}{1+|C\Gamma(\alpha)t^\alpha|},$$

where A_1 and A_2 are positive constants. Hence

$$\begin{aligned}\mathbb{E}\|x(t)-y(t)\|^2 \le C\mathbb{E}\|x_0-y_0\|^2 e^{-2\sigma t} + C\mathbb{E}\|x_0-y_0\|^2 e^{-\sigma t}\frac{A_2}{1+|C\Gamma(\alpha)t^\alpha|} \\ + C\mathbb{E}\|x_0-y_0\|^2 A_1 \exp(((C\Gamma(\alpha))^{\frac{1}{2}}-\sigma)t),\end{aligned} \tag{2.61}$$

this implies that (2.59) and thus the proof is finished. □

Let $\bar{x}(t)$ be the solution of (2.44) in Theorem 2.16 with initial value $\bar{x}(0) = 0$. Then (2.61) gives

$$\mathbb{E}\|x(t)-\bar{x}(t)\|^2 \le C\mathbb{E}\|x_0\|^2 e^{-\sigma t}\left(e^{-\sigma t} + \frac{t^\alpha}{\Gamma^2(\alpha)\alpha}\right),$$

so the solution $\bar{x}(t)$ is mean-square exponentially asymptotically stable (as indeed is any other solution).

Suppose, in addition, that (H_2) also holds in Theorem 2.16. Then Theorem 2.15 also holds and the fractional stochastic lattice system (2.44) has a weak mean-square global attracting set A_w. In this case $A_w = \omega_w(0)$.

Chapter 3

Stochastic fractional impulsive differential equations with delay

There are numerous examples [10, 54, 72] of evolutionary systems that are subjected to rapid changes at certain instants in time. The interest in describing such processes by appropriate mathematical models, which are so-called differential equations with impulsive effects, mainly arose in recent years. In the simulations of such systems it is often convenient to neglect the durations of the rapid changes and to assume that the changes are represented by state jumps, see, e.g., [6, 12, 23, 29, 53]. Therefore, the theory of impulsive differential equations has become an active area due to its wide applications in several fields such as communications, mechanics, electrical engineering, medicine, biology, etc. For the basic theory of impulsive differential equations, we refer the readers to [6, 7, 9, 23, 30, 53, 63, 64, 68] and the references therein.

It is well-known that the deterministic models arising in mathematical finance, climate and weather derivatives, often fluctuate due to the presence of some kind of noise. Hence, the stochastic models driven by a Brownian motion or fractional Brownian motion have attracted the researchers great interest (see, e.g., [9, 12, 13, 28, 66, 65, 77]). On the other hand, the fractional differential equations which are presented in the modeling of many real problems (e.g. in physical phenomena) have been the object of extensive study in order to analyze not only non-random fractional phenomena in physics, but also stochastic processes driven by a fractional Brownian motion, see [3, 23, 55, 58, 57, 70] and references therein. Therefore, our first aim of this chapter is to address the issue of existence, uniqueness and asymptotic behavior of mild solutions to the following fractional stochastic impulsive differential equations with infinite delay,

$$\begin{cases} D_t^\alpha x(t) = Ax(t) + f(t, x_t) + g(t, x_t)\frac{dB(t)}{dt} + h(t)\frac{dB_Q^H(t)}{dt}, \quad t \geq 0, \\ \qquad t \neq t_k, \quad \frac{1}{2} < \alpha < 1, \\ \Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \quad k = 1, 2, \cdots, \\ x(t) = \phi(t), \quad t \in (-\infty, 0], \end{cases} \tag{3.1}$$

where D_t^α is the Caputo fractional derivative of order $\frac{1}{2} < \alpha < 1$.

The theory of local existence of solutions of impulsive non-stochastic or impulsive stochastic differential equations has experienced a good development up to date. In [57], the local existence of mild solutions for neutral impulsive stochastic integro-differential equations has been investigated. Subsequently, using a Banach fixed point theorem, the existence and uniqueness of local solutions for fractional impulsive differential equations with infinite delay are established in [29]. Henceforth, studies of existence of solutions for impulsive stochastic differential equations have been launched as can be seen, for instance, [58, 70, 72, 73]. Notice that, most of the previous research concerns the case of the local existence of solutions for such kind of equations, there has been little regarding the case of the global existence of solutions except for [23, 80], in which the global existence of solutions for fractional impulsive differential equations was obtained.

Motivated by the work in [80], in this chapter, the local and global existence and uniqueness of mild solutions to problem (3.1) are studied by means of a fixed point theorem, and the properties of α-order fractional solution operator $T_\alpha(t)$ and resolvent operator $S_\alpha(t)$. Moreover, the exponential decay to zero of the mild solutions to problem (3.1) is also proved. However, the lack of compactness of the α-order resolvent operator $S_\alpha(t)$ does not allow us to establish the existence and structure of attracting sets, which is a key concept for understanding the dynamical properties. To this respect, in [10] the authors already studied the existence of attractors for impulsive non-autonomous dynamical systems when $\alpha = 1$, since the operator generated by the infinitesimal generator A is a semigroup (see [22, 63] for more details).

Fortunately, to overcome this difficulty in our fractional situation, we can take advantage of the compactness of α-order fractional solution operator $T_\alpha(t)$ which has been proven in [70, 78], and this is one of our motivations to analyze the existence (and eventual uniqueness) of mild solutions, and the global forward attracting set of the following fractional stochastic impulsive differential equations with infinite delay,

$$\begin{cases} D_t^\alpha x(t) = Ax(t) + I_t^{1-\alpha} f(t, x_t) + [I_t^{1-\alpha} g(t, x_t)] \frac{dB(t)}{dt} + [I_t^{1-\alpha} h(t)] \frac{dB_Q^H(t)}{dt}, \\ \qquad t \geq 0, \quad t \neq t_k, \quad 0 < \alpha < 1, \\ \Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \quad t = t_k, \quad k = 1, 2, \cdots, \\ x(t) = \phi(t), \quad t \in (-\infty, 0], \end{cases} \tag{3.2}$$

where D_t^α is the Caputo fractional derivative of order $0 < \alpha < 1$, $I_t^{1-\alpha}$ is the $(1-\alpha)$-order fractional integral operator.

For both models (3.1) and (3.2), $x(\cdot)$ takes the value in the separable Hilbert sapce $\mathbb{H}$. $A : D(A) \subseteq \mathbb{H} \to \mathbb{H}$ is the infinitesimal generator of an α-order fractional compact and analytic operator $T_\alpha(t)(t \geq 0)$, and let $\mathbb{K}$ be another separable Hilbert space. As usual, $B(t)$ and $B_Q^H(t)$ denote, respectively, a $\mathbb{K}$-valued Q-cylindrical Brownian motion and fractional Brownian motion defined on a filtered complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. The fixed time t_k, where the impulses take place, satisfy $0 = t_0 < t_1 < \cdots < t_k \to +\infty$ as $k \to \infty$.

We consider the functions $x_t : (-\infty, 0] \to L^2(\Omega; \mathbb{H})$ defined by $x_t(\theta) = x(t + \theta)$, for all $\theta \in (-\infty, 0]$, which are continuous everywhere except for countable points t_k $(k \in \mathbb{N})$, where impulses take place, at which there exist $x(t_k^+)$ and $x(t_k^-)$, and $x(t_k) = x(t_k^-)$ (for each k, $x(t_k^+) = \lim_{h\to 0} x(t_k + h)$ and $x(t_k^-) = \lim_{h\to 0} x(t_k - h)$ represent the right-hand and left-hand limits of $x(t)$ at $t = t_k$ respectively).

The nonlinear maps $f : \mathbb{R}^+ \times \mathcal{PC} \to \mathbb{H}$, $g : \mathbb{R}^+ \times \mathcal{PC} \to \mathcal{L}(\mathbb{K}, \mathbb{H})$ and $h : \mathbb{R}^+ \to \mathcal{L}(\mathbb{K}, \mathbb{H})$ that satisfies $hQ^{\frac{1}{2}}$ is a Hilbert-Schmidt operator, are appropriate functions which will be specified later, $I_k \in C(\mathbb{H}, \mathbb{H})$ for each $k \in \mathbb{N}$.

It is worth noticing that, the nonlinear terms of the right hand side of problem (3.2) have higher regularity because of the integral operator $I_t^{1-\alpha}$. For this model, by a fractional variation of constants formula, the mild solution to problem (3.2) is only involving the α-order fractional solution operator $T_\alpha(t)$ $(t \geq 0)$ which is compact. We emphasize that the main advantage of model (3.2) is that we can extend the results of model (3.1) with $\alpha \in (\frac{1}{2}, 1)$ to model (3.2) with $\alpha \in (0, 1)$. Also, thanks to the good properties of the α-order fractional solution operator (see Lemma 3.6), we are able to prove the existence of attracting sets and provide interesting information about the dynamics of model (3.2).

3.1 Setting of the phase space, preliminaries

Notice that, in Section 1.2.2, we introduced some basic properties of Brownian motion and fractional Brownian motion, thus in this chapter, we assume that $B(t)$ and $B_Q^H(t)$ appearing in models (3.1) and (3.2) satisfy those hypotheses.

Now we recall some basic definitions concerning the sectorial operator A, α-order fractional solution operator $T_\alpha(t)$ and α-resolvent family $S_\alpha(t)$.

Definition 3.1. *A linear closed densely defined operator A is said to be sectorial if there are constants $\omega \in \mathbb{R}$, $\theta \in [\frac{\pi}{2}, \pi]$, $M > 0$, such that the following two conditions are satisfied,*
(1) $\sigma(A) \subset \sum_{\omega,\theta} = \{\lambda \in \mathbb{C} : \lambda \neq \omega, |\arg(\lambda - \omega)| < \theta\}$;

(2) $\|\mathbb{R}(\lambda, A)\| \leq \frac{M}{|\lambda-\omega|}$, $\lambda \in \sum_{\omega,\theta}$.

Definition 3.2. *Let A be a closed and linear operator with domain $D(A)$ defined on $\mathbb{H}$. Let $\rho(A)$ be the resolvent set of A. We say that A is the generator of an α-resolvent family if there exist $\omega \geq 0$ and a strongly continuous function $S_\alpha : \mathbb{R}^+ \to \mathcal{L}(\mathbb{H})$, such that $\{\lambda^\alpha : Re(\lambda) > \omega\} \subset \rho(A)$ and*

$$(\lambda^\alpha I - A)^{-1} y = \int_0^\infty e^{-\lambda t} S_\alpha(t) y dt, \quad Re(\lambda) > \omega, \quad y \in \mathbb{H},$$

where $S_\alpha(t)$ is called the α-resolvent family generated by A.

Definition 3.3. *A solution operator $T_\alpha(t)$ of (3.1)or (3.2) is called analytic if $T_\alpha(t)$ admits an analytic extension to a sector $\sum_{\theta_0} := \{\lambda \in \mathbb{C}\backslash\{0\} : |\arg \lambda| < \theta_0\}$ for some $\theta_0 \in (0, \frac{\pi}{2}]$. An analytic solution operator is said to be of analyticity type (ω_0, θ_0) if for each $\theta < \theta_0$ and $\omega > \omega_0$, there is a positive constant $M = M(\omega, \theta)$ such that*

$\|T_\alpha(t)\| \le Me^{\omega Re(t)}$, $t \in \sum_\theta := \{t \in \mathbb{C}\backslash\{0\} : |\arg t| < \theta\}$. *Denote* $\mathbb{A}^\alpha(\omega_0, \theta_0) := \{\mathbb{A}$ *generates analytic solution operators* $T_\alpha(t)$ *of type* $(\omega_0, \theta_0)\}$.

Definition 3.4. *A family* $T_\alpha(t) : \mathbb{R}^+ \to \mathcal{L}(\mathbb{H})$ *is called an* α*-order fractional solution operator generated by* A *if the following conditions are satisfied:*
(1) $T_\alpha(t)$ *is strongly continuous for* $t \ge 0$ *and* $T_\alpha(0) = I$*;*
(2) $T_\alpha(t)D(A) \subset D(A)$ *and* $AT_\alpha(t)x = T_\alpha(t)Ax$ *for all* $x \in D(A)$ *and* $t \ge 0$*;*
(3) *for all* $x_0 \in D(A)$*,* $T_\alpha(t)x$ *is a solution of the following operator equation*

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} Ax(s)ds, \quad t \ge 0.$$

Definition 3.5. *An* α*-order fractional solution operator* $T_\alpha(t)(t \ge 0)$ *is called compact if for every* $t > 0$*,* $T_\alpha(t)$ *is a compact operator.*

Arguing as in the proof of Lemma 3.8 in [34], we obtain the continuity of the α-order fractional solution operator $T_\alpha(t)$ and α-resolvent family $S_\alpha(t)$ in the uniform operator topology for $t > 0$.

Lemma 3.6. *Assume* $A \in \mathbb{A}^\alpha(\omega_0,\theta_0)$*, and that the* α*-order fractional solution operator* $T_\alpha(t)(t > 0)$ *and the* α*-resolvent family* $S_\alpha(t)(t > 0)$ *are compact. Then the following properties are fulfilled:*

(1) $\lim_{h\to 0} \|T_\alpha(t+h) - T_\alpha(t)\| = 0$, $\lim_{h\to 0} \|S_\alpha(t+h) - S_\alpha(t)\| = 0$ *for* $t > 0$;

(2) $\lim_{h\to 0^+} \|T_\alpha(t+h) - T_\alpha(h)T_\alpha(t)\| = 0$, $\lim_{h\to 0^+} \|S_\alpha(t+h) - S_\alpha(h)S_\alpha(t)\| = 0$ *for* $t > 0$;

(3) $\lim_{h\to 0^+} \|T_\alpha(t) - T_\alpha(h)T_\alpha(t-h)\| = 0$, $\lim_{h\to 0^+} \|S_\alpha(t) - S_\alpha(h)S_\alpha(t-h)\| = 0$ *for* $t > 0$.

Lemma 3.7. *If* $A \in \mathbb{A}^\alpha(\omega_0,\theta_0)$*, then for every* $\omega > \omega_0$ $(\omega_0 \in \mathbb{R}^+)$*, there exists a constant* $M = M(\omega, \theta)$ *such that, for all* $t > 0$*,*

$$\|T_\alpha(t)\| \le Me^{\omega t} \quad \text{and} \quad \|S_\alpha(t)\| \le Me^{\omega t}(1 + t^{\alpha-1}). \tag{3.3}$$

Furthermore, let $M_T := \sup_{0\le t\le T} \|T_\alpha(t)\|$*,* $N_T := \sup_{0\le t\le T} Me^{\omega t}(1+t^{1-\alpha})$*. Then we obtain that*

$$\|T_\alpha(t)\| \le M_T \quad \text{and} \quad \|S_\alpha(t)\| \le N_T t^{\alpha-1}. \tag{3.4}$$

Next we are going to establish the existence of mild solutions to fractional impulsive stochastic differential equation (3.1) and (3.2). Before doing this, we first present the abstract phase space $\mathcal{PC}$.

Let $L^2(\Omega; \mathbb{H})$ denote the Banach space of all strongly-measurable, square-integrable $\mathbb{H}$-valued random variables equipped with the norm $\|u(\cdot)\|^2_{L^2} = \mathbb{E}\|u(\cdot)\|^2$, where the expectation $\mathbb{E}$ is defined by $\mathbb{E}u = \int_\Omega u(\cdot)d\mathbb{P}$. The abstract phase space $\mathcal{PC}$ is defined by

$$\mathcal{PC} = \Big\{\xi : (-\infty, 0] \to L^2(\Omega; \mathbb{H}) \text{ is } \mathcal{F}_0\text{-adapted and continuous except in at most a countable number of points } \{\theta_k\}, \text{ at which there exist } \xi(\theta_k^+) \text{ and } \xi(\theta_k^-) \text{ with } \xi(\theta_k) = \xi(\theta_k^-), \text{ and } \sup_{\theta\in(-\infty,0]} e^{\gamma\theta}\mathbb{E}\|\xi(\theta)\|^2 < \infty\Big\},$$

for some fixed parameter $\gamma > 0$. If $\mathcal{PC}$ is endowed with the norm

$$\|\xi\|_{\mathcal{PC}} = \left(\sup_{\theta \in (-\infty, 0]} e^{\gamma\theta} \mathbb{E}\|\xi(\theta)\|^2 \right)^{\frac{1}{2}}, \qquad \xi \in \mathcal{PC},$$

then, $(\mathcal{PC}, \|\cdot\|_{\mathcal{PC}})$ is a Banach space.

Now, we introduce the definitions of mild solutions to problems (3.1) and (3.2).

Definition 3.8. *Set $\mathcal{F}_t = \mathcal{F}_0$ for all $t \in (-\infty, 0]$. An $\mathcal{F}_t$-adapted stochastic process $x : (-\infty, T] \to \mathbb{H}$ is called a mild solution of equation (3.1) if $x(t) = \phi(t)$ for $t \in (-\infty, 0]$ with $\phi \in \mathcal{PC}$, and for $t \in [0, T]$, $x(t)$ satisfies the integral equation*

$$x(t) = \begin{cases} T_\alpha(t)\phi(0) + \int_0^t S_\alpha(t-s) f(s, x_s) ds + \int_0^t S_\alpha(t-s) g(s, x_s) dB(s) \\ + \int_0^t S_\alpha(t-s) h(s) dB_Q^H(s), \quad t \in [0, t_1], \\ T_\alpha(t - t_1)(I_1(x(t_1^-)) + x(t_1^-)) + \int_{t_1}^t S_\alpha(t-s) f(s, x_s) ds \\ + \int_{t_1}^t S_\alpha(t-s) g(s, x_s) dB(s) + \int_{t_1}^t S_\alpha(t-s) h(s) dB_Q^H(s), \quad t \in (t_1, t_2], \\ \cdots, \\ T_\alpha(t - t_m)(I_m(x(t_m^-)) + x(t_m^-)) + \int_{t_m}^t S_\alpha(t-s) f(s, x_s) ds \\ + \int_{t_m}^t S_\alpha(t-s) g(s, x_s) dB(s) + \int_{t_m}^t S_\alpha(t-s) h(s) dB_Q^H(s), \quad t \in (t_m, T], \end{cases}$$

where $t_m = \max\{t_k,\ t_k < T,\ k = 0, 1, 2, \cdots\}$,

$$T_\alpha(t) = E_{\alpha,1}(At^\alpha) = \frac{1}{2\pi i} \int_{\Gamma_\theta} e^{\lambda t} \frac{\lambda^{\alpha-1}}{\lambda^\alpha - A} d\lambda,$$

$$S_\alpha(t) = t^{\alpha-1} E_{\alpha,\alpha}(At^\alpha) = \frac{1}{2\pi i} \int_{\Gamma_\theta} e^{\lambda t} \frac{1}{\lambda^\alpha - A} d\lambda,$$

here the integral contour Γ_θ is oriented counter-clockwise.

The definition of mild solution to problem (3.2) is almost the same than Definition 3.8, to be precise, we present the definition as below.

Definition 3.9. *Let $\mathcal{F}_t = \mathcal{F}_0$ for all $t \in (-\infty, 0]$, and let $\phi \in \mathcal{PC}$ be an initial value. An $\mathcal{F}_t$-adapted stochastic process $x : (-\infty, T] \to \mathbb{H}$ is said to be a mild solution of the equation (3.2) if $x(t) = \phi(t)$ for $t \in (-\infty, 0]$, and for $t \in [0, T]$, $x(t)$ satisfies the* *integral equation*

$$x(t) = \begin{cases} T_\alpha(t)\phi(0) + \int_0^t T_\alpha(t-s)f(s,x_s)ds + \int_0^t T_\alpha(t-s)g(s,x_s)dB(s) \\ + \int_0^t T_\alpha(t-s)h(s)dB_Q^H(s), \quad t \in [0,t_1], \\ T_\alpha(t-t_1)(x(t_1^-) + I_1(x(t_1^-))) + \int_{t_1}^t T_\alpha(t-s)f(s,x_s)ds \\ + \int_{t_1}^t T_\alpha(t-s)g(s,x_s)dB(s) + \int_{t_1}^t T_\alpha(t-s)h(s)dB_Q^H(s), \quad t \in (t_1,t_2], \\ \cdots, \\ T_\alpha(t-t_m)(x(t_m^-) + I_m(x(t_m^-))) + \int_{t_m}^t T_\alpha(t-s)f(s,x_s)ds \\ + \int_{t_m}^t T_\alpha(t-s)g(s,x_s)dB(s) + \int_{t_m}^t T_\alpha(t-s)h(s)dB_Q^H(s), \quad t \in (t_m,T], \end{cases}$$

where t_m and $T_\alpha(t)$ are the same as in Definition 3.8.

In order to establish the main result, we impose the following conditions.

(H_1) $f : [0,\infty) \times \mathcal{PC} \to \mathbb{H}$, $g : [0,\infty) \times \mathcal{PC} \to \mathcal{L}(\mathbb{K},\mathbb{H})$ are continuous and there exist two functions $l_1(\cdot)$, $l_2(\cdot) \in L^\infty(\mathbb{R}^+;\mathbb{R}^+)$ such that

$$\mathbb{E}\|f(t,x) - f(t,y)\|^2 \le l_1(t)\|x-y\|^2_{\mathcal{PC}}, \quad \mathbb{E}\|g(t,x) - g(t,y)\|^2 \le l_2(t)\|x-y\|^2_{\mathcal{PC}},$$

for every x, $y \in \mathcal{PC}$, and almost every $t > 0$. Moreover, $g(t,\cdot)$ is measurable relative to $\mathcal{F}_t$ for all $t \in [0,\infty)$ satisfying $\int_0^\infty \mathbb{E}\|g(t,x)\|^2 dt < \infty$.

(H_2) $h : [0,\infty) \to \mathcal{L}(\mathbb{K},\mathbb{H})$ satisfying $hQ^{\frac{1}{2}}$ is a Hilbert-Schmidt operator, and there exist $q \ge 1$ and $\Lambda > 0$ such that, for every $t \in [0,T]$,

$$\int_0^t \|h(s)\|_Q^{2q} ds < \Lambda.$$

(H_3) The functions $I_k : L^2(\Omega;\mathbb{H}) \to L^2(\Omega;\mathbb{H})$ are continuous for each $k \in \mathbb{N}$, and there exist two positive constants b_1, $b_2 > 0$ such that

$$\mathbb{E}\|I_k(x)\|^2 \le b_1\mathbb{E}\|x\|^2 + b_2, \quad \text{for all } \; x \in L^2(\Omega;\mathbb{H}).$$

There exists a positive constant N such that, for all $k \in \mathbb{N}$,

$$\mathbb{E}\|I_k(x) - I_k(y)\|^2 \le N\mathbb{E}\|x-y\|^2, \quad \text{for all } \; x,y \in L^2(\Omega;\mathbb{H}).$$

(H_3') The functions $I_k : L^2(\Omega;\mathbb{H}) \to L^2(\Omega;\mathbb{H})$ are linear and continuous for each $k \in \mathbb{N}$, and there exists $N_k > 0$ with $\sum_{k=1}^\infty N_k < +\infty$, such that

$$\mathbb{E}\|I_k(x)\|^2 \le N_k\mathbb{E}\|x\|^2, \quad \text{for all } \; x \in L^2(\Omega;\mathbb{H}).$$

Notice that these assumptions imply that $N_k \to 0$ as $k \to +\infty$, and there exists a positive constant N such that

$$\mathbb{E}\|I_k(x)\|^2 \le N\mathbb{E}\|x\|^2, \quad \text{for all } \; x \in L^2(\Omega;\mathbb{H}).$$

(H_4) $\beta = \inf_{k\in\mathbb{N}}\{t_k - t_{k-1}\} > 0$, $\eta = \sup_{k\in\mathbb{N}}\{t_k - t_{k-1}\} < \infty$.

Remark 3.10. *Comparing assumptions (H_3) with (H_3'), obviously condition (H_3') is more restrictive, as we need this condition to ensure the existence of attracting set for problem (3.2).*

3.2 Existence results

Theorem 3.11. *Assume (H_1)-(H_4) hold, let $A \in \mathbb{A}^\alpha(\omega_0, \theta_0)$ with $\theta_0 \in (0, \frac{\pi}{2}]$ and $\omega_0 \in \mathbb{R}^+$, and assume that the α-order fractional solution operator $T_\alpha(t)$ $(t > 0)$ and the α-resolvent family $S_\alpha(t)$ $(t > 0)$ are compact. Assume also that there exist two constants p and q which satisfy $\frac{1}{p} + \frac{1}{q} = 1$, where $1 < p < \frac{1}{2(1-\alpha)}$. Then, for every initial value $\phi \in \mathcal{PC}$ and every $T > 0$, the problem (3.1) has at least one mild solution defined on $(-\infty, T]$.*

Proof. We start the proof by defining an abstract phase space $\mathcal{PC}^T$ as follows: for a fixed $T > 0$,

$$\mathcal{PC}^T = \Big\{ x(\cdot,\cdot) : (-\infty, T] \times \Omega \to \mathbb{H} \text{ such that } x(t,\cdot) \text{ is } \mathcal{F}_t\text{-adapted}, \quad x(t,\cdot) \in L^2(\Omega; \mathbb{H}) \text{ for all } t \le T, \quad x|_{J_k} \in C(J_k; L^2(\Omega;\mathbb{H})) \text{ and } \sup_{t\in(-\infty,T]} e^{\gamma t}\mathbb{E}\|x(t)\|^2 < \infty \Big\},$$

where $x|_{J_k}$ is the restriction of x to $J_k = (t_k, t_{k+1}]$, $k \in \mathbb{N}$. Then the abstract space $\mathcal{PC}^T$ endowed with the norm

$$\|x\|_{\mathcal{PC}^T} = \left(\sup_{t\in(-\infty,T]} e^{\gamma t}\mathbb{E}\|x(t)\|^2 \right)^{\frac{1}{2}}, \qquad x \in \mathcal{PC}^T$$

is a Banach space. Notice that, when considering $T = 0$, we have $\mathcal{PC}^T = \mathcal{PC}^0 = \mathcal{PC}$.

Now, for $\phi \in \mathcal{PC}$, we define

$$\mathcal{PC}^T_\phi = \{x \in \mathcal{PC}^T : \quad x(s) = \phi(s), \quad s \le 0\}.$$

It is clear that $\mathcal{PC}^T_\phi$ is a closed subset of $\mathcal{PC}^T$, and consequently, it is a complete metric subspace of $\mathcal{PC}^T$.

In order to simplify our presentation, let us abbreviate $\|l_i\|_\infty$ by $l_i (i = 1, 2)$ according to the fact $l_i(\cdot) \in L^\infty(\mathbb{R}^+, \mathbb{R}^+)$, and we write $\mathcal{PC}_\phi$ instead of $\mathcal{PC}^T_\phi$ for a fixed $T > 0$, when no confusion is possible.

Let us pick $\psi \in \mathcal{PC}_\phi$ and define

$$\begin{cases} x^0(t) = \psi(t), \\ x^n(t) = \chi_{(-\infty,0]}(t)\psi(t) + \sum_k \chi_{(t_k,t_{k+1}]}(t)\Big\{ t(t-t_k)(I_k(x^{n-1}(t_k^-)) + x^{n-1}(t_k^-)) \\ \quad + \int_{t_k}^t S_\alpha(t-s)f(s,x_s^{n-1})ds + \int_{t_k}^t S_\alpha(t-s)g(s,x_s^{n-1})dB(s) \\ \quad + \int_{t_k}^t S_\alpha(t-s)h(s)dB_Q^H(s)\Big\}, \quad t \in (-\infty, T], \quad k = 0,1,2\cdots, \end{cases} \tag{3.5}$$

where $I_0 = 0$ and $t_k < T$, χ is a characteristic function.

To ensure the existence of mild solutions, we split the proof into several steps.

Step 1. For all $n \in \mathbb{N}$, $x^n(\cdot) \in \mathcal{PC}_\phi$.

First, we claim that $x^n(\cdot)$ is $\mathcal{F}_t$-adapted for all $n \in \mathbb{N}$. Obviously, $x^0(t) = \psi(t) \in \mathcal{PC}_\phi$ implies that $x^0(t)$ is $\mathcal{F}_t$-adapted for all $t \in (-\infty, T]$. With the help of the continuity of f and I_k, as well as the fact that the limit of measurable function is still a measurable function, it is easy to see that $I_k(x^0(t_k^-))$ and $\int_{t_k}^t f(s,x_s^0)ds$ are $\mathcal{F}_t$-adapted. In addition, the stochastic integral $\int_{t_k}^t g(s,x_s^0)dB(s)$ is $\mathcal{F}_t$-adapted according to Definition 5.3 in [61]. For the last term $\int_{t_k}^t h(s)dB_Q^H(s)$, which is $\mathcal{F}_t$-adapted naturally. In conclusion, thanks to the Picard iterations technique, we obtain that $x^1(t)$ is also $\mathcal{F}_t$-adapted. By induction, $x^n(t)$ is $\mathcal{F}_t$-adapted for all $t \in (-\infty, T]$ and $n \in \mathbb{N}$.

Next, we want to prove $x^n(t,\cdot) \in L^2(\Omega;\mathbb{H})$. Since $x^n(t) = \psi(t)$ on $(-\infty,0]$, then $x^n(t) = \phi(t)$ for $t \in (-\infty, 0]$, and as $\phi \in \mathcal{PC}$, then we have $x^n(t,\cdot) \in L^2(\Omega;\mathbb{H})$ for all $t \in (-\infty,0]$. Furthermore, for every $t \in [0,t_1]$, by (1.6), (1.9), (3.4), (H_1)-(H_4) and Hölder's inequality,

$$\begin{aligned} \mathbb{E}\|x^1(t)\|^2 &\le 4M_T^2\|\psi\|_{\mathcal{PC}_\phi}^2 + 8N_T^2(\eta l_1 + Tr(Q)l_2)\|\psi\|_{\mathcal{PC}_\phi}^2 \frac{t^{2\alpha-1}}{2\alpha-1} \\ &\quad + 8N_T^2(\eta \sup_{s\in[0,t_1]}\|f(s,0)\|^2 + Tr(Q)\sup_{s\in[0,t_1]}\|g(s,0)\|^2)\frac{t^{2\alpha-1}}{2\alpha-1} \\ &\quad + 4N_T^2 cH(2H-1)t^{2H-1}\int_0^t (t-s)^{2\alpha-2}\|h(s)\|_Q^2 ds \\ &:= 4M_T^2\|\psi\|_{\mathcal{PC}_\phi}^2 + C_1\|\psi\|_{\mathcal{PC}_\phi}^2\frac{t^{2\alpha-1}}{2\alpha-1} + C_2\frac{t^{2\alpha-1}}{2\alpha-1} + C_3 t^{2H-1}, \end{aligned} \tag{3.6}$$

where we have used the notation

$$C_1 = 8N_T^2(\eta l_1 + Tr(Q)l_2), \quad C_2 = 8N_T^2(\eta \sup_{s\in[0,T]}\|f(s,0)\|^2 + Tr(Q)\sup_{s\in[0,T]}\|g(s,0)\|^2),$$

and

$$C_3 \le 4N_T^2 cH(2H-1)\Big(\frac{\eta^{2(\alpha-1)p+1}}{2(\alpha-1)p+1}\Big)^{\frac{1}{p}}\Big(\int_0^t \|h(s)\|_Q^{2q}ds\Big)^{\frac{1}{q}} < \infty.$$

Notice that, if $t+\theta \in [0,t_1]$ (where $\theta \in (-\infty,0]$), in view of $\gamma > 0$, then it follows that

$$e^{\gamma\theta}\mathbb{E}\|x^1(t+\theta)\|^2 \leq 4M_T^2\|\psi\|^2_{\mathcal{PC}_\phi} + C_1\|\psi\|^2_{\mathcal{PC}_\phi}\frac{t^{2\alpha-1}}{2\alpha-1} + C_2\frac{t^{2\alpha-1}}{2\alpha-1} + C_3 t^{2H-1}, \quad (3.7)$$

if $t+\theta < 0$, we have

$$e^{\gamma\theta}\mathbb{E}\|x^1(t+\theta)\|^2 \leq e^{-\gamma t}e^{\gamma(t+\theta)}\mathbb{E}\|\psi(t+\theta)\|^2 \leq e^{-\gamma t}\|\psi\|^2_{\mathcal{PC}_\phi}. \quad (3.8)$$

Combining (3.7) and (3.8),

$$\begin{aligned}\|x_t^1\|^2_{\mathcal{PC}} &= \sup_{\theta\in(-\infty,0]} e^{\gamma\theta}\mathbb{E}\|x^1(t+\theta)\|^2 \\ &\leq \left(\|\psi\|^2_{\mathcal{PC}_\phi} + C_2\frac{\eta^{2\alpha-1}}{2\alpha-1} + C_3\eta^{2H-1}\right) \times \sum_{i=0}^{1}\left(4M_T^2 + C_1\frac{\eta^{2\alpha-1}}{2\alpha-1}\right)^i.\end{aligned}$$

By induction on n, for $t \in [0,t_1]$, we derive that

$$\|x_t^n\|^2_{\mathcal{PC}} \leq \left(\|\psi\|^2_{\mathcal{PC}_\phi} + C_2\frac{\eta^{2\alpha-1}}{2\alpha-1} + C_3\eta^{2H-1}\right) \times \sum_{i=0}^{n}\left(4M_T^2 + C_1\frac{\eta^{2\alpha-1}}{2\alpha-1}\right)^i. \quad (3.9)$$

For $t \in (t_1, t_2 \wedge T]$, similar to (3.6). By (1.6), (1.9), (3.4), (H_1)-(H_4) and Hölder's inequality,

$$\begin{aligned}\mathbb{E}\|x^1(t)\|^2 &\leq 8M_T^2((b_1+1)\|\psi\|^2_{\mathcal{PC}_\phi} + b_2) + C_1\frac{(t-t_1)^{2\alpha-1}}{2\alpha-1}\|\psi\|^2_{\mathcal{PC}_\phi} \\ &\quad + C_2\frac{(t-t_1)^{2\alpha-1}}{2\alpha-1} + C_3(t-t_1)^{2H-1} \\ &\leq \left(\|\psi\|^2_{\mathcal{PC}_\phi} + 8M_T^2 b_2 + C_2\frac{(t-t_1)^{2\alpha-1}}{2\alpha-1} + C_3(t-t_1)^{2H-1}\right) \\ &\quad \times \sum_{i=0}^{1}\left(8M_T^2(b_1+1) + C_1\frac{(t-t_1)^{2\alpha-1}}{2\alpha-1}\right)^i.\end{aligned} \quad (3.10)$$

Using the same argument as in (3.7) and (3.8), together with (3.9) and (3.10), for $t \in (t_1, t_2 \wedge T]$, we have the following estimate,

$$\begin{aligned}\|x_t^1\|^2_{\mathcal{PC}} \leq &\left(\|\psi\|^2_{\mathcal{PC}_\phi} + 8M_T^2 b_2 + C_2\frac{\eta^{2\alpha-1}}{2\alpha-1} + C_3\eta^{2H-1}\right) \\ &\times \sum_{i=0}^{1}\left(8M_T^2(b_1+1) + C_1\frac{\eta^{2\alpha-1}}{2\alpha-1}\right)^i.\end{aligned}$$

By induction on n, for $t \in (t_1, t_2 \wedge T]$, we deduce

$$\begin{aligned}\|x_t^n\|^2_{\mathcal{PC}} \leq &\left(\|\psi\|^2_{\mathcal{PC}_\phi} + 8M_T^2 b_2 + C_2\frac{\eta^{2\alpha-1}}{2\alpha-1} + C_3\eta^{2H-1}\right) \\ &\times \sum_{i=0}^{n}\left(8M_T^2(b_1+1) + C_1\frac{\eta^{2\alpha-1}}{2\alpha-1}\right)^i.\end{aligned} \quad (3.11)$$

In a similar way, combining (3.9) with (3.11), for each fixed $n \in \mathbb{N}$, for all $t \in [0,T]$, we find that

$$\begin{aligned} \|x_t^n\|_{\mathcal{PC}}^2 \leq & \left(\|\psi\|_{\mathcal{PC}_\phi}^2 + 8M_T^2 b_2 + C_2 \frac{\eta^{2\alpha-1}}{2\alpha-1} + C_3 \eta^{2H-1} \right) \\ & \times \sum_{i=0}^{n} \left(8M_T^2(b_1+1) + C_1 \frac{\eta^{2\alpha-1}}{2\alpha-1} \right)^i < \infty. \end{aligned} \tag{3.12}$$

Taking into account that $\mathbb{E}\|x^n(t)\|^2 \leq \|x_t^n\|_{\mathcal{PC}}^2$, then (3.12) implies that $x^n(t,\cdot) \in L^2(\Omega;\mathbb{H})$ for all $t \in [0,T]$.

Finally, since $x^0(\cdot) = \psi \in \mathcal{PC}_\phi$, it is easy to see that $x^n(t) = \phi(t)$ on $(-\infty, 0]$. Now we only need to prove that $x^n(\cdot) \in \mathcal{PC}^T$ for all $n \in \mathbb{N}$.

Let us now check $x^1(\cdot) \in \mathcal{PC}^T$. Because the proof of the case $k=0$ is similar, here we assume that $k \geq 1$. To do that, let us consider $\sigma > 0$ small enough, such that for t, $t+\sigma \in (-\infty, T] \cap (t_k, t_{k+1}]$, then

$$\begin{aligned} \mathbb{E}\|x^1(t+\sigma) - x^1(t)\|^2 \leq & \, 7\|T_\alpha(t+\sigma-t_k) - T_\alpha(t-t_k)\|^2 \mathbb{E}\|I_k(x^0(t_k^-)) + x^0(t_k^-)\|^2 \\ & + 7\mathbb{E}\left\| \int_{t_k}^{t} (S_\alpha(t+\sigma-s) - S_\alpha(t-s)) f(s, x_s^0) ds \right\|^2 \\ & + 7\mathbb{E}\left\| \int_{t}^{t+\sigma} S_\alpha(t+\sigma-s) f(s, x_s^0) ds \right\|^2 \\ & + 7\mathbb{E}\left\| \int_{t_k}^{t} (S_\alpha(t+\sigma-s) - S_\alpha(t-s)) g(s, x_s^0) dB(s) \right\|^2 \\ & + 7\mathbb{E}\left\| \int_{t}^{t+\sigma} S_\alpha(t+\sigma-s) g(s, x_s^0) dB(s) \right\|^2 \\ & + 7\mathbb{E}\left\| \int_{t_k}^{t} (S_\alpha(t+\sigma-s) - S_\alpha(t-s)) h(s) dB_Q^H(s) \right\|^2 \\ & + 7\mathbb{E}\left\| \int_{t}^{t+\sigma} S_\alpha(t+\sigma-s) h(s) dB_Q^H(s) \right\|^2 \\ & := I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7. \end{aligned} \tag{3.13}$$

As $T_\alpha(t)$ is compact for $t > 0$, by Lemma 3.6(1) and condition (H_3), we obtain that

$$I_1 \leq 14\|T_\alpha(t+\sigma-t_k) - T_\alpha(t-t_k)\|^2((b_1+1)\|\psi\|_{\mathcal{PC}_\phi}^2 + b_2) \to 0 \quad \text{as} \quad \sigma \to 0.$$

By (3.4), (H_1), (H_4), Hölder's inequality, Lebesgue's Theorem, Lemma 3.6(1) and for a fixed but sufficiently small $\epsilon > 0$,

$$
\begin{aligned}
I_2 &\leq 14\eta \int_{t_k}^{t} \|S_\alpha(t+\sigma-s) - S_\alpha(t-s)\|^2 (l_1(s)\|\psi\|^2_{\mathcal{PC}_\phi} + \sup_{s\in(t_k,t_{k+1}\wedge T]} \|f(s,0)\|^2) ds \\
&\leq 14\eta (l_1\|\psi\|^2_{\mathcal{PC}_\phi} + \sup_{s\in(t_k,t_{k+1}\wedge T]} \|f(s,0)\|^2) \bigg(\int_{t_k}^{t-\epsilon} \|S_\alpha(t+\sigma-s) - S_\alpha(t-s)\|^2 ds \\
&\quad + 2\int_{t-\epsilon}^{t} \left(\|S_\alpha(t+\sigma-s)\|^2 + \|S_\alpha(t-s)\|^2\right) ds \bigg) \\
&\leq 14\eta (l_1\|\psi\|^2_{\mathcal{PC}_\phi} + \sup_{s\in(t_k,t_{k+1}\wedge T]} \|f(s,0)\|^2) \bigg(\int_{t_k}^{t-\epsilon} \|S_\alpha(t+\sigma-s) - S_\alpha(t-s)\|^2 ds \\
&\quad + 2N_T^2 \frac{(\sigma+\epsilon)^{2\alpha-1}}{2\alpha-1} + 2N_T^2 \frac{\epsilon^{2\alpha-1}}{2\alpha-1} \bigg).
\end{aligned}
\tag{3.14}
$$

Taking now limits when σ goes to zero, we obtain

$$
\lim_{\sigma\to 0} I_2 \leq 28N_T^2\eta \left(l_1\|\psi\|^2_{\mathcal{PC}_\phi} + \sup_{s\in(t_k,t_{k+1}\wedge T]} \|f(s,0)\|^2 \right) \frac{\epsilon^{2\alpha-1}}{2\alpha-1},
$$

and as ϵ is arbitrarily small, then $I_2 \to 0$ as $\sigma \to 0$.

It is worth noticing that property (i) of Lemma 3.6 is not valid for $t = 0$, therefore we split the integral of (3.14) into two parts to avoid the singularity. A similar argument will be used in I_4 and I_6.

According to (1.6), (3.4), (H_1), (H_4), Lebegue's Theorem, Lemma 3.6(1) and, for a fixed but sufficiently small $\epsilon > 0$, we have

$$
\begin{aligned}
I_4 &\leq 14Tr(Q)(l_2\|\psi\|^2_{\mathcal{PC}_\phi} + \sup_{s\in(t_k,t_{k+1}\wedge T]} \|g(s,0)\|^2) \\
&\quad \times \bigg(\int_{t_k}^{t-\epsilon} \|S_\alpha(t+\sigma-s) - S_\alpha(t-s)\|^2 ds \\
&\quad\quad + 2\int_{t-\epsilon}^{t} \left(\|S_\alpha(t+\sigma-s)\|^2 + \|S_\alpha(t-s)\|^2\right) ds \bigg) \\
&\leq 14Tr(Q)(l_2\|\psi\|^2_{\mathcal{PC}_\phi} + \sup_{s\in(t_k,t_{k+1}\wedge T]} \|g(s,0)\|^2) \\
&\quad \times \bigg(\int_{t_k}^{t-\epsilon} \|S_\alpha(t+\sigma-s) - S_\alpha(t-s)\|^2 ds \\
&\quad\quad + 2N_T^2 \frac{(\sigma+\epsilon)^{2\alpha-1}}{2\alpha-1} + 2N_T^2 \frac{\epsilon^{2\alpha-1}}{2\alpha-1} \bigg).
\end{aligned}
\tag{3.15}
$$

Arguing again as in (3.14), we deduce that $I_4 \to 0$ as $\sigma \to 0$.

For I_6, there exist two constants p and q which are given in the theorem, by (1.9), (3.4), (H_2), (H_4), Lemma 3.6(1), Lebesgue's Theorem, Hölder's inequality and, once more, for a fixed but sufficiently small $\epsilon > 0$,

$$
\begin{aligned}
I_6 &\le 7cH(2H-1)\eta^{2H-1}\int_{t_k}^{t-\epsilon}\|S_\alpha(t+\sigma-s)-S_\alpha(t-s)\|^2\|h(s)\|_Q^2 ds \\
&\quad + 14cH(2H-1)\epsilon^{2H-1}\int_{t-\epsilon}^{t}(\|S_\alpha(t+\sigma-s)\|^2+\|S_\alpha(t-s)\|^2)\|h(s)\|_Q^2 ds \\
&\le 7cH(2H-1)\eta^{2H-1}\int_{t_k}^{t-\epsilon}\|S_\alpha(t+\sigma-s)-S_\alpha(t-s)\|^2\|h(s)\|_Q^2 ds \\
&\quad + 14N_T^2 cH(2H-1)\epsilon^{2H-1}\Bigg(\Bigg(\int_{t-\epsilon}^{t}(t+\sigma-s)^{2(\alpha-1)p}ds\Bigg)^{\frac{1}{p}} \\
&\quad + \Bigg(\int_{t-\epsilon}^{t}(t-s)^{2(\alpha-1)p}ds\Bigg)^{\frac{1}{p}}\Bigg)\times\Bigg(\int_{t-\epsilon}^{t}\|h(s)\|_Q^{2q}ds\Bigg)^{\frac{1}{q}} \\
&\le 7cH(2H-1)\eta^{2H-1}\int_{t_k}^{t-\epsilon}\|S_\alpha(t+\sigma-s)-S_\alpha(t-s)\|^2\|h(s)\|_Q^2 ds \\
&\quad + 14N_T^2 cH(2H-1)\epsilon^{2H-1}\Bigg(\Bigg(\frac{(\sigma+\epsilon)^{2(\alpha-1)p+1}}{2(\alpha-1)p+1}\Bigg)^{\frac{1}{p}} \\
&\quad + \Bigg(\frac{\epsilon^{2(\alpha-1)p+1}}{2(\alpha-1)p+1}\Bigg)^{\frac{1}{p}}\Bigg)\times\Bigg(\int_{t-\epsilon}^{t}\|h(s)\|_Q^{2q}ds\Bigg)^{\frac{1}{q}},
\end{aligned}
\tag{3.16}
$$

and, as in the previous cases, we deduce $I_6 \to 0$ as $\sigma \to 0$.

For I_3, by (3.4), (H_1) and Hölder's inequality, we find that

$$
\begin{aligned}
I_3 &\le 14\sigma N_T^2\int_t^{t+\sigma}(t+\sigma-s)^{2\alpha-2}(l_1(s)\|\psi\|^2_{\mathcal{PC}_\phi}+\sup_{s\in(t_k,t_{k+1}\wedge T]}\|f(s,0)\|^2)ds \\
&\le 14\sigma N_T^2(l_1\|\psi\|^2_{\mathcal{PC}_\phi}+\sup_{s\in(t_k,t_{k+1}\wedge T]}\|f(s,0)\|^2)\frac{\sigma^{2\alpha-1}}{2\alpha-1}\to 0 \quad \text{as} \quad \sigma\to 0.
\end{aligned}
\tag{3.17}
$$

As for I_5, from (1.6), (3.4) and (H_1), we deduce

$$
\begin{aligned}
I_5 &\le 14Tr(Q)N_T^2\int_t^{t+\sigma}(t+\sigma-s)^{2\alpha-2}(l_2(s)\|\psi\|^2_{\mathcal{PC}_\phi}+\sup_{s\in(t_k,t_{k+1}\wedge T]}\|g(s,0)\|^2)ds \\
&\le 14Tr(Q)N_T^2(l_2\|\psi\|^2_{\mathcal{PC}_\phi}+\sup_{s\in(t_k,t_{k+1}\wedge T]}\|g(s,0)\|^2)\frac{\sigma^{2\alpha-1}}{2\alpha-1}\to 0 \quad \text{as} \quad \sigma\to 0.
\end{aligned}
\tag{3.18}
$$

Finally, for I_7, by (1.9), (3.4), (H_2) and Hölder's inequality, taking the same constants p, q as for I_6, we obtain that

$$
\begin{aligned}
I_7 &\le 7N_T^2 cH(2H-1)\sigma^{2H-1}\Bigg(\int_t^{t+\sigma}(t+\sigma-s)^{2(\alpha-1)p}ds\Bigg)^{\frac{1}{p}}\Bigg(\int_t^{t+\sigma}\|h(s)\|_Q^{2q}ds\Bigg)^{\frac{1}{q}} \\
&\le 7N_T^2 cH(2H-1)\sigma^{2H-1}\Bigg(\frac{\sigma^{2(\alpha-1)p+1}}{2(\alpha-1)p+1}\Bigg)^{\frac{1}{p}}\Bigg(\int_t^{t+\sigma}\|h(s)\|_Q^{2q}ds\Bigg)^{\frac{1}{q}}\to 0
\end{aligned}
\tag{3.19}
$$

as $\sigma \to 0$.

Therefore, $\mathbb{E}\|x^1(t+\sigma) - x^1(t)\|^2$ tends to zero as $\sigma \to 0$, which implies that $x^1(\cdot) \in C((-\infty, T] \cap (t_k, t_{k+1}]; L^2(\Omega; \mathbb{H}))$. An induction argument shows $x^n(\cdot) \in C((-\infty, T] \cap (t_k, t_{k+1}]; L^2(\Omega; \mathbb{H}))$ for $n \in \mathbb{N}$.

In conclusion, for all $n \in \mathbb{N}$, the assertion $x^n(\cdot) \in \mathcal{PC}_\phi$ holds true.

Step 2. We now show that $\{x^n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{PC}_\phi$.

By the construction of successive approximations, it is a straighforward consequence that $x^n(t) = x^{n-1}(t)$ on $(-\infty, 0]$. On the other hand, for $t \in [0, t_1]$ and $n \geq 1$, by (1.6), (3.4), (H_1), (H_4) and Hölder's inequality, we have for some $0 < p < 1$,

$$\begin{aligned}
\mathbb{E}\|x^{n+1}(t) - x^n(t)\|^2 &\leq 2\eta \int_0^t \|S_\alpha(t-s)\|^2 \mathbb{E}\|f(s, x_s^n) - f(s, x_s^{n-1})\|^2 ds \\
&+ 2Tr(Q) \int_0^t \|S_\alpha(t-s)\|^2 \mathbb{E}\|g(s, x_s^n) - g(s, x_s^{n-1})\|^2 ds \qquad (3.20)\\
&\leq 2N_T^2(\eta l_1 + Tr(Q) l_2) \Big(\int_0^t (t-s)^{\frac{2(\alpha-1)}{1-p}} ds \Big)^{1-p} \Big(\int_0^t \|x_s^n - x_s^{n-1}\|_{\mathcal{PC}}^{\frac{2}{p}} ds \Big)^p.
\end{aligned}$$

Consequently, for all $t \in [0, t_1]$, we have

$$\sup_{t\in[0,t_1]} e^{\gamma t}\mathbb{E}\|x^{n+1}(t) - x^n(t)\|^2 \leq e^{\gamma T} C_4 \Big(\int_0^{t_1} \|x_s^n - x_s^{n-1}\|_{\mathcal{PC}}^{\frac{2}{p}} ds \Big)^p, \qquad (3.21)$$

where we have used the notation

$$C_4 = 2N_T^2(\eta l_1 + Tr(Q) l_2) \Big(\frac{1-p}{2\alpha - 1 - p} \Big)^{1-p} \eta^{2\alpha-1-p}.$$

Hence,

$$\Big(\sup_{t\in[0,t_1]} e^{\gamma t}\mathbb{E}\|x^{n+1}(t) - x^n(t)\|^2 \Big)^{\frac{1}{p}} \leq e^{\frac{\gamma T}{p}} C_4^{\frac{1}{p}} \int_0^{t_1} \|x_s^n - x_s^{n-1}\|_{\mathcal{PC}}^{\frac{2}{p}} ds. \qquad (3.22)$$

By repeating iterations of (3.24), for all $n \in \mathbb{N}$,

$$\Big(\sup_{t\in[0,t_1]} e^{\gamma t}\mathbb{E}\|x^{n+1}(t) - x^n(t)\|^2 \Big)^{\frac{1}{p}} \leq \frac{(e^{\frac{\gamma T}{p}} C_4^{\frac{1}{p}} \eta)^n}{n!} \times \|x_t^1 - x_t^0\|_{\mathcal{PC}}^{\frac{2}{p}}. \qquad (3.23)$$

Using a similar argument to the one we used with (3.6), for all $t \in [0, t_1]$, we have

$$\begin{aligned}
\|x_t^1 - x_t^0\|_{\mathcal{PC}}^2 \leq \sup_{t\in[0,t_1]} \mathbb{E}\|x^1(t) - x^0(t)\|^2 \leq 8(M_T^2 + 1)\|\psi\|_{\mathcal{PC}_\phi} \\
+ C_1\|\psi\|_{\mathcal{PC}_\phi}^2 \frac{t_1^{2\alpha-1}}{2\alpha-1} + C_2 \frac{t_1^{2\alpha-1}}{2\alpha-1} + C_3 t_1^{2H-1} := C_5.
\end{aligned} \qquad (3.24)$$

Replacing (3.24) into (3.23), for all $t \in [0, t_1]$,

$$\sup_{t\in[0,t_1]} e^{\gamma t}\mathbb{E}\|x^{n+1}(t) - x^n(t)\|^2 \leq C_5 \frac{(e^{\gamma T} C_4 t_1^p)^n}{(n!)^p} \leq C \frac{(e^{\gamma T} C_4 t_1^p)^n}{(n!)^p}. \qquad (3.25)$$

For $t \in (t_1, t_2 \wedge T]$, on the one hand, similar to (3.20), by (1.6), (3.4), (H_1)-(H_4), (3.25) and Hölder's inequality, we obtain

$$
\begin{aligned}
&\mathbb{E}\|x^{n+1}(t) - x^n(t)\|^2 \\
&\le 6M_T^2(N+1)\mathbb{E}\|x^n(t_1^-) - x^{n-1}(t_1^-)\|^2 + C_4\bigg(\int_{t_1}^t \|x_s^n - x_s^{n-1}\|_{\mathcal{PC}}^{\frac{2}{p}} ds\bigg)^p \\
&\le 6CM_T^2(N+1)\frac{(e^{\gamma T}C_4 t_1^p)^{n-1}}{((n-1)!)^p} + C_4\bigg(\int_{t_1}^t \|x_s^n - x_s^{n-1}\|_{\mathcal{PC}}^{\frac{2}{p}} ds\bigg)^p,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\bigg(\sup_{t\in(t_1,t_2\wedge T]} e^{\gamma t}\mathbb{E}\|x^{n+1}(t) - x^n(t)\|^2\bigg)^{\frac{1}{p}} \\
&\le e^{\frac{\gamma T}{p}} C_6^{\frac{1}{p}} \frac{(e^{\gamma T}C_4^{\frac{1}{p}} t_1)^{n-1}}{(n-1)!} + e^{\frac{\gamma T}{p}} C_7^{\frac{1}{p}} \int_{t_1}^{t_2\wedge T} \|x_s^n - x_s^{n-1}\|_{\mathcal{PC}}^{\frac{2}{p}} ds,
\end{aligned}
\tag{3.26}
$$

where we have used the notation

$$
C_6 = 2^{1-p}(6CM_T^2(N+1)), \qquad C_7 = 2^{1-p}C_4.
$$

On the other hand, for $t \in (t_1, t_2 \wedge T]$,

$$
\begin{aligned}
\|x_t^1 - x_t^0\|_{\mathcal{PC}}^2 \le \sup_{t\in(t_1,t_2\wedge T]} \mathbb{E}\|x^1(t) - x^0(t)\|^2 &\le (16M_T^2 b_1 + 16M_T^2 + 8)\|\phi\|_{\mathcal{PC}_\phi}^2 \\
&+ 16M_T^2 b_2 + C_1\|\psi\|_{\mathcal{PC}_\phi}^2 \frac{((t_2\wedge T) - t_1)^{2\alpha-1}}{2\alpha - 1} \\
&+ C_2\frac{((t_2\wedge T) - t_1)^{2\alpha-1}}{2\alpha-1} + C_3((t_2\wedge T) - t_1)^{2H-1}.
\end{aligned}
$$

Combining this with (3.26), by induction on n, we obtain

$$
\begin{aligned}
&\sup_{t\in(t_1,t_2\wedge T]} e^{\gamma t}\mathbb{E}\|x^{n+1}(t) - x^n(t)\|^2 \le C\frac{n^p(e^{\gamma T}C(t_2\wedge T)^p)^{n-1}}{((n-1)!)^p} \\
&+ C\frac{(e^{\gamma T}C(t_2\wedge T)^p)^n}{(n!)^p} \le C\frac{(e^{\gamma T}C(t_2\wedge T)^p)^n}{(n!)^p}.
\end{aligned}
\tag{3.27}
$$

By repeating this procedure and induction, combining (3.27) and (3.25), for all $n \in \mathbb{N}$, $t \in [0, T]$, we deduce that

$$
\|x^{n+1} - x^n\|_{\mathcal{PC}^T}^2 \le C\frac{(e^{\gamma T}CT^p)^n}{(n!)^p}. \tag{3.28}
$$

Therefore, for any $0 < n < m$, we deduce

$$
\|x^m - x^n\|_{\mathcal{PC}^T} \le \sum_{r=n}^{m-1}\bigg(C\frac{(e^{\gamma T}CT^p)^r}{(r!)^p}\bigg)^{\frac{1}{2}} \to 0 \ \text{ as } \ n\to\infty, \tag{3.29}
$$

which implies that $\{x^n(\cdot)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{PC}^T$. Therefore, the sequence $x^n(\cdot)$ possesses a limit function denoted by $\hat{x}(\cdot)$, such that, $x^n(\cdot) \to \hat{x}(\cdot)$ in $\mathcal{PC}^T$ as $n \to \infty$. Let us define

$$x(t,\omega) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ \hat{x}(t,\omega), & t \in [0,T]. \end{cases}$$

It is easy to see that $x(t)$ is $\mathcal{F}_t$-adapted, $x^n(\cdot) \to x(\cdot)$ in $\mathcal{PC}_\phi$ as $n \to \infty$.

Step 3. We check the limit x of the sequence $\{x^n\}_{n\in\mathbb{N}}$ is a solution of (3.1). Taking into account condition (H_3), for every $k \geq 1$,

$$\mathbb{E}\|I_k(x^{n-1}(t_k^-)) - I_k(x(t_k^-))\|^2 \leq N\mathbb{E}\|x^{n-1}(t_k^-) - x(t_k^-)\|^2 \to 0 \quad \text{as} \quad n \to \infty.$$

Moreover, in view of (1.6), (3.4), (H_1) and (H_4), when $n \to \infty$, we have for $t \in (t_k, t_{k+1})$,

$$\begin{aligned} &\mathbb{E}\left\| \int_{t_k}^t S_\alpha(t-s)(f(s,x_s^{n-1}) - f(s,x_s))ds \right\|^2 \\ &\quad \leq \eta l_1 N_T^2 \int_{t_k}^t (t-s)^{2\alpha-2}\|x_s^{n-1} - x_s\|^2_{\mathcal{PC}} ds \to 0, \end{aligned}$$

and

$$\begin{aligned} &\mathbb{E}\left\| \int_{t_k}^t S_\alpha(t-s)(g(s,x_s^{n-1}) - g(s,x_s))dB(s) \right\|^2 \\ &\quad \leq Tr(Q) l_2 N_T^2 \int_{t_k}^t (t-s)^{2\alpha-2}\|x_s^{n-1} - x_s\|^2_{\mathcal{PC}} ds \to 0. \end{aligned}$$

Therefore, x is a solution of problem (3.1). This completes the proof. □

Theorem 3.12. *Let $\alpha \in (0,1)$, $A \in \mathbb{A}^\alpha(\omega_0,\theta_0)$ with $\theta_0 \in (0, \frac{\pi}{2}]$ and $\omega_0 \in \mathbb{R}^+$. Assume (H_1), (H_2), (H_3') and (H_4) hold, and the α-order fractional solution operator $T_\alpha(t)$ $(t \geq 0)$ is compact. Then, for every initial data $\phi \in \mathcal{PC}$ and every $T > 0$, the problem (3.2) has at least one mild solution defined on $(-\infty, T]$.*

Proof. By slightly modifying the proof of Theorem 3.11, we can first prove that $x^n(\cdot) \in \mathcal{PC}_\phi^T$ for all $n \geq 1$. Then, one can prove that it is a Cauchy sequence in $\mathcal{PC}_\phi^T$ and its limit is the solution to problem (3.2). □

Remark 3.13. *Note that the main aim of this chapter is to investigate a class of fractional impulsive stochastic differential equations, therefore, it is more reasonable to derive the real-value result of theorems 3.11 and 3.12 with $T > t_1$, which ensures there is at least one impulse taking place in $[0,T]$.*

In what follows, a general result on the continuous dependence of mild solutions on initial value will be proved. In particular, we obtain the uniqueness of mild solutions to problem (3.1) by means of the conclusion below.

Theorem 3.14. *Under assumptions of Theorem 3.11, the mild solution of (3.1) is continuous with respect to the initial value* $\phi \in \mathcal{PC}$. *In particular, if* $x(t)$, $y(t)$ *are the corresponding mild solutions, in the interval* $(-\infty, T]$, *to the initial data* ϕ *and* φ, *then the following estimate holds,*

$$\|x_t - y_t\|_{\mathcal{PC}}^2 \le 2^{1-p}(3M_T^2+1)\|\phi-\varphi\|_{\mathcal{PC}}^2 e^{(A_2 p + \frac{\ln A_3}{\beta})t}, \qquad \forall t \in [0,T], \tag{3.30}$$

where A_2 *and* A_3 *are constants depending on* η.

Proof. It follows from (3.3) that, for any $t \in [0, t_1]$, in view of (H_1), (1.6), (3.4) and Hölder's inequality, arguing as in the proof of (3.21),

$$\begin{aligned} \mathbb{E}\|x(t)-y(t)\|^2 &\le 3M_T^2\mathbb{E}\|\phi(0)-\varphi(0)\|^2 + 3N_T^2(l_1\eta \\ &\quad + Tr(Q)l_2)\int_0^t (t-s)^{2\alpha-2}\|x_s-y_s\|_{\mathcal{PC}}^2 ds \\ &\le 3M_T^2\|\phi-\varphi\|_{\mathcal{PC}}^2 + 3N_T^2(l_1\eta + Tr(Q)l_2) \\ &\quad \times \Big(\int_0^t (t-s)^{\frac{2(\alpha-1)}{1-p}} ds\Big)^{1-p}\Big(\int_0^t \|x_s-y_s\|_{\mathcal{PC}}^{\frac{2}{p}} ds\Big)^p \\ &\le 3M_T^2\|\phi-\varphi\|_{\mathcal{PC}}^2 + A_1\Big(\int_0^t \|x_s-y_s\|_{\mathcal{PC}}^{\frac{2}{p}} ds\Big)^p, \end{aligned} \tag{3.31}$$

where we have used the notation

$$A_1 = 3N_T^2(l_1\eta + Tr(Q)l_2)\Big(\frac{1-p}{2\alpha-p-1}\Big)^{1-p}\eta^{2\alpha-p-1}.$$

Observe that $\gamma > 0$, if $t+\theta \in [0, t_1]$ (where $\theta \in (-\infty, 0]$), then

$$e^{\gamma\theta}\mathbb{E}\|x(t+\theta)-y(t+\theta)\|^2 \le 3M_T^2\|\phi-\varphi\|_{\mathcal{PC}}^2 + A_1\Big(\int_0^{t+\theta} \|x_s-y_s\|_{\mathcal{PC}}^{\frac{2}{p}} ds\Big)^p,$$

on the other hand, if $t+\theta < 0$, it follows

$$\begin{aligned} &e^{\gamma\theta}\mathbb{E}\|x(t+\theta)-y(t+\theta)\|^2 \\ &= e^{-\gamma t}e^{\gamma(t+\theta)}\mathbb{E}\|\phi(t+\theta)-\varphi(t+\theta)\|^2 \le e^{-\gamma t}\|\phi-\varphi\|_{\mathcal{PC}}^2. \end{aligned}$$

Therefore,

$$\begin{aligned} \|x_t-y_t\|_{\mathcal{PC}}^2 &= \sup_{\theta\in(-\infty,0]} e^{\gamma\theta}\mathbb{E}\|x(t+\theta)-y(t+\theta)\|^2 \\ &\le (3M_T^2+1)\|\phi-\varphi\|_{\mathcal{PC}}^2 + A_1\Big(\int_0^t \|x_s-y_s\|_{\mathcal{PC}}^{\frac{2}{p}} ds\Big)^p, \end{aligned}$$

and

$$\|x_t-y_t\|_{\mathcal{PC}}^{\frac{2}{p}} \le 2^{\frac{1-p}{p}}(3M_T^2+1)^{\frac{1}{p}}\|\phi-\varphi\|_{\mathcal{PC}}^{\frac{2}{p}} + A_2\int_0^t \|x_s-y_s\|_{\mathcal{PC}}^{\frac{2}{p}} ds,$$

where

$$A_2 = 2^{\frac{1-p}{p}} A_1^{\frac{1}{p}}.$$

As a consequence of Gronwall's inequality, for $t \in [0, t_1]$,

$$\|x_t - y_t\|_{\mathcal{PC}}^2 \leq 2^{1-p}(3M_T^2 + 1)\|\phi - \varphi\|_{\mathcal{PC}}^2 e^{A_2 pt}, \tag{3.32}$$

and thus

$$\mathbb{E}\|x(t_1) - y(t_1)\|^2 \leq 2^{1-p}(3M_T^2 + 1)\|\phi - \varphi\|_{\mathcal{PC}}^2 e^{A_2 pt_1} = B_1. \tag{3.33}$$

For $t \in (t_1, t_2 \wedge T]$, similar to (3.31) and (3.32), by (H_1), (H_5), (1.6), (3.4) and Hölder's inequality, we find for $t + \theta > t_1$ ($\theta \in (-\infty, 0]$) that

$$\begin{aligned} \mathbb{E}\|x(t) - y(t)\|^2 &\leq 3M_T^2 \mathbb{E}\|x(t_1^-) - y(t_1^-) + I_1(x(t_1^-)) - I_1(y(t_1^-))\|^2 \\ &+ 3N_T^2(l_1\eta + Tr(Q)l_2)\bigg(\int_{t_1}^t (t-s)^{\frac{2(\alpha-1)}{1-p}} ds\bigg)^{1-p}\bigg(\int_{t_1}^t \|x_s - y_s\|_{\mathcal{PC}}^{\frac{2}{p}} ds\bigg)^p \\ &\leq 6M_T^2(N+1)\mathbb{E}\|x(t_1^-) - y(t_1^-)\|^2 + A_1\bigg(\int_{t_1}^t \|x_s - y_s\|_{\mathcal{PC}}^{\frac{2}{p}} ds\bigg)^p. \end{aligned} \tag{3.34}$$

Replacing t by $t+\theta$ in (3.34), in view of $\gamma > 0$, if $t + \theta \in (t_1, t_2 \wedge T]$ ($\theta \in (-\infty, 0]$), we have

$$\begin{aligned} &e^{\gamma\theta}\mathbb{E}\|x(t+\theta) - y(t+\theta)\|^2 \\ &\leq 6M_T^2(N+1)E\|x(t_1^-) - y(t_1^-)\|^2 + A_1\bigg(\int_{t_1}^{t+\theta} \|x_s - y_s\|_{\mathcal{PC}}^{\frac{2}{p}} ds\bigg)^p. \end{aligned} \tag{3.35}$$

It follows from (3.32) and (3.33) that for $t \in (t_1, t_2 \wedge T]$ and $t + \theta < t_1$,

$$\begin{aligned} e^{\gamma\theta}\mathbb{E}\|x(t+\theta) - y(t+\theta)\|^2 &\leq 2^{1-p}(3M_T^2+1)\|\phi - \varphi\|_{\mathcal{PC}}^2 e^{A_2 p(t+\theta)} \\ &= 2^{1-p}(3M_T^2+1)\|\phi - \varphi\|_{\mathcal{PC}}^2 e^{A_2 p((t+\theta-t_1)+t_1)} \leq B_1 e^{A_2 p\eta}. \end{aligned} \tag{3.36}$$

Combining (3.35) and (3.36),

$$\|x_t - y_t\|_{\mathcal{PC}}^{\frac{2}{p}} \leq 2^{\frac{1-p}{p}}(6M_T^2(N+1) + e^{A_2 p\eta})^{\frac{1}{p}} B_1^{\frac{1}{p}} + A_2 \int_{t_1}^t \|x_s - y_s\|_{\mathcal{PC}}^{\frac{2}{p}} ds.$$

Applying Gronwall's inequality, we derive for $t \in (t_1, t_2 \wedge T]$,

$$\|x_t - y_t\|_{\mathcal{PC}}^2 \leq 2^{1-p}(6M_T^2(N+1) + e^{A_2 p\eta})B_1 e^{A_2 p(t-t_1)}. \tag{3.37}$$

If $T < t_2$, this shows the assertion. We assume that $T > t_2$, and consequently,

$$\mathbb{E}\|x(t_2) - y(t_2)\|^2 \leq 2^{1-p}(6M_T^2(N+1) + e^{A_2 p\eta})B_1 e^{A_2 p(t_2-t_1)} = B_2.$$

Arguing similarly, we find that for $t \in (t_k, t_{k+1} \wedge T]$ with $k \geq 2$,

$$\|x_t - y_t\|_{\mathcal{PC}}^2 \leq 2^{1-p}(6M_T^2(N+1) + e^{A_2 p\eta})B_k e^{A_2 p(t-t_k)}, \tag{3.38}$$

we may as well consider the case $t_{k+1} < T$, because the estimate (3.30) holds true when $t_{k+1} > T$ with $k \geq 2$. Thus

$$\mathbb{E}\|x(t_{k+1}) - y(t_{k+1})\|^2 \leq 2^{1-p}(6M_T^2(N+1) + e^{A_2 p\eta})B_k e^{A_2 p(t_{k+1}-t_k)} = B_{k+1}.$$

For the sake of convenience, let $A_3 = 2^{1-p}(6M_T^2(N+1) + e^{A_2 p\eta})$, we deduce the following result by mathematical induction for $k \geq 2$,

$$B_k \leq A_3 B_{k-1} e^{A_2 p(t_k - t_{k-1})} \leq A_3^{k-1} B_1 e^{A_2 p(t_k - t_1)}. \tag{3.39}$$

Hence, it follows from (3.38) and (3.39), and for $t \in (t_k, t_{k+1} \wedge T]$,

$$\|x_t - y_t\|_{\mathcal{PC}}^2 \leq A_3 B_k e^{A_2 p(t-t_k)} \leq A_3^k B_1 e^{A_2 p(t-t_1)}. \tag{3.40}$$

In view of the fact that condition (H_4) implies that $k\beta < t < (k+1)\eta$ for $t \in (t_k, t_{k+1}]$, taking into account (3.40), (3.32) and (3.38), we deduce for all $t \in (0, T]$,

$$\begin{aligned}\|x_t - y_t\|_{\mathcal{PC}}^2 &\leq 2^{1-p}(3M_T^2 + 1)\|\phi - \varphi\|_{\mathcal{PC}}^2 e^{A_2 pt} e^{k \ln A_3} \\ &\leq 2^{1-p}(3M_T^2 + 1)\|\phi - \varphi\|_{\mathcal{PC}}^2 e^{(A_2 p + \frac{\ln A_3}{\beta})t}.\end{aligned}$$

The proof is complete. □

Theorem 3.15. *Assume the hypotheses of Theorem 3.12. Then, the mild solution to problem (3.2) is continuous with respect to the initial value ϕ. That is, if $x(t)$, $y(t)$ are the corresponding mild solutions to the initial data ϕ and φ on $[0, T]$, we have*

$$\|x_t - y_t\|_{\mathcal{PC}}^2 \leq (3M_T^2 + 1)\|\phi - \varphi\|_{\mathcal{PC}}^2 e^{\left(\mathcal{A}_1 + \frac{\ln \mathcal{A}_2}{\beta}\right)t}, \qquad \forall t \in [0, T],$$

where $\mathcal{A}_1 = 3M_T^2(l_1\eta + l_2 Tr(Q))$ and $\mathcal{A}_2 = 6M_T^2(N+1) + e^{\mathcal{A}_1 \eta}$.

3.3 Asymptotic behavior to problem (3.1)

We first prove the global existence of mild solutions to stochastic impulsive differential equations (3.1) before studying the exponential asymptotic behavior of mild solutions.

To start off we state some conditions which will be imposed later.

(C_1) The closed and linear sectorial operator $A \in \mathbb{A}^\alpha(\omega_0, \theta_0)$ with $\theta_0 \in (0, \frac{\pi}{2}]$ and $\omega_0 \in \mathbb{R}$, generates an α-order fractional solution operator $T_\alpha(t)$ and an α-resolvent family $S_\alpha(t)$, on the separable Hilbert space $\mathbb{H}$ with

$$\|T_\alpha(t)\| \leq Me^{-\mu t}, \quad \|S_\alpha(t)\| \leq Me^{-\mu t}(1 + t^{\alpha-1}), \quad \forall t > 0,$$

where $M \geq 1$, Re $(\mu) \in \mathbb{R}^+$.

(C_2) There exist two positive constants C_f and C_g, such that, for any $x, y \in \mathcal{PC}$ and for all $t > 0$,

$$\mathbb{E}\|f(t,x) - f(t,y)\|^2 \leq C_f\|x-y\|^2_{\mathcal{PC}}, \quad \mathbb{E}\|g(t,x) - g(t,y)\|^2 \leq C_g\|x-y\|^2_{\mathcal{PC}},$$

and

$$\int_0^\infty e^{2\mu qs}\|f(s,0)\|^{2q}ds < \infty, \quad \int_0^\infty e^{2\mu qs}\|g(s,0)\|^{2q}ds < \infty,$$

where $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

(C_3) In addition to assumption (H_2), also suppose that

$$\int_0^\infty e^{2\mu qs}\|h(s)\|^{2q}_Q ds < \infty.$$

(C_4) Under condition (H_3), we impose an additional assumption on I_k, that is, for all $k \in \mathbb{N}$,

$$\mathbb{E}\|I_k(x)\|^2 \leq b_1 E\|x\|^2, \quad \text{for all } \ x \in L^2(\Omega;\mathbb{H}).$$

Now we state the global existence of mild solution to problem (3.1).

Theorem 3.16. *Assume the conditions of Theorem 3.11 and* (C_1)*, then for every initial value* $\phi \in \mathcal{PC}$*, there exists a unique solution to problem (3.1), in the sense of Definition 3.8, defined on* $[0,\infty)$*.*

Proof. We derive the local existence and uniqueness of mild solution to (3.1) by means of the estimates (3.4) in Theorem 3.11, where M_T and N_T are constants depending on T which is finite. In order to extend the results to $[0,\infty)$, the constants of the estimates (3.4) must be independent of T. For this aim, we modify the estimates (3.4) slightly by condition (C_1), that is,

$$M' = \sup_{t\in[0,\infty)} \|t(t)\| < \infty \ \text{ and } \ N' = \sup_{t\in[0,\infty)} Me^{-\mu t}(1 + t^{1-\alpha}) < \infty.$$

Replacing M_T and N_T by M' and N' in Theorem 3.11 and Theorem 3.14, respectively, the results still hold. Now, we are ready to prove the global existence and uniqueness of mild solutions to (3.4).

By theorems 3.11 and 3.14, we deduce that there exists a unique solution $x^{(1)}(t)$ to the initial value problem (3.1) such that

$$x^{(1)}(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ T_\alpha(t)\phi(0) + \int_0^t S_\alpha(t-s)f(s,x^{(1)}_s)ds + \int_0^t S_\alpha(t-s)g(s,x^{(1)}_s)dB(s) \\ + \int_0^t S_\alpha(t-s)h(s)dB^H_Q(s), & t \in [0,t_1]. \end{cases}$$

Arguing as in the proof of Theorem 3.11, we derive the existence of $x^{(2)}(t)$ satisfying

$$x^{(2)}(t) = \begin{cases} x^{(1)}(t), \qquad t \in (-\infty, t_1], \\ T_\alpha(t-t_1)(I_1(x^{(1)}(t_1^-)) + x^{(1)}(t_1^-)) + \int_{t_1}^t S_\alpha(t-s) f(s, x_s^{(2)}) ds \\ + \int_{t_1}^t S_\alpha(t-s) g(s, x_s^{(2)}) dB(s) + \int_{t_1}^t S_\alpha(t-s) h(s) dB_Q^H(s), \quad t \in (t_1, t_2]. \end{cases}$$

Continuing the procedure in this way, we obtain a unique global solution to problem (3.1) in the sense of Definition 3.8. This completes the proof. □

Motivated by the work of Caraballo et al. in [13], we turn our attention in this subsection to prove the exponential asymptotic behavior of the mild solutions to problem (3.1) in $\mathcal{PC}$.

Theorem 3.17. *Let conditions* (H_4) *and* (C_1)*-*(C_4) *hold, assume that*

$$\gamma > 2\mu,$$

and there exist two positive constants L *and* w_1 *(which will be explicitly written in the proof) such that*

$$2\mu q - L - \frac{\ln w_1}{\beta} > 0. \tag{3.41}$$

Then, every mild solution $x(\cdot)$ *of system (3.1) with the initial value* $x_0 = \phi \in \mathcal{PC}$ *satisfies*

$$\|x_t\|_{\mathcal{PC}}^{2q} \le C(1 + \|\phi\|_{\mathcal{PC}}^{2q}) e^{-(2\mu q - L - \frac{\ln w_1}{\beta})t}, \quad \forall t \ge 0. \tag{3.42}$$

Proof. We split the proof into three steps.

Step 1. By Definition 3.8, (1.6), (1.9), (H_4) and (C_1)-(C_3), we obtain for $t \in [0, t_1]$,

$$\begin{aligned} \mathbb{E}\|x(t)\|^2 &\le 4M^2 e^{-2\mu t}\|\phi\|_{\mathcal{PC}}^2 + 16\eta M^2 e^{-2\mu t}\int_0^t (1+(t-s)^{2\alpha-2}) e^{2\mu s} \\ &\times (C_f \|x_s\|_{\mathcal{PC}}^2 + \|f(s,0)\|^2) ds + 16 Tr(Q) M^2 e^{-2\mu t} \\ &\times \int_0^t (1+(t-s)^{2\alpha-2}) e^{2\mu s}(C_g\|x_s\|_{\mathcal{PC}}^2 + \|g(s,0)\|^2) ds \\ &+ 8cH(2H-1)\eta^{2H-1} M^2 e^{-2\mu t}\int_0^t (1+(t-s)^{2\alpha-2}) e^{2\mu s}\|h(s)\|_Q^2 ds \\ &= 4M^2 e^{-2\mu t}\|\phi\|_{\mathcal{PC}}^2 + 16 M^2 e^{-2\mu t}(\eta C_f + Tr(Q) C_g) \\ &\times \int_0^t (1+(t-s)^{2\alpha-2}) e^{2\mu s}\|x_s\|_{\mathcal{PC}}^2 ds \\ &+ 16 M^2 e^{-2\mu t}\int_0^t (1+(t-s)^{2\alpha-2}) e^{2\mu s}(\eta\|f(s,0)\|^2 + Tr(Q)\|g(s,0)\|^2) ds \\ &+ 8cH(2H-1)\eta^{2H-1} M^2 e^{-2\mu t}\int_0^t (1+(t-s)^{2\alpha-2}) e^{2\mu s}\|h(s)\|_Q^2 ds. \end{aligned} \tag{3.43}$$

In view of condition (C_3), which ensures the existence of a positive constant L_1, using Hölder's inequality, we derive

$$\begin{aligned} & cH(2H-1)\eta^{2H-1}\int_0^t (1+(t-s)^{2\alpha-2})e^{2\mu s}\|h(s)\|_Q^2 ds \\ & \leq cH(2H-1)\eta^{2H-1}\Big(\int_0^t (1+(t-s)^{2\alpha-2})^p ds\Big)^{\frac{1}{p}}\Big(\int_0^t e^{2\mu q s}\|h(s)\|_Q^{2q} ds\Big)^{\frac{1}{q}} \\ & \leq L_1. \end{aligned} \tag{3.44}$$

Observe that condition (C_2) ensures the existence of a positive constant L_2, by Hölder's inequality, one has

$$\begin{aligned} & \int_0^t (1+(t-s)^{2\alpha-2})e^{2\mu s}\|f(s,0)\|^2 ds \\ & \leq \Big(\int_0^\eta (1+(t-s)^{2\alpha-2})^p ds\Big)^{\frac{1}{p}}\Big(\int_0^\infty e^{2\mu q s}\|f(s,0)\|^{2q} ds\Big)^{\frac{1}{q}} \quad (3.45) \\ & \leq L_2. \quad (3.46) \end{aligned}$$

Analogously, one can prove that the following estimate holds true, thanks to condition (C_2), with a positive constant L_3,

$$\begin{aligned} & \int_0^t (1+(t-s)^{2\alpha-2})e^{2\mu s}\|g(s,0)\|^2 ds \\ & \leq \Big(\int_0^\eta (1+(t-s)^{2\alpha-2})^p ds\Big)^{\frac{1}{p}}\Big(\int_0^\infty e^{2\mu q s}\|g(s,0)\|^{2q} ds\Big)^{\frac{1}{q}} \quad (3.47) \\ & \leq L_3. \quad (3.48) \end{aligned}$$

Replacing (3.44)-(3.48) into (3.43), by Hölder's inequality,

$$\begin{aligned} & \mathbb{E}\|x(t)\|^2 \leq 4M^2e^{-2\mu t}\|\phi\|_{\mathcal{PC}}^2 + 16M^2e^{-2\mu t}(\eta L_2 + Tr(Q)L_3) + 8M^2e^{-2\mu t}L_1 \\ & \quad + 16M^2e^{-2\mu t}(\eta C_f + Tr(Q)C_g)\Big(\int_0^t (1+(t-s)^{2\alpha-2})^p ds\Big)^{\frac{1}{p}}\Big(\int_0^t e^{2\mu q s}\|x_s\|_{\mathcal{PC}}^{2q} ds\Big)^{\frac{1}{q}} \\ & \leq 4M^2e^{-2\mu t}\|\phi\|_{\mathcal{PC}}^2 + 16M^2e^{-2\mu t}L_4 + 16M^2e^{-2\mu t}L_5\Big(\int_0^t e^{2\mu q s}\|x_s\|_{\mathcal{PC}}^{2q} ds\Big)^{\frac{1}{q}}, \end{aligned} \tag{3.49}$$

where we have used the notation

$$L_4 = \frac{L_1}{2} + \eta L_2 + Tr(Q)L_3,$$

and

$$L_5 \leq 2^{\frac{p-1}{p}}(\eta C_f + Tr(Q)C_g)\Big(\eta + \frac{\eta^{2(\alpha-1)p+1}}{2(\alpha-1)p+1}\Big)^{\frac{1}{p}} < \infty.$$

Note that $e^{(\gamma-2\mu)\theta} < 1$ (for $\theta < 0$) since $\gamma > 2\mu$. For $t+\theta \in [0, t_1]$, multiplying by $e^{\gamma\theta}$ and replacing t by $t+\theta$ in (3.49), one has

$$\begin{aligned} e^{\gamma\theta}\mathbb{E}\|x(t+\theta)\|^2 &\le 4M^2 e^{-2\mu(t+\theta)} e^{\gamma\theta}\|\phi\|^2_{\mathcal{PC}} + 16M^2 e^{-2\mu(t+\theta)} e^{\gamma\theta} L_4 \\ &\quad + 16M^2 e^{-2\mu(t+\theta)} e^{\gamma\theta} L_5 \left(\int_0^{t+\theta} e^{2\mu q s}\|x_s\|^{2q}_{\mathcal{PC}} ds\right)^{\frac{1}{q}} \\ &\le 4M^2 e^{-2\mu t}\|\phi\|^2_{\mathcal{PC}} + 16M^2 e^{-2\mu t} L_4 \\ &\quad + 16M^2 e^{-2\mu t} L_5 \left(\int_0^{t} e^{2\mu q s}\|x_s\|^{2q}_{\mathcal{PC}} ds\right)^{\frac{1}{q}}, \end{aligned} \tag{3.50}$$

and

$$\begin{aligned} e^{\gamma\theta}\mathbb{E}\|x(t+\theta)\|^2 &= e^{-\gamma t} e^{\gamma(t+\theta)}\mathbb{E}\|\phi(t+\theta)\|^2 \\ &\le e^{-\gamma t}\|\phi\|^2_{\mathcal{PC}} \le e^{-2\mu t}\|\phi\|^2_{\mathcal{PC}}, \quad \text{if} \quad t+\theta < 0. \end{aligned} \tag{3.51}$$

Therefore, for $t \in [0, t_1]$, (3.50) and (3.51) lead to

$$\begin{aligned} \|x_t\|^2_{\mathcal{PC}} &\le 4M^2 e^{-2\mu t}\|\phi\|^2_{\mathcal{PC}} + 16M^2 e^{-2\mu t} L_4 \\ &\quad + 16M^2 e^{-2\mu t} L_5 \left(\int_0^{t} e^{2\mu q s}\|x_s\|^{2q}_{\mathcal{PC}} ds\right)^{\frac{1}{q}}, \end{aligned} \tag{3.52}$$

thus

$$\begin{aligned} \|x_t\|^{2q}_{\mathcal{PC}} &\le 3^{q-1}4^q M^{2q} e^{-2\mu q t}\|\phi\|^{2q}_{\mathcal{PC}} + 3^{q-1}4^{2q} M^{2q} e^{-2\mu q t} L_4^q \\ &\quad + 3^{q-1}4^{2q} M^{2q} e^{-2\mu q t} L_5^q \int_0^{t} e^{2\mu q s}\|x_s\|^{2q}_{\mathcal{PC}} ds. \end{aligned} \tag{3.53}$$

Multiplying both sides of (3.53) by $e^{2\mu q t}$,

$$\begin{aligned} e^{2\mu q t}\|x_t\|^{2q}_{\mathcal{PC}} &\le 3^{q-1}4^q M^{2q}\|\phi\|^{2q}_{\mathcal{PC}} + 3^{q-1}4^{2q} M^{2q} L_4^q \\ &\quad + 3^{q-1}4^{2q} M^{2q} L_5^q \int_0^{t} e^{2\mu q s}\|x_s\|^{2q}_{\mathcal{PC}} ds. \end{aligned} \tag{3.54}$$

Gronwall's inequality implies

$$\|x_t\|^{2q}_{\mathcal{PC}} \le (3^{q-1}4^q M^{2q}\|\phi\|^{2q}_{\mathcal{PC}} + L_6) e^{-(2\mu q - L)t}, \tag{3.55}$$

where we have used the notation

$$L_6 = 3^{q-1}4^{2q} M^{2q} L_4^q, \qquad L = 3^{q-1}4^{2q} M^{2q} L_5^q,$$

and, consequently,

$$\mathbb{E}\|x(t_1)\|^2 \le (3^{q-1}4^q M^{2q}\|\phi\|^{2q}_{\mathcal{PC}} + L_6)^{\frac{1}{q}} e^{-\frac{2\mu q - L}{q} t_1} = (B_1^*)^{\frac{1}{q}}. \tag{3.56}$$

Step 2. By Definition 3.8, (1.6), (1.9), (H_4) and (C_1)-(C_4), similar to (3.43), we obtain for $t \in (t_1, t_2]$ that

$$\begin{aligned}
\mathbb{E}\|x(t)\|^2 \leq\ & 8M^2 e^{-2\mu(t-t_1)}(1+b_1)\mathbb{E}\|x(t_1^-)\|^2 + 16M^2(\eta C_f + Tr(Q)C_g)e^{-2\mu(t-t_1)} \\
& \times \int_{t_1}^{t}(1+(t-s)^{2\alpha-2})e^{2\mu(s-t_1)}\|x_s\|_{\mathcal{PC}}^2 ds + 16M^2 e^{-2\mu(t-t_1)} \\
& \times \int_{t_1}^{t}(1+(t-s)^{2\alpha-2})e^{2\mu(s-t_1)}(\eta\|f(s,0)\|^2 + Tr(Q)\|g(s,0)\|^2)ds \\
& + 8cH(2H-1)\eta^{2H-1}M^2 e^{-2\mu(t-t_1)}\int_{t_1}^{t}(1+(t-s)^{2\alpha-2})e^{2\mu(s-t_1)}\|h(s)\|_Q^2 ds.
\end{aligned} \tag{3.57}$$

In order to prove the exponential decay to zero of solutions to problem (3.1), we need the results to the last two terms of (3.57) including $e^{-(2\mu-\frac{L}{q})t_1}$. For this purpose, we do the estimates separately as follows. On the one hand, using the same argument as in (3.44), condition (C_3) ensures the existence of a positive constant which is still denoted by L_1 for simplicity, such that

$$\begin{aligned}
& cH(2H-1)\eta^{2H-1}\int_{t_1}^{t}(1+(t-s)^{2\alpha-2})e^{(2\mu-\frac{L}{q})(s-t_1)+\frac{L}{q}(s-t_1)}\|h(s)\|_Q^2 ds \\
& \leq cH(2H-1)\eta^{2H-1}e^{-(2\mu-\frac{L}{q})t_1}\int_{t_1}^{t}(1+(t-s)^{2\alpha-2})e^{2\mu s}\|h(s)\|_Q^2 ds \\
& \leq cH(2H-1)\eta^{2H-1}e^{-(2\mu-\frac{L}{q})t_1}\left(\int_{t_1}^{t}(1+(t-s)^{2\alpha-2})^p ds\right)^{\frac{1}{p}}\left(\int_{t_1}^{t}e^{2\mu q s}\|h(s)\|_Q^{2q}ds\right)^{\frac{1}{q}} \\
& \leq e^{-(2\mu-\frac{L}{q})t_1}L_1.
\end{aligned} \tag{3.58}$$

On the other hand, similar to (3.46) and (3.48), using the same technique as in (3.58), by condition (C_2), there are two positive constants still denoted by L_2 and L_3 respectively, such that

$$\begin{aligned}
& \int_{t_1}^{t}(1+(t-s)^{2\alpha-2})e^{2\mu(s-t_1)}\|f(s,0)\|^2 ds \\
& \leq \left(\int_{t_1}^{t}(1+(t-s)^{2\alpha-2})^p ds\right)^{\frac{1}{p}} \times \left(\int_{t_1}^{t}e^{(2\mu q-L)(s-t_1)}e^{L(s-t_1)}\|f(s,0)\|^{2q}ds\right)^{\frac{1}{q}} \\
& \leq \left(\int_{t_1}^{t}(1+(t-s)^{2\alpha-2})^p ds\right)^{\frac{1}{p}} \times e^{-\frac{2\mu q-L}{q}t_1}\left(\int_{t_1}^{t}e^{2\mu q s}\|f(s,0)\|^{2q}ds\right)^{\frac{1}{q}} \\
& \leq L_2 e^{-\frac{2\mu q-L}{q}t_1},
\end{aligned} \tag{3.59}$$

and

$$\begin{aligned}
&\int_{t_1}^{t}(1+(t-s)^{2\alpha-2})e^{2\mu(s-t_1)}\|g(s,0)\|^2ds\\
&\le\bigg(\int_{t_1}^{t}(1+(t-s)^{2\alpha-2})^pds\bigg)^{\frac{1}{p}}\times\bigg(\int_{t_1}^{t}e^{(2\mu q-L)(s-t_1)}e^{L(s-t_1)}\|g(s,0)\|^{2q}ds\bigg)^{\frac{1}{q}}\\
&\le\bigg(\int_{t_1}^{t}(1+(t-s)^{2\alpha-2})^pds\bigg)^{\frac{1}{p}}\times e^{-\frac{2\mu q-L}{q}t_1}\bigg(\int_{t_1}^{t}e^{2\mu qs}\|g(s,0)\|^{2q}ds\bigg)^{\frac{1}{q}}\\
&\le L_3e^{-\frac{2\mu q-L}{q}t_1}.
\end{aligned}\tag{3.60}$$

Substituting (3.58)-(3.60) into (3.57), and using Hölder's inequality,

$$\begin{aligned}
\mathbb{E}\|x(t)\|^2&\le 8M^2e^{-2\mu(t-t_1)}(1+b_1)\mathbb{E}\|x(t_1^-)\|^2+16M^2(\eta C_f+Tr(Q)C_g)e^{-2\mu(t-t_1)}\\
&\quad\times\bigg(\int_{t_1}^{t}(1+(t-s)^{2\alpha-2})^pds\bigg)^{\frac{1}{p}}\bigg(\int_{t_1}^{t}e^{2\mu q(s-t_1)}\|x_s\|_{\mathcal{PC}}^{2q}ds\bigg)^{\frac{1}{q}}\\
&\quad+16M^2e^{-2\mu(t-t_1)}(\eta L_2+Tr(Q)L_3)e^{-\frac{2\mu q-L}{q}t_1}+8M^2e^{-2\mu(t-t_1)}L_1e^{-\frac{2\mu q-L}{q}t_1}\\
&\le 8M^2e^{-2\mu(t-t_1)}(1+b_1)\mathbb{E}\|x(t_1^-)\|^2+16M^2e^{-2\mu(t-t_1)}L_4e^{-\frac{2\mu q-L}{q}t_1}\\
&\quad+16M^2e^{-2\mu(t-t_1)}L_5\bigg(\int_{t_1}^{t}e^{2\mu q(s-t_1)}\|x_s\|_{\mathcal{PC}}^{2q}ds\bigg)^{\frac{1}{q}}.
\end{aligned}\tag{3.61}$$

Arguing as in the proof of Step 1, due to the fact that $e^{(\gamma-2\mu)\theta}<1$, we obtain for $t+\theta\in(t_1,t_2]$ (where $\theta\in(-\infty,0]$)

$$\begin{aligned}
e^{\gamma\theta}\mathbb{E}\|x(t+\theta)\|^2&\le 8M^2e^{-2\mu(t+\theta-t_1)}e^{\gamma\theta}(1+b_1)\mathbb{E}\|x(t_1^-)\|^2+16M^2e^{-2\mu(t+\theta-t_1)}e^{\gamma\theta}\\
&\quad\times L_4e^{-\frac{2\mu q-L}{q}t_1}+16M^2e^{-2\mu(t+\theta-t_1)}e^{\gamma\theta}\\
&\quad\times L_5\bigg(\int_{t_1}^{t+\theta}e^{2\mu q(s-t_1)}\|x_s\|_{\mathcal{PC}}^{2q}ds\bigg)^{\frac{1}{q}}\\
&\le 8M^2e^{-2\mu(t-t_1)}(1+b_1)\mathbb{E}\|x(t_1^-)\|^2+16M^2e^{-2\mu(t-t_1)}L_4e^{-\frac{2\mu q-L}{q}t_1}\\
&\quad+16M^2e^{-2\mu(t-t_1)}L_5\bigg(\int_{t_1}^{t}e^{2\mu q(s-t_1)}\|x_s\|_{\mathcal{PC}}^{2q}ds\bigg)^{\frac{1}{q}}.
\end{aligned}\tag{3.62}$$

It follows from (3.55) and (3.56) that, for $t\in(t_1,t_2]$ and $t+\theta\le t_1$,

$$\begin{aligned}
e^{\gamma\theta}\mathbb{E}\|x(t+\theta)\|^2&\le(3^{q-1}4^qM^{2q}\|\phi\|_{\mathcal{PC}}^{2q}+L_6)^{\frac{1}{q}}e^{-\frac{2\mu q-L}{q}t}\\
&=(3^{q-1}4^qM^{2q}\|\phi\|_{\mathcal{PC}}^{2q}+L_6)^{\frac{1}{q}}e^{-\frac{2\mu q-L}{q}t_1}e^{-\frac{2\mu q-L}{q}(t-t_1)}\\
&\le(B_1^*)^{\frac{1}{q}}e^{\frac{L}{q}\eta}e^{-2\mu(t-t_1)}.
\end{aligned}\tag{3.63}$$

Hence, (3.62) and (3.63) imply that for all $t \in (t_1, t_2]$,

$$\|x_t\|^2_{\mathcal{PC}} \leq \left(8M^2(1+b_1) + e^{\frac{L}{q}\eta}\right)(B_1^*)^{\frac{1}{q}} e^{-2\mu(t-t_1)} + 16M^2 e^{-2\mu(t-t_1)} L_4 e^{-\frac{2\mu q - L}{q}t_1} + 16M^2 e^{-2\mu(t-t_1)} L_5 \left(\int_{t_1}^t e^{2\mu q(s-t_1)} \|x_s\|^{2q}_{\mathcal{PC}} ds\right)^{\frac{1}{q}}. \tag{3.64}$$

Thus

$$\|x_t\|^{2q}_{\mathcal{PC}} \leq 3^{q-1}\left(8M^2(1+b_1) + e^{\frac{L}{q}\eta}\right)^q B_1^* e^{-2\mu q(t-t_1)} + 3^{q-1}4^{2q}M^{2q}e^{-2\mu q(t-t_1)} \times L_4^q e^{-(2\mu q - L)t_1} + 3^{q-1}4^{2q}M^{2q}e^{-2\mu q(t-t_1)} L_5^q \int_{t_1}^t e^{2\mu q(s-t_1)}\|x_s\|^{2q}_{\mathcal{PC}} ds. \tag{3.65}$$

Multiplying both sides of (3.65) by $e^{2\mu q(t-t_1)}$,

$$e^{2\mu q(t-t_1)}\|x_t\|^{2q}_{\mathcal{PC}} \leq 3^{q-1}\left(8M^2(1+b_1) + e^{\frac{L}{q}\eta}\right)^q B_1^* + 3^{q-1}4^{2q}M^{2q}L_4^q e^{-(2\mu q - L)t_1} + 3^{q-1}4^{2q}M^{2q}L_5^q \int_{t_1}^t e^{2\mu q(s-t_1)}\|x_s\|^{2q}_{\mathcal{PC}} ds. \tag{3.66}$$

Solving the above Gronwall inequality yields

$$\|x_t\|^{2q}_{\mathcal{PC}} \leq \left(3^{q-1}(8M^2(1+b_1) + e^{\frac{L}{q}\eta})^q B_1^* + L_6 e^{-(2\mu q - L)t_1}\right) e^{-(2\mu q - L)(t-t_1)}, \tag{3.67}$$

and, consequently,

$$\mathbb{E}\|x(t_2)\|^2 \leq \left(3^{q-1}(8M^2(1+b_1) + e^{\frac{L}{q}\eta})^q B_1^* + L_6 e^{-(2\mu q - L)t_1}\right)^{\frac{1}{q}} \times e^{-\frac{2\mu q - L}{q}(t_2 - t_1)} = (B_2^*)^{\frac{1}{q}}. \tag{3.68}$$

Step 3. The same reasoning as above implies, for $t \in (t_k, t_{k+1}]$ with $k \geq 2$,

$$\|x_t\|^{2q}_{\mathcal{PC}} \leq \left(3^{q-1}(8M^2(1+b_1) + e^{\frac{L}{q}\eta})^q B_k^* + L_6 e^{-(2\mu q - L)t_k}\right) e^{-(2\mu q - L)(t-t_k)}, \tag{3.69}$$

and

$$\mathbb{E}\|x(t_{k+1})\|^2 \leq \left(3^{q-1}(8M^2(1+b_1) + e^{\frac{L}{q}\eta})^q B_k^* + L_6 e^{-(2\mu q - L)t_k}\right)^{\frac{1}{q}} \times e^{-\frac{2\mu q - L}{q}(t_{k+1} - t_k)} = (B_{k+1}^*)^{\frac{1}{q}}. \tag{3.70}$$

For convenience, let $w_1 = 3^{q-1}(8M^2(1+b_1) + e^{\frac{L}{q}\eta})^q$. It is obvious $w_1 > 2$ such that $\sum_{j=0}^{k-2} w_1^j \leq \frac{w_1^{k-1}}{w_1 - 1} \leq \frac{w_1^{k-1}}{w_1 - \frac{w_1}{2}} \leq 2w_1^{k-2}$. In addition, condition (H_4) implies that $k - 1 \leq \frac{t_k - t_1}{\beta}$ and $k\beta < t_k$. Then, for $k \geq 2$, the mathematical induction method furnishes that

$$\begin{aligned}
B_k^* &\leq w_1 B_{k-1}^* e^{-(2\mu q-L)(t_k-t_{k-1})} + L_6 e^{-(2\mu q-L)t_k} \\
&\leq w_1^{k-1} B_1^* e^{-(2\mu q-L)(t_k-t_1)} + L_6 e^{-(2\mu q-L)t_k} \sum_{j=0}^{k-2} w_1^j \\
&\leq B_1^* e^{(t_k-t_1)\frac{\ln w_1}{\beta}} e^{-(2\mu q-L)(t_k-t_1)} + 2L_6 e^{-(2\mu q-L)t_k} w_1^{k-2} \\
&\leq B_1^* e^{-(2\mu q-L-\frac{\ln w_1}{\beta})(t_k-t_1)} + 2L_6 e^{-(2\mu q-L)t_k} e^{(k-2)\ln w_1} \\
&\leq B_1^* e^{-(2\mu q-L-\frac{\ln w_1}{\beta})(t_k-t_1)} + 2L_6 e^{-(2\mu q-L)t_k} e^{\frac{t_k}{\beta}\ln w_1} \\
&\leq B_1^* e^{-(2\mu q-L-\frac{\ln w_1}{\beta})(t_k-t_1)} + 2L_6 e^{-(2\mu q-L-\frac{\ln w_1}{\beta})t_k}.
\end{aligned} \tag{3.71}$$

Therefore, by (3.56), (3.69) and (3.71) we deduce that, for $t \in (t_k, t_{k+1}]$ with $k \geq 2$,

$$\begin{aligned}
\|x_t\|_{\mathcal{PC}}^{2q} &\leq w_1 B_k^* e^{-(2\mu q-L-\frac{\ln w_1}{\beta})(t-t_k)} + L_6 e^{-(2\mu q-L)t} \\
&\leq w_1 \left(B_1^* e^{-(2\mu q-L-\frac{\ln w_1}{\beta})(t_k-t_1)} + 2L_6 e^{-(2\mu q-L-\frac{\ln w_1}{\beta})t_k} \right) \\
&\quad \times e^{-(2\mu q-L-\frac{\ln w_1}{\beta})(t-t_k)} + L_6 e^{-(2\mu q-L-\frac{\ln w_1}{\beta})t} \\
&\leq w_1 (3^{q-1} 4^q M^{2q} \|\phi\|_{\mathcal{PC}}^{2q} + 3L_6) e^{-(2\mu q-L-\frac{\ln w_1}{\beta})t} \\
&\leq C(1+\|\phi\|_{\mathcal{PC}}^{2q}) e^{-(2\mu q-L-\frac{\ln w_1}{\beta})t},
\end{aligned}$$

which, thanks to (3.67) and (3.55), implies that, for all $t \geq 0$,

$$\|x_t\|_{\mathcal{PC}}^{2q} \leq C(1+\|\phi\|_{\mathcal{PC}}^{2q}) e^{-(2\mu q-L-\frac{\ln w_1}{\beta})t}.$$

This completes the proof. □

3.4 Asymptotic behavior to problem (3.2)

Now we study the long time behavior of the global mild solutions to problem (3.2), first we enumerate some assumptions which will be imposed in our further analysis. These assumptions here are similar than the above section of this chapter, but we prefer to include them here to make this section more readable.

(C_1') $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ is the infinitesimal generator of an α-order fractional compact and analytic solution operator $T_\alpha(t)$ $(t \geq 0)$ on a separable Hilbert space $\mathbb{H}$ with

$$\|T_\alpha(t)\| \leq Me^{-\mu t}, \qquad \forall t \geq 0, \quad M \geq 1, \quad \mu \in \mathbb{R}^+.$$

(C_2') There exist two nonnegative continuous functions k_1, $k_2 \in L^1(\mathbb{R}^+)$, such that the continuous function $f : \mathbb{R}^+ \times \mathcal{PC} \to \mathbb{H}$ satisfies

$$\mathbb{E}\|f(t,x)\|^2 \leq k_1(t) + k_2(t)\|x\|^2_{\mathcal{PC}},$$

for $t \in [0,\infty)$ and every $x \in \mathcal{PC}$.

(C_3') There exist two nonnegative continuous functions k_3, $k_4 \in L^1(\mathbb{R}^+)$, such that the continuous function $g : \mathbb{R}^+ \times \mathcal{PC} \to \mathcal{L}(\mathbb{K},\mathbb{H})$ satisfies

$$\mathbb{E}\|g(t,x)\|^2 \leq k_3(t) + k_4(t)\|x\|^2_{\mathcal{PC}},$$

for $t \in [0,\infty)$ and every $x \in \mathcal{PC}$.

At this point some remarks are in order.

Remark 3.18. *i) Notice that, under the Lipschitz condition (H_1), we can obtain the local existence and uniqueness of mild solution to problem (3.2) (see Theorem 3.12). However, in this section we are interested in analyzing the asymptotic behavior of mild solutions to equation (3.2) no matter how many solutions the problem may have for each initial condition. Therefore, our analysis can be carried out without imposing (H_1). Instead, in order to guarantee that we have mild solutions globally defined in time, it is enough to assume conditions (C_2') and (C_3') as above. A well-known conclusion is that conditions (C_2') and (C_3') hold automatically once we assume condition (H_1) holds true. Henceforth, throughout this paper, we will assume either condition (H_1) (when we need uniqueness of solution) or (C_2)-(C_3).*

ii) Due to the fact that the continuous functions $k_i \in L^1(\mathbb{R}^+)$ appearing in conditions (C_2') and (C_3') are nonnegative, we will denote in the sequel

$$\int_0^\infty k_i(s)ds := K_i < \infty, \quad i = 1,2,3,4,$$

where K_i are positive constants.

Theorem 3.19. *Assume hypotheses of Theorem 3.12 and (C_1') hold. Then for every initial value $\phi \in \mathcal{PC}$, the initial value problem (3.2) has a unique solution defined on $[0,\infty)$ in the sense of Definition 3.9.*

Proof. Thanks to assumption (C_1'), the estimates which are necessary to prove Theorem 3.12 are independent of T. This implies that the solution is defined in $(-\infty, T]$ for all $T > 0$. More details can be found in Theorem 3.3. □

Next, we shall obtain the estimate of solutions which will imply that the solutions are bounded uniformly with respect to bounded sets of initial conditions and positive values of time. This also implies the existence of an absorbing set for the solutions which is also a property on the ultimate boundedness of solutions.

Theorem 3.20. *Assume (H_2), (H_3'), (H_4), (C_1')-(C_3') and*

$$\gamma > 2\mu, \tag{3.72}$$

also, let

$$2\mu - \frac{\ln \mathcal{K}_1}{\beta} > 0, \tag{3.73}$$

where

$$\mathcal{K}_1 = 8M^2(N+1)(1+L_3),$$
$$L_3 = 4M^2(\eta K_2 + Tr(Q)K_4)\exp\left(4M^2(\eta K_2 + Tr(Q)K_4)\right).$$

Then every solution $x(\cdot)$ *of problem* (3.2) *with* $x_0 = \phi \in \mathcal{PC}$*, defined globally in time, verifies*

$$\|x_t\|^2_{\mathcal{PC}} \leq C\|\phi\|^2_{\mathcal{PC}} e^{-\left(2\mu - \frac{\ln \mathcal{K}_1}{\beta}\right)t} + C, \quad \forall t \geq 0.$$

Proof. For the sake of convenience, we split the proof into three steps.

Step 1. By (1.6), (1.9), (3.4), Definition 3.9, the Cauchy-Schwarz inequality, (H_2), (H_3'), (H_4) and (C_1')-(C_3'), we obtain that for $t \in [0, t_1]$,

$$\begin{aligned}
\mathbb{E}\|x(t)\|^2 &\leq 4M^2 e^{-2\mu t}\mathbb{E}\|\phi(0)\|^2 + 4M^2\eta \int_0^t e^{-2\mu(t-s)}\left(k_1(s) + k_2(s)\|x_s\|^2_{\mathcal{PC}}\right) ds \\
&\quad + 4M^2 Tr(Q)\int_0^t e^{-2\mu(t-s)}\left(k_3(s) + k_4(s)\|x_s\|^2_{\mathcal{PC}}\right) ds \\
&\quad + 4cH(2H-1)t^{2H-1}\int_0^t \|T_\alpha(t-s)\|^2\|h(s)\|^2_Q ds \\
&\leq 4M^2 e^{-2\mu t}\|\phi\|^2_{\mathcal{PC}} + 4M^2(\eta K_1 + Tr(Q)K_3) + L_1 \\
&\quad + 4M^2\int_0^t \left(\eta k_2(s) + Tr(Q)k_4(s)\right) e^{-2\mu(t-s)}\|x_s\|^2_{\mathcal{PC}} ds,
\end{aligned} \tag{3.74}$$

where we have used the notation

$$L_1 = 4M^2 cH(2H-1)\eta^{2H-1}\int_0^\eta \|h(s)\|^2_Q ds \leq 4M^2 cH(2H-1)\eta^{2H-1}\Lambda.$$

By assumption (3.72), we have $2\mu < \gamma$, then $e^{-(2\mu-\gamma)\theta} \leq 1$ holds immediately for any $\theta \leq 0$. Multiplying (3.74) by $e^{\gamma\theta}$ and replacing t by $t+\theta$, it follows

$$\begin{aligned}
&\sup_{\theta\in(-t,0]} e^{\gamma\theta}\mathbb{E}\|x(t+\theta)\|^2 \\
&\leq 4M^2 e^{-2\mu(t+\theta)}e^{\gamma\theta}\|\phi\|^2_{\mathcal{PC}} + 4M^2(\eta K_1 + Tr(Q)K_3)e^{\gamma\theta} + L_1 e^{\gamma\theta} \\
&\quad + 4M^2\int_0^{t+\theta}\left(\eta k_2(s) + Tr(Q)k_4(s)\right) e^{-2\mu(t+\theta-s)}e^{\gamma\theta}\|x_s\|^2_{\mathcal{PC}} ds \\
&\leq 4M^2 e^{-2\mu t}\|\phi\|^2_{\mathcal{PC}} + 4M^2(\eta K_1 + Tr(Q)K_3) + L_1 \\
&\quad + 4M^2\int_0^t \left(\eta k_2(s) + Tr(Q)k_4(s)\right) e^{-2\mu(t-s)}\|x_s\|^2_{\mathcal{PC}} ds.
\end{aligned}$$

Note that

$$e^{\gamma\theta}\mathbb{E}\|x(t+\theta)\|^2 = e^{-\gamma t}e^{\gamma(t+\theta)}\mathbb{E}\|x(t+\theta)\|^2 \\ \leq e^{-\gamma t}\|\phi\|^2_{\mathcal{PC}} \leq e^{-2\mu t}\|\phi\|^2_{\mathcal{PC}}, \qquad \forall\theta \in (-\infty, -t].$$

Therefore,

$$e^{2\mu t}\|x_t\|^2_{\mathcal{PC}} \leq 4M^2\|\phi\|^2_{\mathcal{PC}} + 4M^2\left(\eta K_1 + Tr(Q)K_3\right)e^{2\mu t} + L_1 e^{2\mu t} \\ + 4M^2\int_0^t \left(\eta k_2(s) + Tr(Q)k_4(s)\right)e^{2\mu s}\|x_s\|^2_{\mathcal{PC}}ds.$$

Applying the Gronwall inequality, we have for $t \in [0, t_1]$ that

$$\|x_t\|^2_{\mathcal{PC}} \leq 4M^2\|\phi\|^2_{\mathcal{PC}}(1+L_3)e^{-2\mu t} + L_2 + L_2L_3, \tag{3.75}$$

where we used the notation

$$L_2 = 4M^2(\eta K_1 + Tr(Q)K_3) + L_1, \\ L_3 = 4M^2(\eta K_2 + Tr(Q)K_4)\exp\left(4M^2(\eta K_2 + Tr(Q)K_4)\right).$$

In particular,

$$\mathbb{E}\|x(t_1)\|^2 \leq 4M^2\|\phi\|^2_{\mathcal{PC}}(1+L_3)e^{-2\mu t_1} + L_2 + L_3L_3 := D_1^*. \tag{3.76}$$

Step 2. Similar to (3.74) and in view of (H_3'), we obtain for $t \in (t_1, t_2]$,

$$\mathbb{E}\|x(t)\|^2 \leq 4\mathbb{E}\|T_\alpha(t-t_1)(x(t_1^-) + I_1(x(t_1^-)))\|^2 \\ + 4\eta\int_{t_1}^t \|T_\alpha(t-s)\|^2\mathbb{E}\|f(s,x_s)\|^2ds \\ + 4Tr(Q)\int_{t_1}^t \|T_\alpha(t-s)\|^2\mathbb{E}\|g(s,x_s)\|^2ds \\ + 4cH(2H-1)(t-t_1)^{2H-1}\int_{t_1}^t \|T_\alpha(t-s)\|^2\|h(s)\|^2_Q ds \\ \leq 8M^2e^{-2\mu(t-t_1)}(N+1)E\|x(t_1^-)\|^2 + 4M^2(\eta K_1 + Tr(Q)K_3) \\ + 4M^2\int_{t_1}^t \left(\eta k_2(s) + Tr(Q)k_4(s)\right)e^{-2\mu(t-s)}\|x_s\|^2_{\mathcal{PC}}ds + L_1.$$

Arguing as in Step 1, we derive for $t + \theta > t_1$ (where $\theta \in (-\infty, 0]$) that

$$\sup_{\theta\in(t_1-t,0]} e^{\gamma\theta}\mathbb{E}\|x(t+\theta)\|^2 \\ \leq 8M^2e^{-2\mu(t+\theta-t_1)}e^{\gamma\theta}(N+1)\mathbb{E}\|x(t_1^-)\|^2 + 4M^2(\eta K_1 + Tr(Q)K_3)e^{\gamma\theta} \\ + L_1e^{\gamma\theta} + 4M^2\int_{t_1}^{t+\theta}\left(\eta k_2(s) + Tr(Q)k_4(s)\right)e^{-2\mu(t+\theta-s)}e^{\gamma\theta}\|x_s\|^2_{\mathcal{PC}}ds \\ \leq 8M^2e^{-2\mu(t-t_1)}(N+1)\mathbb{E}\|x(t_1^-)\|^2 + 4M^2(\eta K_1 + Tr(Q)K_3) + L_1 \\ + 4M^2\int_{t_1}^t \left(\eta k_2(s) + Tr(Q)k_4(s)\right)e^{-2\mu(t-s)}\|x_s\|^2_{\mathcal{PC}}ds.$$

It follows from (3.75) and (3.76) that for $t \in (t_1, t_2]$ such that $t + \theta < t_1$,

$$\begin{aligned} e^{\gamma\theta}\mathbb{E}\|x(t+\theta)\|^2 &\leq 4M^2\|\phi\|^2_{\mathcal{PC}}(1+L_3)e^{-2\mu t_1}e^{-2\mu(t+\theta-t_1)} + L_2 + L_2L_3 \\ &\leq (4M^2\|\phi\|^2_{\mathcal{PC}}(1+L_3)e^{-2\mu t_1} + L_2 + L_2L_3)e^{-2\mu(t+\theta-t_1)} \\ &= D_1^* e^{-2\mu(t+\theta-t_1)}. \end{aligned}$$

Hence,

$$\begin{aligned} e^{2\mu(t-t_1)}\|x_t\|^2_{\mathcal{PC}} &\leq 8M^2(N+1)D_1^* + 4M^2(\eta K_1 + Tr(Q)K_3)e^{2\mu(t-t_1)} \\ &\quad + L_1 e^{2\mu(t-t_1)} + 4M^2\int_{t_1}^{t}(\eta k_2(s) + Tr(Q)k_4(s))\, e^{2\mu(s-t_1)}\|x_s\|^2_{\mathcal{PC}}ds. \end{aligned}$$

Thanks to the Gronwall inequality, we deduce for $t \in (t_1, t_2]$

$$\|x_t\|^2_{\mathcal{PC}} \leq 8M^2(N+1)D_1^*(1+L_3)e^{-2\mu(t-t_1)} + L_2 + L_2L_3, \tag{3.77}$$

and, consequently,

$$\mathbb{E}\|x(t_2)\|^2 \leq 8M^2(N+1)D_1^*(1+L_3)e^{-2\mu(t_2-t_1)} + L_2 + L_2L_3 := D_2^*. \tag{3.78}$$

Step 3: In a similar way, for $t \in (t_k, t_{k+1}]$ with $k \geq 2$, we have

$$\|x_t\|^2_{\mathcal{PC}} \leq 8M^2(N+1)D_k^*(1+L_3)e^{-2\mu(t-t_k)} + L_2 + L_2L_3, \tag{3.79}$$

and

$$\mathbb{E}\|x(t_{k+1})\|^2 \leq 8M^2(N+1)D_k^*(1+L_3)e^{-2\mu(t_{k+1}-t_k)} + L_2 + L_2L_3 := D_{k+1}^*. \tag{3.80}$$

For convenience, let $\mathcal{K}_1 = 8M^2(N+1)(1+L_3)$, $\mathcal{K}_2 = L_2 + L_2L_3$. Then, by using the mathematical induction method, we obtain for $k \geq 2$ that

$$\begin{aligned} D_k^* &= \mathcal{K}_1 D_{k-1}^* e^{-2\mu(t_k - t_{k-1})} + \mathcal{K}_2 \\ &\leq \mathcal{K}_1^{k-1} D_1^* e^{-2\mu(t_k-t_1)} + \mathcal{K}_2 \sum_{j=0}^{k-2} \mathcal{K}_1^k e^{-2\mu(t_k - t_{k-j})}. \end{aligned} \tag{3.81}$$

Noticing that (H_4) implies that $k-1 \leq \frac{t_k - t_1}{\beta}$ and $k \leq \frac{t_k - t_{k-j}}{\beta}$, it then follows from (3.73) and (3.81) that

$$\begin{aligned} D_k^* &\leq \mathcal{K}_1^{\frac{t_k-t_1}{\beta}} e^{-2\mu(t_k-t_1)} D_1^* + \mathcal{K}_2 \sum_{j=0}^{k-2} C_1^{\frac{t_k-t_{k-j}}{\beta}} e^{-2\mu(t_k-t_{k-j})} \\ &= e^{\frac{t_k-t_1}{\beta}\ln\mathcal{K}_1} e^{-2\mu(t_k-t_1)} D_1^* + \mathcal{K}_2 \sum_{j=0}^{k-2} e^{\frac{t_k-t_{k-j}}{\beta}\ln\mathcal{K}_1} e^{-2\mu(t_k-t_{k-j})} \\ &\leq e^{-\left(2\mu - \frac{\ln\mathcal{K}_1}{\beta}\right)(t_k-t_1)} D_1^* + \mathcal{K}_2 \sum_{j=0}^{k-2} e^{-\left(2\mu-\frac{\ln\mathcal{K}_1}{\beta}\right)(t_k-t_{k-j})} \\ &\leq e^{-\left(2\mu - \frac{\ln\mathcal{K}_1}{\beta}\right)(t_k-t_1)} D_1^* + \mathcal{K}_2 \frac{e^{\left(2\mu-\frac{\ln\mathcal{K}_1}{\beta}\right)\beta}}{e^{\left(2\mu-\frac{\ln\mathcal{K}_1}{\beta}\right)\beta} - 1}. \end{aligned} \tag{3.82}$$

Therefore, by (3.76), (3.79) and (3.82), we deduce that for all $t \in (t_k, t_{k+1}]$ with $k \geq 2$,

$$\begin{aligned}
\|x_t\|^2_{\mathcal{PC}} &\leq \mathcal{K}_1 e^{-2\mu(t-t_k)} D^*_k + \mathcal{K}_2 \\
&\leq \mathcal{K}_1 e^{-\left(2\mu - \frac{\ln \mathcal{K}_1}{\beta}\right)(t-t_1)} D^*_1 + \mathcal{K}_1\mathcal{K}_2 \frac{e^{(2\mu\beta - \ln \mathcal{K}_1)}}{e^{(2\mu\beta-\ln\mathcal{K}_1)} - 1} + \mathcal{K}_2 \\
&\leq \mathcal{K}_1^2 e^{-\left(2\mu - \frac{\ln \mathcal{K}_1}{\beta}\right)(t-t_1)} e^{-2\mu t_1} \|\phi\|^2_{\mathcal{PC}} + C \\
&\leq \mathcal{K}_1^2 e^{-\left(2\mu - \frac{\ln \mathcal{K}_1}{\beta}\right)t} \|\phi\|^2_{\mathcal{PC}} + C \\
&:= C\|\phi\|^2_{\mathcal{PC}} e^{-\left(2\mu - \frac{\ln \mathcal{K}_1}{\beta}\right)t} + C,
\end{aligned}$$

which, on account of (3.75) and (3.77), implies

$$\|x_t\|^2_{\mathcal{PC}} \leq C\|\phi\|^2_{\mathcal{PC}} e^{-\left(2\mu - \frac{\ln \mathcal{K}_1}{\beta}\right)t} + C, \quad \text{for all } t \geq 0.$$

The proof is finished. □

Remark 3.21. *We emphasize that under assumptions (C_2')-(C_3') (instead of Lipschitz condition (H_1)), we may only have a general result ensuring that there exists at least one mild solution to problem (3.2), i.e., as Theorem 3.15 may not hold, the uniqueness of solutions in Theorem 3.12 is not ensured. Hence, in Theorem 3.20 we have proved that for any solution corresponding to the initial value $\phi \in \mathcal{PC}$, this a priori estimate holds true.*

3.4.1 Existence of global attracting set: General case

A general result concerning the existence of a minimal compact set in $\mathcal{PC}$ which is globally attracting for the solutions of our problem will be proved in this subsection. To that end we first need the following compactness conclusion.

Lemma 3.22. *Assume the conditions of Theorem 3.20. Then for any bounded subset D of $\mathcal{PC}$, any sequence $\{\tau_n\}$ with $\tau_n \to \infty$ $(n \to \infty)$, $\{\phi_n\}$ with $\phi_n \in D$, and any sequence of solutions $\{x^n(\cdot)\}$ of problem (3.2) with $x^n_0 = \phi_n \in D$, the sequence $\{x^n_{\tau_n}\}$ is relatively compact in $\mathcal{PC}$.*

Proof. Without loss of generality, we assume that $\|\phi\|_{\mathcal{PC}} \leq d$ for all $\phi \in D$. For any $\phi_n \in D$, we define $u^n_{\tau_n}(\cdot) : (-\infty, 0] \to \mathbb{H}$ by

$$u^n(\tau_n + \theta) = \begin{cases} \phi_n(\tau_n + \theta), & \tau_n + \theta \in (-\infty, 0], \\ T_\alpha(\tau_n + \theta)\phi(0), & \tau_n + \theta \in [0, t_1], \\ T_\alpha(\tau_n + \theta - t_k)(u^n(t_k^-) + I_k(u^n(t_k^-))), & \tau_n + \theta \in (t_k, t_{k+1}], \end{cases}$$

for all $k = 1, 2, \cdots$.

Let us do estimates likewise as in the proof of Theorem 3.20 , by (C_1') and (H_3') we find that

$$\|u^n_{\tau_n}\|_{\mathcal{PC}} \leq Ce^{-\mu\tau_n}\|\phi\|_{\mathcal{PC}}. \tag{3.83}$$

Next we define the function $z^n_{\tau_n}(\cdot) : (-\infty, 0] \to \mathbb{H}$ by

$$z^n(\tau_n+\theta) = \begin{cases} 0, & \tau_n+\theta \in (-\infty,0], \\ \int_0^{\tau_n+\theta} T_\alpha(\tau_n+\theta-s)f(s,x^n_s)ds + \int_0^{\tau_n+\theta} T_\alpha(\tau_n+\theta-s)g(s,x^n_s)dB(s) \\ \quad + \int_0^{\tau_n+\theta} T_\alpha(\tau_n+\theta-s)h(s)dB^H_Q(s), & \tau_n+\theta \in [0,t_1], \\ T_\alpha(\tau_n+\theta-t_k)(z^n(t_k^-)+I_k(z^n(t_k^-))) + \int_{t_k}^{\tau_n+\theta} T_\alpha(\tau_n+\theta-s)f(s,x^n_s)ds \\ \quad + \int_{t_k}^{\tau_n+\theta} T_\alpha(\tau_n+\theta-s)g(s,x^n_s)dB(s) \\ \quad + \int_{t_k}^{\tau_n+\theta} T_\alpha(\tau_n+\theta-s)h(s)dB^H_Q(s), & \tau_n+\theta \in (t_k,t_{k+1}], \; k=1,2,\cdots. \end{cases} \tag{3.84}$$

It is important to observe that if $x^n_{\tau_n}(\cdot)$ satisfies the format of the mild solution to problem (3.2), then $x^n_{\tau_n} = u^n_{\tau_n} + z^n_{\tau_n}$ for $\tau_n \in [0,\infty)$ since the impulse functions I_k $(k \in \mathbb{N})$ are linear. In order to prove that $\{x^n_{\tau_n}\}$ is relatively compact in $\mathcal{PC}$, by the decomposition of $x^n_{\tau_n}$ and (3.83), it is enough to state $\{z^n_{\tau_n}\}$ is compact in $\mathcal{PC}$ as $\tau_n \to \infty$.

Initially, we show that $\{z^n(\tau_n+\cdot)\}_{n=0}^\infty$ is equicontinuous on $[-T^*,0]$ for any fixed $T^*>0$. For such T^* fixed, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the points $\tau_n+\theta > r$ for all $\theta \in [-T^*,0]$, for some $r>0$. Then, to prove the equicontinuity in the interval $[-T^*,0]$, it is sufficient to assume that for each $n \geq n_0$, and θ_1,$\theta_2 \in [-T^*,0]$ with $\theta_1<\theta_2$ (with θ_2-θ_1 sufficiently small), we have that $\tau_n+\theta_1$, $\tau_n+\theta_2 \in (t_k,t_{k+1}] \cap [r,+\infty)$, for some $k \in \mathbb{N}$ and $r>0$. Once we have proved the equicontinuity for this case, the possibility that the points $\tau_n+\theta_1$, $\tau_n+\theta_2$ may belong to different intervals can be handled by comparing with the values of the solution in the impulse time t_k, by using Theorem 3.20, estimate (3.83) and the properties of the impulsive linear function I_k, in particular the fact that $N_k \to 0$ as $k \to \infty$. Consequently, given $\varepsilon>0$, we assume that $\tau_n+\theta_2 \in (t_k,t_{k+1}] \cap [r,+\infty)$ for all $\theta_1<\theta_2$ in the interval $[-T^*,0]$, with θ_2-θ_1 sufficiently small as we will determine below. From (3.84) we can deduce, for all $n \geq n_0$,

$$\begin{aligned}
&\mathbb{E}\|z^n(\tau_n+\theta_2)-z^n(\tau_n+\theta_1)\|^2 \\
&\leq 7\|T_\alpha(\tau_n+\theta_2-t_k)-T_\alpha(\tau_n+\theta_1-t_k)\|^2\mathbb{E}\|z^n(t_k^-)+I_k(z^n(t_k^-))\|^2 \\
&\quad+7\mathbb{E}\left\|\int_{t_k}^{\tau_n+\theta_1}(T_\alpha(\tau_n+\theta_2-s)-T_\alpha(\tau_n+\theta_1-s))f(s,x_s^n)ds\right\|^2 \\
&\quad+7\mathbb{E}\left\|\int_{\tau_n+\theta_1}^{\tau_n+\theta_2}T_\alpha(\tau_n+\theta_2-s)f(s,x_s^n)ds\right\|^2 \\
&\quad+7\mathbb{E}\left\|\int_{t_k}^{\tau_n+\theta_1}(T_\alpha(\tau_n+\theta_2-s)-T_\alpha(\tau_n+\theta_1-s))g(s,x_s^n)dB(s)\right\|^2 \\
&\quad+7\mathbb{E}\left\|\int_{\tau_n+\theta_1}^{\tau_n+\theta_2}T_\alpha(\tau_n+\theta_2-s)g(s,x_s^n)dB(s)\right\|^2 \\
&\quad+7\mathbb{E}\left\|\int_{t_k}^{\tau_n+\theta_1}(T_\alpha(\tau_n+\theta_2-s)-T_\alpha(\tau_n+\theta_1-s))h(s)dB_Q^H(s)\right\|^2 \\
&\quad+7\mathbb{E}\left\|\int_{\tau_n+\theta_1}^{\tau_n+\theta_2}T_\alpha(\tau_n+\theta_2-s)h(s)dB_Q^H(s)\right\|^2 \\
&:=\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3+\mathcal{I}_4+\mathcal{I}_5+\mathcal{I}_6+\mathcal{I}_7.
\end{aligned} \tag{3.85}$$

Hereafter, we assume that $k \geq 1$, since the proof of the case $k = 0$ is similar.

Since $\phi_n \in D$, by Theorem 3.20 we find that for all $s \geq 0$ and $n \in \mathbb{N}$,

$$\|x_s^n\|_{\mathcal{PC}}^2 \leq C\|\phi\|_{\mathcal{PC}}^2 e^{-\left(2\mu-\frac{\ln \mathcal{K}_1}{\beta}\right)s}+C \leq C\left(1+e^{-\left(2\mu-\frac{\ln \mathcal{K}_1}{\beta}\right)s}\right). \tag{3.86}$$

For every $0 < \varepsilon < 1$, in view of (H_2) and Remark 3.18 *ii*), we obtain from the absolute continuity of the integral that there exists $0 < \sigma < \varepsilon$ such that

$$\sup_{t\geq 0}\int_t^{t+\sigma} k_i(s)ds < \varepsilon, \quad i=1,2,3,4, \qquad \text{and} \qquad \sup_{t\geq 0}\int_t^{t+\sigma}\|h(s)\|_Q^2 ds < \varepsilon. \tag{3.87}$$

Moreover, by Lemma 3.6 (i) and (C_1'), we deduce that $T_\alpha(s)$ is uniformly continuous for $s \in [\sigma,\infty)$, i.e., there exists $0 < \delta < \sigma$ such that for all $s_1,\ s_2 \in [\sigma,\infty)$ with $|s_1 - s_2| < \delta$, we have

$$\|T_\alpha(s_2)-T_\alpha(s_1)\| < \varepsilon. \tag{3.88}$$

Hence, by (3.85)-(3.87) and the fact that $x_{\tau_n}^n = u_{\tau_n}^n + z_{\tau_n}^n$, for all $|\theta_1-\theta_2| < \delta$ and $n > n_0$ such that $\tau_n+\theta_1,\ \tau_n+\theta_2 \in (t_k,t_{k+1}] \cap [\sigma,\infty)$ (we choose $r = \sigma$ here), it follows that

$$\mathcal{I}_1 \leq C\|T_\alpha(\tau_n+\theta_2-t_k)-T_\alpha(\tau_n+\theta_1-t_k)\|^2 e^{-\left(2\mu-\frac{\ln \mathcal{K}_1}{\beta}\right)t_k} \leq C\varepsilon. \tag{3.89}$$

For the term $\mathcal{I}_2$, by (H_4), (C_1')-(C_2'), (3.86)-(3.88), and the Cauchy-Schwarz inequality, we have for all $n \geq n_0$ and $|\theta_2 - \theta_1| < \delta$,

$$\begin{aligned}
\mathcal{I}_2 &\leq C\eta \int_{t_k}^{\tau_n+\theta_1-\sigma} \|T_\alpha(\tau_n+\theta_2-s) - T_\alpha(\tau_n+\theta_1-s)\|^2 \left(k_1(s) + k_2(s)\|x_s^n\|_{\mathcal{PC}}^2\right) ds \\
&\quad + C\sigma \int_{\tau_n+\theta_1-\sigma}^{\tau_n+\theta_1} \left(e^{-2\mu(\tau_n+\theta_2-s)} + e^{-2\mu(\tau_n+\theta_1-s)}\right) \left(k_1(s) + k_2(s)\|x_s^n\|_{\mathcal{PC}}^2\right) ds \\
&\leq C \int_{t_k}^{\tau_n+\theta_1-\sigma} \|T_\alpha(\tau_n+\theta_2-s) - T_\alpha(\tau_n+\theta_1-s)\|^2 \Bigg(k_1(s) + Ck_2(s) \\
&\quad \times \left(1 + e^{-\left(2\mu - \frac{\ln \mathcal{K}_1}{\beta}\right)s}\right) \Bigg) ds + C\sigma \int_{\tau_n+\theta_1-\sigma}^{\tau_n+\theta_1} (k_1(s) + k_2(s)) ds \leq C\varepsilon.
\end{aligned} \tag{3.90}$$

Now, for $\mathcal{I}_3$, thanks to (C_1')-(C_2'), (3.86), and the Cauchy-Schwarz inequality, we find that for all $n \geq n_0$ and $|\theta_1 - \theta_2| < \delta$,

$$\begin{aligned}
\mathcal{I}_3 &\leq C(\theta_2 - \theta_1) \int_{\tau_n+\theta_1}^{\tau_n+\theta_2} \|T_\alpha(\tau_n+\theta_2-s)\|^2 \mathbb{E}\|f(s, x_s^n)\|^2 ds \\
&\leq C(\theta_2 - \theta_1) \int_{\tau_n+\theta_1}^{\tau_n+\theta_2} e^{-2\mu(\tau_n+\theta_2-s)} \left(k_1(s) + Ck_2(s) \left(1 + e^{-\left(2\mu - \frac{\ln \mathcal{K}_1}{\beta}\right)s}\right)\right) ds \\
&\leq C\varepsilon.
\end{aligned} \tag{3.91}$$

Analogously, (C_1'), (C_3'), (1.6), (3.86)-(3.88) and the Cauchy-Schwarz inequality imply, for all $n \geq n_0$ and $|\theta_2 - \theta_1| < \delta$,

$$\begin{aligned}
\mathcal{I}_4 &\leq C \int_{t_k}^{\tau_n+\theta_1-\sigma} \|T_\alpha(\tau_n+\theta_2-s) - T_\alpha(\tau_n+\theta_1-s)\|^2 \Bigg(k_3(s) + Ck_4(s) \\
&\quad \times \left(1 + e^{-\left(2\mu - \frac{\ln \mathcal{K}_1}{\beta}\right)s}\right)\Bigg) ds + C \int_{\tau_n+\theta_1-\sigma}^{\tau_n+\theta_1} (k_3(s) + k_4(s)) \leq C\varepsilon.
\end{aligned} \tag{3.92}$$

Using the same arguments as for $\mathcal{I}_3$, by (1.6), (C_1'), (C_3') and (3.85) we find that, for all $n \geq n_0$ and $|\theta_2 - \theta_1| < \delta$,

$$\begin{aligned}
\mathcal{I}_5 &\leq CTr(Q) \int_{\tau_n+\theta_1}^{\tau_n+\theta_2} \|T_\alpha(\tau_n+\theta_2-s)\|^2 \left(k_3(s) + Ck_4(s)\left(1 + e^{-\left(2\mu - \frac{\ln \mathcal{K}_1}{\beta}\right)s}\right)\right) ds \\
&\leq C \int_{\tau_n+\theta_1}^{\tau_n+\theta_2} e^{-2\mu(\tau_n+\theta_2-s)} (k_3(s) + k_4(s)) ds \leq C\varepsilon.
\end{aligned} \tag{3.93}$$

To estimate $\mathcal{I}_6$, we can see that (1.6), (H_4), (C_1'), (3.87)-(3.88) imply that, for all

$n \geq n_0$ and $|\theta_2 - \theta_1| < \delta$,

$$\begin{aligned}\mathcal{I}_6 &\leq C\eta \int_{t_k}^{\tau_n+\theta_1-\sigma} \|T_\alpha(\tau_n+\theta_2-s) - T_\alpha(\tau_n+\theta_1-s)\|^2 \|h(s)\|_Q^2 ds \\ &\quad + C\sigma^{2H-1} \int_{\tau_n+\theta_1-\sigma}^{\tau_n+\theta_1} \|T_\alpha(\tau_n+\theta_2-s) - T_\alpha(\tau_n+\theta_1-s)\|^2 \|h(s)\|_Q^2 ds \\ &\leq C \int_{t_k}^{\tau_n+\theta_1-\sigma} \|T_\alpha(\tau_n+\theta_2-s) - T_\alpha(\tau_n+\theta_1-s)\|^2 \|h(s)\|_Q^2 ds \\ &\quad + C\sigma^{2H-1} \int_{\tau_n+\theta_1-\sigma}^{\tau_n+\theta_1} \left(e^{-2\mu(\tau_n+\theta_1-s)} + e^{-2\mu(\tau_n+\theta_2-s)}\right) \|h(s)\|_Q^2 ds \\ &\leq C \int_{t_k}^{\tau_n+\theta_1-\sigma} \|T_\alpha(\tau_n+\theta_2-s) - T_\alpha(\tau_n+\theta_1-s)\|^2 \|h(s)\|_Q^2 ds \\ &\quad + C\sigma^{2H-1} \int_{\tau_n+\theta_1-\sigma}^{\tau_n+\theta_1} \|h(s)\|_Q^2 ds \leq C\varepsilon.\end{aligned} \tag{3.94}$$

As for the last term $\mathcal{I}_7$, by (1.6), (3.87) and (C_1'), we obtain that, for all $n \geq n_0$ and $|\theta_2 - \theta_1| < \delta$,

$$\begin{aligned}\mathcal{I}_7 &\leq C(\theta_2-\theta_1)^{2H-1} \int_{\tau_n+\theta_1}^{\tau_n+\theta_2} \|T_\alpha(\tau_n+\theta_2-s)\|^2 \|h(s)\|_Q^2 ds \\ &\leq C(\theta_2-\theta_1)^{2H-1} \int_{\tau_n+\theta_1}^{\tau_n+\theta_2} e^{-2\mu(\tau_n+\theta_2-s)} \|h(s)\|_Q^2 ds \leq C\varepsilon.\end{aligned} \tag{3.95}$$

Therefore, (3.89)-(3.95) show the sequence $\{z^n(\tau_n+\cdot) : n \in \mathbb{N}\}$ is equicontinous on $[-T^*, 0]$.

Next, we state that the sequence $\{z^n(\tau_n+\theta)\}_{n=1}^{\infty}$ is relatively compact in $L^2(\Omega; \mathbb{H})$ for each fixed $\theta \in [-T^*, 0]$. Then, for such fixed $\theta \in [-T^*, 0]$, there exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$, $0 < \lambda < \tau_n + \theta$. Now, for a fixed $n \geq n_0$ there exists $k \in \mathbb{N}$ such that $\tau_n + \theta - \lambda \in (t_k, t_{k+1}] \cap [\sigma, \infty)$, and we can define

$$\begin{aligned}z_\lambda^n(\tau_n+\theta) &= T_\alpha(\lambda)\left(T_\alpha(\tau_n+\theta-\lambda-t_k)(z^n(t_k^-) + I_k(z^n(t_k^-)))\right) \\ &\quad + T_\alpha(\lambda)\Bigg(\int_{t_k}^{\tau_n+\theta-\lambda} T_\alpha(\tau_n+\theta-\lambda-s) f(s, x_s^n) ds \\ &\quad + \int_{t_k}^{\tau_n+\theta-\lambda} T_\alpha(\tau_n+\theta-\lambda-s) g(s, x_s^n) dB(s) \\ &\quad + \int_{t_k}^{\tau_n+\theta-\lambda} T_\alpha(\tau_n+\theta-\lambda-s) h(s) dB_Q^H(s)\Bigg) \\ &:= T_\alpha(\lambda)\mathcal{Z}_1^n(\tau_n+\theta-\lambda) + T_\alpha(\lambda)\mathcal{Z}_2^n(\tau_n+\theta-\lambda).\end{aligned}$$

On the one hand, if $t_k = 0$, then this implies $z^n(0) = 0$ and $\mathcal{Z}_1^n = 0$. On the other hand, if $t_k > 0$, then by (H_4), (C_1') and (3.86), we deduce

$$\begin{aligned}\mathbb{E}\|\mathcal{Z}_1^n(\tau_n+\theta-\lambda)\|^2 &\le \|T_\alpha(\tau_n+\theta-\lambda-t_k)\|^2\mathbb{E}\|z^n(t_k^-)+I_k(z^n(t_k^-))\|^2\\ &\le CM^2e^{-2\mu(\tau_n+\theta-\lambda-t_k)}\left(1+e^{-\left(2\mu-\frac{\ln\mathcal{K}_1}{\beta}\right)t_k}\right)\le C,\end{aligned}$$

and by (1.6)-(1.9), (H_2), (H_4), (C_1')-(C_3'), (3.86) and the Cauchy-Schwarz inequality,

$$\begin{aligned}&\mathbb{E}\|\mathcal{Z}_2^n(\tau_n+\theta-\lambda)\|^2 \le 3\eta\int_{t_k}^{\tau_n+\theta-\lambda}\|T_\alpha(\tau_n+\theta-\lambda-s)\|^2\mathbb{E}\|f(s,x_s^n)\|^2ds\\ &\quad+3Tr(Q)\int_{t_k}^{\tau_n+\theta-\lambda}\|T_\alpha(\tau_n+\theta-\lambda-s)\|^2\mathbb{E}\|g(s,x_s^n)\|^2ds\\ &\quad+3cH(2H-1)\eta^{2H-1}\int_{t_k}^{\tau_n+\theta-\lambda}\|T_\alpha(\tau_n+\theta-\lambda-s)\|^2\|h(s)\|_Q^2ds\\ &\le CM^2\int_{t_k}^{\tau_n+\theta-\lambda}e^{-2\mu(\tau_n+\theta-\lambda-s)}\left(k_1(s)+Ck_2(s)\left(1+e^{-\left(2\mu-\frac{\ln\mathcal{K}_1}{\beta}\right)s}\right)\right)ds\\ &\quad+CM^2\int_{t_k}^{\tau_n+\theta-\lambda}e^{-2\mu(\tau_n+\theta-\lambda-s)}\left(k_3(s)+Ck_4(s)\left(1+e^{-\left(2\mu-\frac{\ln\mathcal{K}_1}{\beta}\right)s}\right)\right)ds\\ &\quad+CM^2\int_{t_k}^{\tau_n+\theta-\lambda}e^{-2\mu(\tau_n+\theta-\lambda-s)}\|h(s)\|_Q^2ds\\ &\le C\int_{t_k}^{\tau_n+\theta-\lambda}\left(k_1(s)+k_2(s)+k_3(s)+k_4(s)+\|h(s)\|_Q^2\right)ds\\ &\le C.\end{aligned}$$

By assumption, $T_\alpha(t)$ is compact for every $t > 0$, the set $\{z_\lambda^n(\tau_n+\theta)\}_{n=1}^\infty$ is relatively compact in $L^2(\Omega;\mathbb{H})$ for every $0 < \lambda < \tau_n+\theta$. Moreover, for all $n \ge n_0$, there exists a constant $\lambda > 0$ such that $\tau_n+\theta$, $\tau_n+\theta-\lambda \in (t_k, t_{k+1}]\cap[\sigma,\infty)$ $(k\in\mathbb{N})$, thus we have

$$\begin{aligned}\mathbb{E}\|z_\lambda^n(\tau_n+\theta)-z^n(\tau_n+\theta)\|^2 &\le 2\mathbb{E}\|T_\alpha(\lambda)\mathcal{Z}_1^n(\tau_n+\theta-\lambda)-\mathcal{Z}_1^n(\tau_n+\theta)\|^2\\ &\quad+2\mathbb{E}\|T_\alpha(\lambda)\mathcal{Z}_2^n(\tau_n+\theta-\lambda)-\mathcal{Z}_2^n(\tau_n+\theta)\|^2\\ &:=\mathcal{G}_1+\mathcal{G}_2.\end{aligned}\tag{3.96}$$

Using the same argument as for (3.87), by Lemma 3.6 (ii), (3.88) and (C_1'), we find that there exists $0<\delta^*<\delta$ such that for all $0\le\lambda<\delta^*$, we have

$$\sup_{t\in[\sigma,\infty)}\|T_\alpha(\lambda)T_\alpha(t-\lambda)-T_\alpha(t)\|<C\varepsilon,\tag{3.97}$$

where σ and δ are given in (3.87) and (3.88). When $\tau_n+\theta\in[0,t_1]$, it is obvious that $\mathcal{Z}_1^n(\tau_n+\theta-\lambda)=\mathcal{Z}_1^n(\tau_n+\theta)=0$ and $\mathcal{G}_1=0$. When $\tau_n+\theta$, $\tau_n+\theta-\lambda\in(t_k,t_{k+1}]\cap[\sigma,\infty)$ $(k\ge1)$, by (3.86) and (3.97), for all $n\ge n_0$ and $0<\lambda<\delta^*$, we have

$$\begin{aligned}\mathcal{G}_1 &\leq C\|T_\alpha(\lambda)T_\alpha(\tau_n+\theta-\lambda-t_k)-T_\alpha(\tau_n+\theta-t_k)\|^2 E\|z^n(t_k^-)+I_k(z^n(t_k^-))\|^2\\ &\leq C\|T_\alpha(\lambda)T_\alpha(\tau_n+\theta-\lambda-t_k)-T_\alpha(\tau_n+\theta-t_k)\|^2\left(1+e^{-\left(2\mu-\frac{\ln\mathcal{K}_1}{\beta}\right)t_k}\right)\\ &\leq C\varepsilon,\end{aligned} \tag{3.98}$$

and

$$\begin{aligned}\mathcal{G}_2 &\leq 6E\Big\|T_\alpha(\lambda)\int_{t_k}^{\tau_n+\theta-\lambda}T_\alpha(\tau_n+\theta-\lambda-s)f(s,x_s^n)ds\\ &\quad-\int_{t_k}^{\tau_n+\theta}T_\alpha(\tau_n+\theta-s)f(s,x_s^n)ds\Big\|^2\\ &\quad+6E\Big\|T_\alpha(\lambda)\int_{t_k}^{\tau_n+\theta-\lambda}T_\alpha(\tau_n+\theta-\lambda-s)g(s,x_s^n)dB(s)\\ &\quad-\int_{t_k}^{\tau_n+\theta}T_\alpha(\tau_n+\theta-s)g(s,x_s^n)dB(s)\Big\|^2\\ &\quad+6E\Big\|T_\alpha(\lambda)\int_{t_k}^{\tau_n+\theta-\lambda}T_\alpha(\tau_n+\theta-\lambda-s)h(s)dB_Q^H(s)\\ &\quad-\int_{t_k}^{\tau_n+\theta}T_\alpha(\tau_n+\theta-s)h(s)dB_Q^H(s)\Big\|^2\\ &:=\mathcal{G}_{21}+\mathcal{G}_{22}+\mathcal{G}_{23}.\end{aligned} \tag{3.99}$$

In what follows, we will do estimates for (3.99) one by one. By (3.86), (3.97), (H_4), (C_1')-(C_2') and the Cauchy-Schwarz inequality, we obtain for all $n\geq n_0$ and $0<\lambda<\delta^*$,

$$\begin{aligned}\mathcal{G}_{21} &\leq C\Bigg(\int_{t_k}^{\tau_n+\theta-\sigma}\|T_\alpha(\lambda)T_\alpha(\tau_n+\theta-\lambda-s)-T_\alpha(\tau_n+\theta-s)\|^2\\ &\quad+(\sigma-\lambda)\int_{\tau_n+\theta-\sigma}^{\tau_n+\theta-\lambda}\left(e^{-2\mu\lambda}e^{-2\mu(\tau_n+\theta-\lambda-s)}+e^{-2\mu(\tau_n+\theta-s)}\right)\Bigg)\\ &\quad\times\left(k_1(s)+Ck_2(s)\left(1+e^{-\left(2\mu-\frac{\ln\mathcal{K}_1}{\beta}\right)s}\right)\right)ds\\ &\quad+C\lambda\int_{\tau_n+\theta-\lambda}^{\tau_n+\theta}e^{-2\mu(\tau_n+\theta-s)}\left(k_1(s)+Ck_2(s)\left(1+e^{-\left(2\mu-\frac{\ln\mathcal{K}_1}{\beta}\right)s}\right)\right)ds\\ &\leq C\varepsilon.\end{aligned} \tag{3.100}$$

For $\mathcal{G}_{22}$, in a similar way, by (1.6), (3.86), (3.97), (C_1') and (C_3'), we find that, for all $n \geq n_0$ and $0 < \lambda < \delta^*$,

$$
\begin{aligned}
\mathcal{G}_{22} \leq & CTr(Q) \int_{t_k}^{\tau_n+\theta-\sigma} \|T_\alpha(\lambda)T_\alpha(\tau_n+\theta-\lambda-s) - T_\alpha(\tau_n+\theta-s)\|^2 \\
& \times \left(k_3(s) + Ck_4(s)\left(1 + e^{-\left(2\mu - \frac{\ln \mathcal{K}_1}{\beta}\right)s}\right)\right) ds \\
& + CTr(Q) \int_{\tau_n+\theta-\sigma}^{\tau_n+\theta-\lambda} \left(e^{-2\mu\lambda} e^{-2\mu(\tau_n+\theta-\lambda-s)} + e^{-2\mu(\tau_n+\theta-s)}\right) \\
& \times \left(k_3(s) + Ck_4(s)\left(1 + e^{-\left(2\mu - \frac{\ln \mathcal{K}_1}{\beta}\right)s}\right)\right) ds \\
& + CTr(Q) \int_{T\wedge\tau_n+\theta-\lambda}^{\tau_n+\theta} e^{-2\mu(\tau_n+\theta-s)} \left(k_3(s) + Ck_4(s)\left(1 + e^{-\left(2\mu - \frac{\ln \mathcal{K}_1}{\beta}\right)s}\right)\right) ds \\
\leq & C\varepsilon + C\int_{\tau_n+\theta-\sigma}^{\tau_n+\theta-\lambda} (k_3(s) + k_4(s))ds + C\int_{\tau_n+\theta-\lambda}^{\tau_n+\theta} (k_3(s) + k_4(s))ds \\
\leq & C\varepsilon.
\end{aligned}
\tag{3.101}
$$

For the last term $\mathcal{G}_{23}$, it follows from (1.6), (3.97), (H_2), (H_4) and (C_1') that, for all $n \geq n_0$ and $0 < \lambda < \delta^*$,

$$
\begin{aligned}
\mathcal{G}_{23} \leq & C\eta^{2H-1} \int_{t_k}^{\tau_n+\theta-\sigma} \|T_\alpha(\lambda)T_\alpha(\tau_n+\theta-\lambda-s) - T_\alpha(\tau_n+\theta-s)\|^2 \|h(s)\|_Q^2 ds \\
& + C(\sigma-\lambda)^{2H-1} \int_{\tau_n+\theta-\sigma}^{\tau_n+\theta-\lambda} \left(e^{-2\mu\lambda} e^{-2\mu(\tau_n+\theta-\lambda-s)} + e^{-2\mu(\tau_n+\theta-s)}\right) \|h(s)\|_Q^2 ds \\
& + C\lambda^{2H-1} \int_{\tau_n+\theta-\lambda}^{\tau_n+\theta} e^{-2\mu(\tau_n+\theta-s)} \|h(s)\|_Q^2 ds \\
\leq & C\varepsilon + C(\sigma-\lambda)^{2H-1} \int_{\tau_n+\theta-\sigma}^{\tau_n+\theta-\lambda} \|h(s)\|_Q^2 ds + C\lambda^{2H-1} \int_{\tau_n+\theta-\lambda}^{\tau_n+\theta} \|h(s)\|_Q^2 ds \\
\leq & C\varepsilon.
\end{aligned}
\tag{3.102}
$$

Thus, (3.98)-(3.102) imply

$$
\mathbb{E}\|z_\lambda^n(\tau_n+\theta) - z^n(\tau_n+\theta)\|^2 \to 0
$$

as $\lambda \to 0$, uniformly for n. Hence, $\{z_{\tau_n}^n(\theta)\}_{n=1}^\infty$ is precompact in $L^2(\Omega;\mathbb{H})$ for any $\theta \in [-T^*, 0]$. By the Arzelà-Ascoli theorem there exists a subsequence $\{z_{\tau_{n'}}^{n'}(\theta)\}_{n=1}^\infty$ and a function $\xi : \mathbb{R}^- \to L^2(\Omega;\mathbb{H})$ which is the uniform limit of $\{z_{\tau_{n'}}^{n'}(\cdot)\}$ on every interval $[-T^*, 0]$.

Eventually, let us show that $x_{\tau_{n'}}^{n'}(\cdot)$ converges to ξ in $L^2(\Omega;\mathbb{H})$. To do so, we choose some $n \geq n_0$ such that $\tau_n + \theta \in (t_k, t_{k+1}] \cap [\sigma, \infty)$. In view of (1.6)-(1.9), (H_2), (H_3'), (H_4), (C_1')-(C_3'), (3.86) and the Cauchy-Schwarz inequality, together with the fact that $x^n_{\tau_n} = u^n_{\tau_n} + z^n_{\tau_n}$, the following a priori estimate holds,

$$\begin{aligned}
&\mathbb{E}\|z^n(\tau_n+\theta)\|^2 \le CM^2(N+1)\left(1+e^{-\left(2\mu-\frac{\ln \mathcal{K}_1}{\beta}\right)t_k}\right)\\
&+ C\eta M^2 \int_{t_k}^{\tau_n+\theta} e^{-2\mu(\tau_n+\theta-s)}\left(k_1(s)+Ck_2(s)\left(1+e^{-\left(2\mu-\frac{\ln \mathcal{K}_1}{\beta}\right)s}\right)\right)ds\\
&+ CTr(Q)M^2 \int_{t_k}^{\tau_n+\theta} e^{-2\mu(\tau_n+\theta-s)}\left(k_3(s)+Ck_4(s)\left(1+e^{-\left(2\mu-\frac{\ln \mathcal{K}_1}{\beta}\right)s}\right)\right)ds\\
&+ C\eta^{2H-1}M^2 \int_{t_k}^{\tau_n+\theta} e^{-2\mu(\tau_n+\theta-s)}\|h(s)\|_Q^2 ds \le C, \qquad \theta \le 0,
\end{aligned} \tag{3.103}$$

where we assumed that $k \ge 1$, since the proof of the case $k = 1$ is similar. From (3.103), we know that

$$\sup_{\theta\in[-T^*,0]} e^{\gamma\theta}\mathbb{E}\|z^{n'}_{\tau_{n'}}(\theta)\|^2 = \sup_{\theta\in[-T^*,0]} e^{\gamma\theta}\mathbb{E}\|z^{n'}(\tau_{n'}+\theta)\|^2 \le C,$$

and thus for every $T^* > 0$,

$$\sup_{\theta\in[-T^*,0]} e^{\gamma\theta}\mathbb{E}\|\xi(\theta)\|^2 = \lim_{n'\to\infty} \sup_{\theta\in[-T^*,0]} e^{\gamma\theta}\mathbb{E}\|z^{n'}_{\tau_{n'}}(\theta)\|^2 \le C, \tag{3.104}$$

which implies that $\xi \in \mathcal{PC}$ on $[-T^*, 0]$ and $\|\xi\|_{\mathcal{PC}} \le C$.

What we want to prove is $z^{n'}_{\tau_{n'}}(\cdot)$ converges to ξ on $(-\infty, 0]$. That is, we need to check that for every $\varepsilon > 0$, there exists $N(\varepsilon)$ such that

$$\|z^{n'}_{\tau_{n'}} - \xi\|^2_{\mathcal{PC}} = \sup_{\theta\in(-\infty,0]} e^{\gamma\theta}\mathbb{E}\|z^{n'}_{\tau_{n'}}(\theta) - \xi(\theta)\|^2 \le \varepsilon, \qquad n' > N(\varepsilon). \tag{3.105}$$

Thanks to (3.104), we have that $z^{n'}_{\tau_{n'}}(\cdot)$ converges to ξ on $[-T^*, 0]$ for arbitrarily fixed $T^* > 0$. Therefore, we choose $T^* \ge \tau_{n'}$, where $n' > N(\varepsilon)$ defined in (3.105). Obviously in order to prove (3.105), it only remains to show that

$$\sup_{\theta\in(-\infty,-T^*]} e^{\gamma\theta}\mathbb{E}\|z^{n'}_{\tau_{n'}}(\theta) - \xi(\theta)\|^2 \le \varepsilon, \qquad n' > N(\varepsilon), \;\; T^* \ge \tau_{n'}. \tag{3.106}$$

Observe that for $n' > N(\varepsilon)$, $T^* \ge \tau_{n'}$, combining with (3.103)-(3.104), we have

$$\begin{aligned}
&\sup_{\theta\in(-\infty,-T^*]} e^{\gamma\theta}\mathbb{E}\|z^{n'}_{\tau_{n'}}(\theta) - \xi(\theta)\|^2\\
&\le \sup_{\theta\in(-\infty,-T^*]} e^{\gamma\theta}\mathbb{E}\|z^{n'}_{\tau_{n'}}(\theta)\|^2 + \sup_{\theta\in(-\infty,-T^*]} e^{\gamma\theta}\mathbb{E}\|\xi(\theta)\|^2\\
&\le Ce^{-\gamma T^*} + \lim_{n'\to\infty} \sup_{\theta\in(-\infty,-T^*]} e^{\gamma\theta}\mathbb{E}\|z^{n'}_{\tau_{n'}}(\theta)\|^2\\
&\le C\varepsilon.
\end{aligned} \tag{3.107}$$

On account of (3.105) and (3.107), the convergence of $\{z^{n'}_{\tau_{n'}}(\cdot)\}$ to ξ in $\mathcal{PC}$ immediately follows. Recall that the previous decomposition shows $x^n_{\tau_n} = u^n_{\tau_n} + z^n_{\tau_n}$. Moreover, (3.83) implies that

$$\lim_{\tau_n\to\infty} \|u^n_{\tau_n}\|_{\mathcal{PC}} = 0,$$

since $\phi_n \in D$. Thus we have found the convergence of $x^{n'}_{\tau_{n'}}$ to ξ in $\mathcal{PC}$, and the proof of this lemma is finished. □

Now we analyze the properties of the omega limit sets for our problem.

Theorem 3.23. *Assume the conditions of Theorem 3.20. Then for any bounded subset D of $\mathcal{PC}$, the set*

$$\omega(D) = \{x : \exists \tau_n \to \infty, \phi_n \in D \text{ and a sequence of solutions } x^n(\cdot) \text{ of problem } (3.2) \text{ with } x^n_0 = \phi_n \in D \text{ such that } x^n_{\tau_n} \to x \text{ in } \mathcal{PC}\}.$$

Proof. The definition of omega limit set and Lemma 3.22 imply that $\omega(D)$ is nonempty and compact immediately. Now we show that $\omega(D)$ attracts D. We argue by contradiction, then if the result were not true, there would exist $\varepsilon > 0$ and sequences $\{\tau_n\}$ with $\tau_n \to \infty$ $(n \to \infty)$, $\{\phi_n\}$ with $\phi_n \in D$ and solutions $\{x^n(\cdot)\}$ of (3.2) with initial values $x^n_0 = \phi_n$ such that

$$dist(x^n_{\tau_n}, \omega(D)) > \varepsilon, \quad \forall n \in \mathbb{N}, \tag{3.108}$$

where $\text{dist}(\cdot,\cdot)$ is the metric for the topology of $\mathcal{PC}$. By Lemma 3.22, we can ensure that $x^n_{\tau_n}$ is relatively compact and possesses at least one cluster point $z \in \mathcal{PC}$. Obviously $z \in \omega(D)$, and this contradicts (3.108). Thus this theorem is completed. □

We are now ready to state the following key result.

Theorem 3.24. *Assume the conditions in Theorem 3.20. Then the set*

$$A = \overline{\bigcup \{\omega(D) : \; D \subset \mathcal{PC}, \; D \text{ bounded}\}}$$

is compact in $\mathcal{PC}$, and, moreover, is the minimal closed set that attracts all bounded subsets of $\mathcal{PC}$ in the topology of $\mathcal{PC}$. In other words, for any bounded set $D \subset \mathcal{PC}$ and any $\varepsilon > 0$, there exists $t(D,\varepsilon) > 0$ such that for any $\phi \in D$ and any solution $x(\cdot)$ of (3.2) *with initial value ϕ, it holds*

$$dist(x_t, A) \le \varepsilon, \quad \text{for all } t \ge t(D,\varepsilon).$$

Proof. Let us denote

$$\widetilde{A} = \{\omega(D) : D \subset \mathcal{PC}, \; D \text{ bounded}\},$$

and let us first prove that $\widetilde{A}$ is relatively compact, which will imply the compactness of A.

Indeed, let $\{\xi^n\}_{n=1}^{\infty}$ be a sequence in $\widetilde{A}$ with $\xi^n \in \omega(D_n)$, and $\|D_n\|_{\mathcal{PC}} = \sup_{\phi_n \in D_n} \|\phi_n\|_{\mathcal{PC}} \le d_n$. Thanks to the definition of $\omega(D)$, there exist sequences $\{\tau_n\}$ with $\tau_n \to +\infty$ and

$$\max\left\{\frac{d_n}{e^{2\mu\tau_n}}, \; \frac{d_n^2}{e^{(2\mu - \ln \mathcal{K}_1/\beta)\tau_n}}\right\} \to 0 \qquad \text{as} \qquad n \to +\infty, \tag{3.109}$$

and $\{\phi_n\}$ with $\phi_n \in D_n$ whose solutions $\{x^n(\cdot)\}$ of problem (3.2) corresponding to the initial datum $x_0^n = \phi_n \in D_n$ satisfy, for all $n \in \mathbb{N}$,

$$\|x_{\tau_n}^n - \xi^n\|_{\mathcal{PC}} < \frac{1}{n}. \tag{3.110}$$

Arguing as in the proof of Lemma 3.22, taking into account (3.109), one can prove that $\{x_{\tau_n}^n\}_{n=0}^{\infty}$ is relatively compact in $\mathcal{PC}$. Therefore, this result and (3.110) imply that $\{\xi^n\}_{n=1}^{\infty}$ is relatively compact in $\mathcal{PC}$, and thus A is compact in $\mathcal{PC}$.

Finally we show that A is the minimal closed set attracting any bounded set $D \subset \mathcal{PC}$. To prove this, notice that if A' is another closed subset which attracts any bounded set $D \subset \mathcal{PC}$, then by the definition of $\omega(D)$, we have that $\omega(D) \subset A'$, and thus $\bigcup\{\omega(D) : D \subset \mathcal{PC}, D \text{ bounded}\}$ belongs to A'. Since A' is closed,

$$A = \overline{\bigcup\{\omega(B) : B \subset \mathcal{PC},\ B \text{ bounded}\}} \subseteq A',$$

and the proof is complete. □

3.4.2 Existence of the singleton global attracting set in the case of uniqueness of solutions

In general, we cannot obtain much more information about the attracting set just proved to exist in the previous section. In fact, such attracting sets may have a complex structure, even of fractal nature as the vast literature on the theory of global attractors has shown over the last decades. However, in the case of uniqueness of solutions, we can provide more details of the geometrical structure of this set. In fact, we will be able to prove in this subsection that it becomes a singleton, which means the solutions are attracted by a single point in $\mathcal{PC}$, which is not in general an equilibrium point of the problem.

We start this section with the following a priori estimate.

Theorem 3.25. *Let $A \in \mathbb{A}^{\alpha}(\omega_0, \theta_0)$ with $\theta_0 \in (0, \frac{\pi}{2}]$. Assume that (H_1), (H_3'), (H_4) and (C_1')hold and, in addition,*

$$\gamma > 2\mu > \mathcal{R} + \frac{\ln w_1}{\beta}, \tag{3.111}$$

(C_4')

$$\int_0^{\infty} e^{2\mu s}\mathbb{E}\|f(s,0)\|^2 ds < \infty, \quad \int_0^{\infty} e^{2\mu s}\mathbb{E}\|g(s,0)\|^2 ds < \infty,$$

$$\int_0^{\infty} e^{2\mu s}\|h(s)\|_Q^2 ds < \infty.$$

Then, for every $\phi \in \mathcal{PC}$, there exists a unique mild solution to problem (3.2) fulfilling

$$\|x_t\|_{\mathcal{PC}}^2 \le C(1 + \|\phi\|_{\mathcal{PC}}^2) e^{-\left(2\mu - \mathcal{R} - \frac{\ln w_1}{\beta}\right)t},$$

where

$$\mathcal{R} = 8M^2(\eta l_1 + Tr(Q) l_2), \qquad w_1 = 8M^2(1+N) + e^{\mathcal{R}\eta}.$$

Proof. The proof of this theorem follows the lines of the corresponding Theorem 3.17. Thus we only sketch it. The proof is split into three steps.

Step 1: From Definition 1.6, (1.9), (H_1), (C_1') and the Cauchy-Schwarz inequality, we obtain for $t \in [0, t_1]$,

$$\begin{aligned}\mathbb{E}\|x(t)\|^2 \leq 4M^2 e^{-2\mu t}\|\phi\|^2_{\mathcal{PC}} + 8M^2 e^{-2\mu t}(\eta l_1 + Tr(Q) l_2)\int_0^t e^{2\mu s}\|x_s\|^2_{\mathcal{PC}} ds \\ + 8M^2 e^{-2\mu t}\int_0^t e^{2\mu s}\left(\eta \mathbb{E}\|f(s,0)\|^2 + Tr(Q)\mathbb{E}\|g(s,0)\|^2\right) ds \\ + 4cH(2H-1)\eta^{2H-1} M^2 e^{-2\mu t}\int_0^t e^{2\mu s}\|h(s)\|^2_Q ds,\end{aligned} \tag{3.112}$$

and condition (C_4') ensures that there exist three constants $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_3$ such that

$$\begin{aligned} cH(2H-1)\eta^{2H-1}\int_0^t e^{2\mu s}\|h(s)\|^2_Q ds \leq \mathcal{R}_1, \\ \int_0^t e^{2\mu s}\mathbb{E}\|f(s,0)\|^2 ds \leq \mathcal{R}_2, \qquad \int_0^t e^{2\mu s}\mathbb{E}\|g(s,0)\|^2 ds \leq \mathcal{R}_3. \end{aligned} \tag{3.113}$$

Replacing (3.113) into (3.112) implies, for $t \in [0, t_1]$,

$$\mathbb{E}\|x(t)\|^2 \leq 4M^2 e^{-2\mu t}\|\phi\|^2_{\mathcal{PC}} + 8M^2 e^{-2\mu t}\mathcal{R}_4 + 8M^2 e^{-2\mu t}\mathcal{R}_5\int_0^t e^{2\mu s}\|x_s\|^2_{\mathcal{PC}} ds,$$

where we have used the notation

$$\mathcal{R}_4 = (\eta\mathcal{R}_2 + Tr(Q)\mathcal{R}_3) + \frac{\mathcal{R}_1}{2}, \qquad \mathcal{R}_5 = \eta l_1 + Tr(Q) l_2.$$

An analogous argument to the one in Theorem 3.20 to deal with the infinite delay, together with the Gronwall inequality, show that

$$\|x_t\|^2_{\mathcal{PC}} \leq (4M^2\|\phi\|^2_{\mathcal{PC}} + 8M^2\mathcal{R}_4)e^{-(2\mu-\mathcal{R})t},$$

and, consequently,

$$\mathbb{E}\|x(t_1)\|^2 \leq (4M^2\|\phi\|^2_{\mathcal{PC}} + 8M^2\mathcal{R}_4)e^{-(2\mu-\mathcal{R})t_1} := B_1^*. \tag{3.114}$$

Step 2: Similar to (3.112), in view of (H_3'), we find for $t \in (t_1, t_2]$ that

$$\begin{aligned}\mathbb{E}\|x(t)\|^2 \leq 8M^2 e^{-2\mu(t-t_1)}(1+N)\mathbb{E}\|x(t_1^-)\|^2 + 8M^2 e^{-2\mu(t-t_1)}\mathcal{R}_4 e^{-(2\mu-\mathcal{R})t_1} \\ + 8M^2 e^{-2\mu(t-t_1)}\mathcal{R}_5\int_{t_1}^t e^{2\mu(s-t_1)}\|x_s\|^2_{\mathcal{PC}} ds.\end{aligned}$$

We proceed now as in the proof of Step 2 of Theorem 3.17, combing the Gronwall inequality. It then follows, for $t \in (t_1, t_2]$,

$$\|x_t\|^2_{\mathcal{PC}} \leq (8M^2(1+N) + e^{\mathcal{R}\eta})B_1^* + 8M^2\mathcal{R}_4 e^{-(2\mu-\mathcal{R})t_1}e^{-(2\mu-\mathcal{R})(t-t_1)}, \tag{3.115}$$

and, therefore,

$$\mathbb{E}\|x(t_2)\|^2 \le (8M^2(1+N)+e^{\mathcal{R}\eta})B_1^*+8M^2\mathcal{R}_4e^{-(2\mu-\mathcal{R})t_1}e^{-(2\mu-\mathcal{R})(t_2-t_1)} := B_2^*. \quad (3.116)$$

Step 3: The same reasoning as above implies that for $t \in (t_k, t_{k+1}]$ with $k \ge 2$,

$$\|x_t\|_{\mathcal{PC}}^2 \le (8M^2(1+N)+e^{\mathcal{R}\eta})B_k^* + 8M^2\mathcal{R}_4e^{-(2\mu-\mathcal{R})t_k}e^{-(2\mu-\mathcal{R})(t-t_k)}, \quad (3.117)$$

and

$$\begin{aligned}\mathbb{E}\|x(t_k)\|^2 \le {} & (8M^2(1+N)+e^{\mathcal{R}\eta})B_k^* \\ & + 8M^2\mathcal{R}_4e^{-(2\mu-\mathcal{R})t_k}e^{-(2\mu-\mathcal{R})(t_{k+1}-t_k)} := B_{k+1}^*.\end{aligned} \quad (3.118)$$

For convenience, let $w_1 = 8M^2(1+N)+e^{\mathcal{R}\eta}$. It is obvious that $w_1 > 2$ so that $\sum_{k=0}^{k-2} w_1^j \le \frac{w_1^{k-1}}{w_1-\frac{w_1}{2}} \le 2w_1^{k-2}$. In addition, condition (H_4) implies that $k-1 \le \frac{t_k-t_1}{\beta}$ and $k\beta < t_k$. Then for $k \ge 2$, the mathematical induction method furnishes that

$$B_k^* \le B_1^* e^{-\left(2\mu-\mathcal{R}-\frac{\ln w_1}{\beta}\right)}(t_k-t_1) + 16M^2\mathcal{R}_4e^{-\left(2\mu-\mathcal{R}-\frac{\ln w_1}{\beta}\right)t_k}. \quad (3.119)$$

Therefore, by (3.114), (3.117) and (3.119), we deduce that, for $\in (t_k, t_{k+1}]$ with $k \ge 2$,

$$\|x_t\|_{\mathcal{PC}}^2 \le C(1+\|\phi\|_{\mathcal{PC}}^2)e^{-\left(2\mu-\mathcal{R}-\frac{\ln w_1}{\beta}\right)t},$$

which, thanks to (3.116) and (3.119), implies that, for all $t > 0$,

$$\|x_t\|_{\mathcal{PC}}^2 \le C(1+\|\phi\|_{\mathcal{PC}}^2)e^{-\left(2\mu-\mathcal{R}-\frac{\ln w_1}{\beta}\right)t}.$$

This completes the proof. □

In order to show that the global attracting set is a singleton set, we first establish the second moment exponential stability of solutions to problem (3.2).

Lemma 3.26. *Assume the conditions of Theorem 3.25. Then, for any two solutions $x(t)$ and $y(t)$ of problem (3.2) corresponding to initial values ψ and ϕ in $\mathcal{PC}$, we have*

$$\|x_t - y_t\|_{\mathcal{PC}}^2 \le 3M^2\|\psi-\phi\|_{\mathcal{PC}}^2 e^{-\left(2\mu-\mathcal{R}-\frac{\ln w_1}{\beta}\right)t}, \qquad \forall t \ge 0,$$

where $\mathcal{R}$ and w_1 are defined in Theorem 3.25.

Proof. It is straightforward that we are able to obtain global existence (without uniqueness) of mild solutions to problem (3.2) by the conditions of this lemma. An analogous argument to that already applied in the proof of Theorem 3.14 proves the exponential asymptotic behavior with condition (C_4'), so we omit the details here. □

Now we state and prove the main results of this subsection.

Theorem 3.27. *Assume the conditions of Theorem 3.25. Then*

(i) *For any bounded subset D of $\mathcal{PC}$, any sequence $\{\tau_n\}$ with $\tau_n \to +\infty$ $(n \to +\infty)$, $\{\phi_n\}$ with $\phi_n \in D$, and any sequence of solution $\{x^n(\cdot)\}$ of problem (3.2) with $x_0^n = \phi_n \in D$, this last sequence $\{x_{\tau_n}^n\}$ is relatively compact in $\mathcal{PC}$.*

(*ii*) *For any bounded subset D of $\mathcal{PC}$, the set*

$$\omega(D) \ = \{x : \exists \tau_n \to \infty, \phi_n \in D \text{ and a sequence of solutions } x^n(\cdot) \text{of problem } (3.2) \text{with } \ x_0^n = \phi_n \in D \ \text{ such that } \ x_{\tau_n}^n \to x \text{ in } \mathcal{PC}\}$$

is a singleton set and attracts D.

(*iii*) *The set*

$$\mathcal{A} = \bigcup \{\omega(D) : \ D \subset \mathcal{PC}, \ D \text{ bounded } \}$$

is a singleton set, and the minimal set that attracts all bounded subsets of $\mathcal{PC}$.

Proof. (*i*) To prove $\{x_{\tau_n}^n\}_{n=1}^{\infty}$ is precompact in $\mathcal{PC}$, we only need to state that $\{x_{\tau_n}^n\}$ is a Cauchy sequence in $\mathcal{PC}$. Thanks to Theorem 3.25 and Lemma 3.26, we deduce that

$$\begin{aligned} \|x_{\tau_n}^n - x_{\tau_m}^m\|_{\mathcal{PC}}^2 &\le 3\|x_{\tau_n}^n - x_{\tau_n}^m\|_{\mathcal{PC}}^2 + 3\|x_{\tau_n}^m\|_{\mathcal{PC}}^2 + 3\|x_{\tau_m}^m\|_{\mathcal{PC}}^2 \\ &\le C\|\phi_n - \phi_m\|_{\mathcal{PC}}^2 e^{-\left(2\mu - \mathcal{R} - \frac{\ln w_1}{\beta}\right)\tau_n} \\ &\quad + C(1 + \|\phi_m\|_{\mathcal{PC}}^2)\left(e^{-\left(2\mu - \mathcal{R} - \frac{\ln w_1}{\beta}\right)\tau_n} + e^{-\left(2\mu - \mathcal{R} - \frac{\ln w_1}{\beta}\right)\tau_m}\right). \end{aligned} \tag{3.120}$$

Furthermore, as D is a bounded subset of $\mathcal{PC}$ then

$$\|D\|_{\mathcal{PC}} := \sup_{\phi \in D} \|\phi\|_{\mathcal{PC}} \le d, \tag{3.121}$$

and hence (3.120) and (3.121) imply that $\{x_{\tau_n}^n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{PC}$ as n, $m \to \infty$.

(*ii*) Now we need to prove that $\omega(D)$ is a singleton set. If this were not the case, then there would exist x, $y \in \omega(D)$ such that $x \neq y$. By the definition of $\omega(D)$, we see there exist sequences $\{\tau_n\}$ and $\{s_m\}$ with $\tau_n \to$ $(n \to \infty)$ and $s_m \to \infty$ $(m \to \infty)$, $\{\psi_n\}$ and $\{\phi_m\}$ with ψ_n, $\phi_m \in D$, the solutions $\{x^n(\cdot)\}$ and $\{y^m(\cdot)\}$ of problem (3.2) with $x_0^n = \psi_n$ and $y_0^m = \phi_m$ such that

$$x_{\tau_n}^n \to x \ (n \to \infty) \quad \text{and} \quad y_{s_m}^m \to y \ (n \to \infty).$$

Taking into account Theorem 3.25 and Lemma 3.26, we derive that

$$\begin{aligned} \|x_{\tau_n}^n - y_{s_m}^m\|_{\mathcal{PC}}^2 &\le 3\|x_{\tau_n}^n - y_{\tau_n}^m\|_{\mathcal{PC}}^2 + 3\|y_{\tau_n}^m\|_{\mathcal{PC}}^2 + 3\|y_{s_m}^m\|_{\mathcal{PC}}^2 \\ &\le C\|\psi_n - \phi_m\|_{\mathcal{PC}}^2 e^{-\left(2\mu - \mathcal{R} - \frac{\ln w_1}{\beta}\right)\tau_n} \\ &\quad + C(1 + \|\phi_m\|_{\mathcal{PC}}^2)\left(e^{-\left(2\mu - \mathcal{R} - \frac{\ln w_1}{\beta}\right)\tau_n} + e^{-\left(2\mu - \mathcal{R} - \frac{\ln w_1}{\beta}\right)s_m}\right), \end{aligned}$$

which implies that

$$\|x_{\tau_n}^n - y_{s_m}^m\|_{\mathcal{PC}}^2 \to 0 \qquad \text{as} \quad n, \ m \to \infty.$$

Hence $\|x - y\|_{\mathcal{PC}} = 0$, and this is a contradiction since $x \neq y$.

(iii) The set $\mathcal{A}$ is a bounded subset of $\mathcal{PC}$ thanks to Theorem 3.25. The assertion (ii) implies that $\omega(\mathcal{B}(0,\rho))$ is a singleton for each $\rho \in \mathbb{R}^+$, where $\mathcal{B}(0,\rho) = \{x \in \mathcal{PC} : \|x\|_{\mathcal{PC}} \leq \rho\}$. From the definition of omega limit set we have that $\omega(\mathcal{B}(0,1)) \subset \omega(\mathcal{B}(0,2)) \subset \cdots \subset \omega(\mathcal{B}(0,n)) \cdots$, and as all of them are singleton sets, all of them must coincide, i.e., $\omega(\mathcal{B}(0,1)) = \omega(\mathcal{B}(0,2)) = \cdots = \omega(\mathcal{B}(0,n)) = \cdots$. Consequently,

$$\mathcal{A} = \bigcup \{\omega(D) : \; D \subset \mathcal{PC}, D \text{ bounded}\} = \bigcup_{\rho \in \mathbb{N}} \Big\{ \omega(\mathcal{B}(0,\rho)) \Big\}$$

is a singleton set. Therefore, $\mathcal{A}$ is the minimal set attracting any bounded set $D \subset \mathcal{PC}$, and we have completed the proof of Theorem 3.27. $\square$

Chapter 4

Stochastic time fractional 2D-Stokes equations with delay

4.1 Motivation and preliminaries

The well-posedness of flow problems in a viscous fluid is crucial for many areas of science and engineering, for example, the automotive and aerospace industries, as well as nanotechnology. In the latter case of microfluidic structures, we often encounter flow problems at moderate viscosities which arise, in the study of the modeling of various devices for the separation and manipulation of particles in microfluids systems [51], in the study of tumor tissue as a porous medium described by Darcy's law [36, 37, 43], etc. In applications such as these, the Stokes equations provide a first approximation of the more general Navier-Stokes equations in situations where the flow is nearly steady and slow, and has small velocity gradients, so the inertial effects can be ignored.

P.M. Carvalho-Neto and G. Planas analyzed in [22] the following Navier-Stokes model with Caputo fractional time derivative,

$$\begin{cases} D_t^\alpha u - \kappa\Delta u + u\cdot\nabla u + \nabla p = f & \text{in } \mathbb{R}^N,\ t\geq 0,\\ \nabla\cdot u = 0 & \text{in } \mathbb{R}^N,\ t\geq 0,\\ u(0,x)=u_0 & \text{in } \mathbb{R}^N, \end{cases} \tag{4.1}$$

where D_t^α is the Caputo fractional time derivative of order $\alpha\in(0,1)$ with respect to t (see Definition 1.4), u is the velocity field of the fluid, $\kappa>0$ is the kinematic viscosity, p is the pressure, f is the external force and u_0 is an appropriate initial value, and $N\geq 2$. The authors in [22] analyzed the well-posedness of the problem, the existence and eventual uniqueness of mild solutions as well as the regularity in time.

However, in order to have a much better description of our model, it is sensible to consider some other features in the formulation of the equations. On the one hand, it is well known and accepted nowadays that, in physical systems of the real world, the different stochastic perturbations that originate from many natural sources are ubiquitous, most often, they cannot be ignored. This leads us to consider some randomness in the model which can be described by some kind of white or colored noise or some other type of stochastic term. On the other hand, it is also obvious that the future evolution of a system does not only depend on its current state, its past history does determine its future behavior too. Also, on those problems in which we intend to apply some control, it is very convenient to consider some delay or memory term in the formulation [17, 18].

Motivated by the previous considerations, our model can be more realistic if we introduce these both features in the formulation. Needless to say, there are many choices in the type of noise, such as, Brownian motion/Wiener processes, fractional Brownian motion, Lévy or Poisson ones, etc. In order to perform our analysis clearly and show how it works, we prefer to consider the classical and standard Brownian motion, because the problem can be easily handled mathematically, and serves as guide for more complicated expressions. Based on these advantages, there has been a growing interest in stochastic time-fractional partial differential equations with delays. For instance, in more recent decades, the stochastic classical/time fractional partial differential equations have been extensively studied theoretically [29, 56, 36, 82, 89]. However, there appear to be fewer studies in the literatures related to the theoretical analysis of stochastic Stokes/Navier-Stokes equations driven by multiplicative noise with time fractional derivative, and as far as we know, no one dealt with delays. This is why we are strongly interested in the following problem

$$\begin{cases} D_t^\alpha u - \kappa\Delta u + u\cdot\nabla u + \nabla p = f(t,u_t) + g(t,u_t)\frac{dW(t)}{dt} & \text{in } \mathbb{R}^N,\ t\geq 0,\\ \nabla\cdot u = 0 & \text{in } \mathbb{R}^N,\ t\geq 0,\\ u(t,x) = \varphi(t,x) & \text{in } \mathbb{R}^N,\ t\in[-h,0], \end{cases} \tag{4.2}$$

where now f and g are external forcing terms containing some hereditary or delay characteristics, and φ is the initial data in the interval of time $t\in[-h,0]$, where h is a fixed positive number, and $W(t)$ is a standard scalar Brownian motion/ Wiener process on an underlying complete filtered probability space $\{\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P}\}$.

Although our final and challenging goal is to analyze the well-posedness of mild solutions and asymptotic behavior of time-fractional stochastic Navier-Stokes model with delay (4.2), there are some difficulties/troubles which suggest us to start by analyzing a linearized version first before we can tackle the complete problem. It is well known that when we deal with the integer time stochastic Navier-Stokes equations in the phase space $L^2(\Omega;C([0,T];X))$, with the help of Itô's isometry and Burkholder-Davis-Gundy's inequality, a priori estimates can be handled smoothly. However, for time fractional stochastic Navier-Stokes equations, if the same phase space were adopted, we would face essential troubles: (a) Itô's isometry only holds true for the integer time derivative rather than time fractional derivative; (b) Burkholder-Davis-Gundy's inequality cannot be used since the integral is not a martingale (the main reason is the singular kernel appearing in the stochastic integral).

For this reason, in this first approach we will analyze the following time fractional stochastic delay incompressible flow problems, i.e., the non-stationary 2D-Stokes equations,

$$\begin{cases} D_t^\alpha u - \kappa\Delta u + \nabla p = f(t,u_t) + g(t,u_t)\frac{dW(t)}{dt} & \text{in } \mathbb{R}^2, \ t \geq 0, \\ \nabla \cdot u = 0 & \text{in } \mathbb{R}^2, \ t \geq 0, \\ u(t,x) = \varphi(t,x) & \text{in } \mathbb{R}^2, \ t \in [-h,0]. \end{cases} \tag{4.3}$$

We point out that in the deterministic case, the concept of weak (or variational) solution for the Navier-Stokes problem without delay was also analyzed [88]. However, the proof of this deterministic problem relies on direct estimates involving the time-fractional derivative as well as the Fourier transform, while the stochastic case cannot be analyzed by similar techniques since the term containing noise only makes sense in integral form. For this reason, we carry out a program based on a fixed point theorem which is different also from the one used in the papers [22, 87]. We highlight that it might be possible to perform the technique in [88] to handle those cases containing a much simpler noisy term in which the stochastic integral does not appear, for instance, when the noise has a special additive form. It is our objective to analyze these problems in future works.

The results of this chapter can be found in [85].

In what follows, we present basic notations related to stochastic theory, collect useful facts on Mittag-Leffler function and establish the definition of the mild solution to problem (4.3), for more details, we refer to [21, 22, 60, 45] and references therein.

To begin we fix a stochastic basis, that is,

$$\mathcal{S} := (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, W),$$

where $\mathbb{P}$ is a probability measure on Ω, $\mathcal{F}$ is a σ-algebra. In order to avoid unnecessary complications below, we may assume that $\{\mathcal{F}_t\}_{t\geq 0}$ is a right-continuous filtration on $(\Omega, \mathcal{F})$ such that $\mathcal{F}_0$ contains all the $\mathbb{P}$-negligible subsets and $W(t) = W(\omega, t)$, $\omega \in \Omega$ is a standard 1-D Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$.

To set our problem (4.3) in the abstract framework, we consider the standard notation L^2_σ to describe the subspace of the divergence-free vector fields in L^2:

$$L^2_\sigma = \{u \in L^2 : \nabla \cdot u = 0 \ \text{ in } \ \mathbb{R}^2\}$$

with norm $\|\cdot\|_{L^2}$, where L^2 denotes the vector-valued Lebesgue space and for $u \in L^2$,

$$\|u\|^2_{L^2} = \sum_{j=1}^{2} \int_{\mathbb{R}^2} |u_j(x)|^2 dx.$$

Besides, let $S \subset \mathbb{R}$ and X be a Banach space. We denote the space of continuous functions from S to X by $C(S;X)$ (equipped with its usual norm). $L^2(S;X)$ denotes the Banach space of L^2 integrable functions $u : S \to X$. $H^1(S;X) = W^{1,2}(S;X)$ is the subspace of $L^2(S;X)$ consisting of functions such that the weak derivative $\frac{\partial u}{\partial t}$ belongs to $L^2(S;X)$. Both spaces $L^2(S;X)$ and $W^{1,2}(S;X)$ are endowed with their standard norms.

Consider a fixed $T > 0$, given $u : [-h, T] \to L^2_\sigma$, for each $t \in [0, T]$, we denote by u_t the function on $[-h, 0]$ via the relation

$$u_t(s) = u(t + s), \qquad s \in [-h, 0],$$

where $h > 0$ denotes the delay, when $h = \infty$, it denotes infinite delay. Furthermore, let $L^2(\Omega; X)$ be a Hilbert space of X-valued random variable with norm $\|u(\cdot)\|^2_{L^2} = \mathbb{E}\|u(\cdot)\|^2$, where the expectation $\mathbb{E}$ is defined by $\mathbb{E}u = \int_\Omega u(\cdot)d\mathbb{P}$.

We now recall some properties of Mainardi function [21], denoted by M_α. This function is a particular case of the Wright type function introduced by Mainardi in [60]. More precisely, for $\alpha \in (0, 1)$, the entire function $M_\alpha : \mathbb{C} \to \mathbb{C}$ is given by

$$M_\alpha(z) := \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(1 - \alpha(1 + n))}.$$

Some basic properties of the Mainardi function will be used further in this chapter to obtain most of the estimates.

Proposition 4.1. *For $\alpha \in (0, 1)$ and $-1 < r < \infty$, when we restrict M_α to the positive real line, it holds that*

$$M_\alpha(t) \geq 0 \text{ for all } t \geq 0, \quad \text{and} \quad \int_0^\infty t^r M_\alpha(t)dt = \frac{\Gamma(r + 1)}{\Gamma(\alpha r + 1)}.$$

The next results are classical computation done in the literature that study the Mittag-Leffler operators, for instance [22]. To do this, let X be a Banach space and $-\mathcal{A} : D(\mathcal{A}) \subset X \to X$ be the infinitesimal generator of an analytic semigroup $\{T(t) : t \geq 0\}$. Then for each $\alpha \in (0, 1)$, we define the Mittag-Leffler families $\{\mathbf{E}_\alpha(-t^\alpha\mathcal{A}) : t \geq 0\}$ and $\{\mathbf{E}_{\alpha,\alpha}(-t^\alpha\mathcal{A}) : t \geq 0\}$ by, respectively,

$$\mathbf{E}_\alpha(-t^\alpha\mathcal{A}) = \int_0^\infty M_\alpha(s)T(st^\alpha)ds,$$

and

$$\mathbf{E}_{\alpha,\alpha}(-t^\alpha\mathcal{A}) = \int_0^\infty \alpha s M_\alpha(s)T(st^\alpha)ds.$$

It is interesting to notice that the Mainardi functions act as a bridge between the fractional and the classical abstract theories, this relation is based on the inversion of certain Laplace transform in order to obtain the fundamental solutions of the fractional diffusion-wave equation. Let us mention e.g., [21, 22, 56] and references therein.

The following lemma compiles the main assertions of the theory of the abstract fractional calculus.

Lemma 4.2. *The operators $\mathbf{E}_\alpha(-t^\alpha\mathcal{A})$ and $\mathbf{E}_{\alpha,\alpha}(-t^\alpha\mathcal{A})$ are well defined from X to X. Moreover, for $x \in X$ it holds,*

(i) $\mathbf{E}_\alpha(-t^\alpha\mathcal{A})x|_{t=0} = x$;

(ii) the vectorial functionst $\to \mathbf{E}_\alpha(-t^\alpha\mathcal{A})x$ and $t \to \mathbf{E}_{\alpha,\alpha}(-t^\alpha\mathcal{A})x$ are analytic from $[0, \infty)$ to X.

Let us rewrite the time fractional stochastic 2D-Stokes delay differential equations (4.3) in an abstract form

$$\begin{cases} D_t^\alpha u = -Au + F(t,u_t) + G(t,u_t)\frac{dW(t)}{dt}, & t>0, \\ u(t) = \varphi(t), & t \in [-h,0], \end{cases} \tag{4.4}$$

where $A = -P\Delta = -\Delta P$, $F(t,u_t) = Pf(t,u_t)$ and $G(t,u_t) = Pg(t,u_t)$. Here, $P : L^2 \to L^2_\sigma$ is the Helmholtz-Leray projector and $A : D(A) \subset L^2_\sigma \to L^2_\sigma$ is the Stokes operator.

We end this subsection by recapitulating the properties of both families of Mittag-Leffler operators, which furnish the essential tools used throughout the whole article, see [22, 46] for more details. Notice that the following lemmas hold true when the dimension is $N \geq 2$.

Lemma 4.3. *Consider $\alpha \in (0,1)$, and r_1, r_2 real numbers satisfying*

$$1 < r_1 \leq r_2 < \infty \quad \text{and} \quad r_2N/(2r_2+N) < r_1.$$

Then, for any $v \in L^{r_1}_\sigma$, there exists a constant $C = C(r_1,r_2,N,\alpha) > 0$ such that

$$(i) \qquad \|\mathbf{E}_\alpha(-t^\alpha A_{r_1})v\|_{L^{r_2}} \leq Ct^{-\alpha(N/r_1-N/r_2)/2}\|v\|_{L^{r_1}}, \quad t>0$$

and

$$(ii) \qquad \|\mathbf{E}_{\alpha,\alpha}(-t^\alpha A_{r_1})v\|_{L^{r_2}} \leq Ct^{-\alpha(N/r_1-N/r_2)/2}\|v\|_{L^{r_1}}, \quad t>0.$$

Remark 4.4. *For simplicity we will consider the case $N = 2$ in our analysis, but the results hold true for $N \geq 2$ (see Remark 4.16 at the end of this chapter).*

4.2 Well-posedness results with bounded delay

First of all, inspired by the arguments in [78] and references therein, we now make precise the notations of the mild solution to problem (4.4), which is given by a fractional variation of constants formula involving the Mittag-Leffler families.

Definition 4.5. (**Mild solution**). *Let $\mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}\}_{t\geq 0})$ be a fixed stochastic basis generated by a standard Brownian motion W, and $T > 0$. Consider $\alpha \in (0,1)$ and an initial function φ, such that $\varphi(t,\cdot)$ is a $\mathcal{F}_0$-measurable random variable for all $t \leq 0$ (relative to $\mathcal{S}$). A mild solution to problem (4.4) on $[-h,T]$ is a stochastic process u such that $u(t) = \varphi(t)$, for $t \in [-h,0]$, fulfilling*

$$\begin{aligned} u(t) = \mathbf{E}_\alpha(-t^\alpha A)\varphi(0) &+ \int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)F(s,u_s)ds \\ &+ \int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)G(s,u_s)dW(s), \ \mathbb{P}\text{-a.s.}, \quad \text{for every } t \in [0,T]. \end{aligned} \tag{4.5}$$

Remark 4.6. *Notice that, the Stokes operator $-A$ is the infinitesimal generator of an analytic semigroup $\{e^{-tA} : t \geq 0\}$. Hence, both Mittag-Leffler families $\mathbf{E}_\alpha(-t^\alpha A)$ and $\mathbf{E}_{\alpha,\alpha}(-t^\alpha A)$ are well defined.*

Remark 4.7. *It is worth mentioning that the analysis in this paper can be easily extended to the case in which system (4.3) is driven by Hilbert valued Brownian motion/Wiener process in infinite dimensions, however we prefer to consider this simpler formulation for the sake of clarity to the reader.*

In this section, the crucial well-posedness of fractional stochastic 2D-Stokes equation with bounded delay

$$\begin{cases} D_t^\alpha u = -Au + F(t,u_t) + G(t,u_t)\frac{dW(t)}{dt}, & t>0, \\ u(t) = \varphi(t), & t \in [-h,0], \end{cases} \tag{4.6}$$

will be justified, where h is a positive fixed constant (finite delay).

In order to apply the previous lemmas successfully, it is necessary to introduce suitable Banach spaces, which aim to capture the essence of the problem.

For any $\alpha \in (0,1)$ and fixed $T>0$, consider the Banach space $\mathcal{X}_2$ which is the set of continuous function $u : [-h,T]\times\Omega \to L^2(\Omega; L^2_\sigma)$ equipped with its natural norm

$$\|u\|_{\mathcal{X}_2} = \left(\sup_{t\in[-h,T]} \mathbb{E}\|u(t)\|^2_{L^2} \right)^{\frac{1}{2}},$$

here we omit T in $\mathcal{X}_2$ but no confusion is possible.

Let us now state the hypotheses imposed on external forcing terms in our problem.

(H_1) There exists a constant $L_f>0$, such that the function $F : [0,\infty)\times C([-h,0]; L^2(\Omega; L^2_\sigma)) \to L^2(\Omega; L^2_\sigma)$ satisfies

$$\int_0^t \mathbb{E}\|F(s,u_s) - F(s,v_s)\|^2_{L^2} ds \le L_f \int_{-h}^t \mathbb{E}\|u(s)-v(s)\|^2_{L^2} ds,$$

for all u, $v \in C([-h,T]; L^2(\Omega; L^2_\sigma))$.

(H_2) There exists a constant $L_g>0$ such that, the function $G : [0,\infty)\times C([-h,0]; L^2(\Omega; L^2_\sigma)) \to L^2(\Omega; L^2_\sigma)$ satisfies

$$\int_0^t \mathbb{E}\|G(s,u_s) - G(s,v_s)\|^2_{L^2} ds \le L_g \int_{-h}^t \mathbb{E}\|u(s)-v(s)\|^2_{L^2} ds,$$

for all u, $v \in C([-h,T]; L^2(\Omega; L^2_\sigma))$.

Initially, we establish the local existence and uniqueness of mild solution to problem (4.6) by a fixed point argument.

Theorem 4.8. *Let $\alpha \in (0,1)$. Assume that (H_1)-(H_2) hold, and initial value $\varphi \in C([-h,0]; L^2(\Omega; L^2_\sigma))$, such that $\varphi(t,\cdot)$ is a $\mathcal{F}_0$-measurable random variable for all $-h \le t \le 0$. Then there exists $T>0$ (small enough) such that problem (4.6) admits a unique mild solution u in the sense of Definition 4.5 on $[-h,T]$.*

Proof. To start off, let us pick up an initial function $\varphi(t) \in C([-h,0];L^2(\Omega;L^2_\sigma))$ such that $\|\varphi\|_{C([-h,0];L^2(\Omega;L^2_\sigma))}$ is small enough compared with R, precisely, we choose R such that

$$(3(C+1)+2ChL_g)\|\varphi\|^2_{C([-h,0];L^2(\Omega;L^2_\sigma))} \leq \frac{R^2}{2}.$$

Define the following space $\mathcal{B}^\varphi_R$ with $\alpha \in (0,1)$ and $R>0$, for every $t \in [0,T]$:

$$\mathcal{B}^\varphi_R = \left\{ u \in C([-h,T];L^2(\Omega;L^2_\sigma)) : u(t) = \varphi(t) \;\; \forall t \in [-h,0], \quad \|u\|_{\mathcal{X}_2} \leq R \right\}.$$

As a preparation for our main result, with the choice of an initial value $\varphi \in C([-h,0];L^2(\Omega;L^2_\sigma))$, let us define the operator $\mathcal{L}$ on $\mathcal{B}^\varphi_R$ as follows,

$$(\mathcal{L}u)(t) = \begin{cases} \varphi(t), \qquad t \in [-h,0], \\ \mathbf{E}_\alpha(-t^\alpha A)\varphi(0) + \int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)F(s,u_s)ds \\ \quad + \int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)G(s,u_s)dW(s), \quad t \in [0,T], \;\; \mathbb{P}\text{-}a.s. \end{cases} \tag{4.7}$$

Assertion 1: $\mathcal{L}u \in C([-h,T];L^2(\Omega;L^2_\sigma))$, for every $u \in C([-h,0];L^2(\Omega;L^2_\sigma))$.

Observe that, if $t \in [-h,0]$, then $(\mathcal{L}u)(t) = \varphi(t)$ and $\varphi \in C([-h,0];L^2(\Omega;L^2_\sigma))$. Therefore, we only need to check the continuity of $\mathcal{L}u$ on $[0,T]$. For any $t_1, t_2 \in [0,T]$, $\delta > 0$ small enough with $0 < |t_2 - t_1| < \delta$. By slightly modifying the proof of the [22, Lemma 11], with the help of the analytical property of the Mittag-Leffler operators in time (see Lemma 4.2(ii)), the result holds immediately.

Assertion 2: $\|\mathcal{L}u\|_{\mathcal{X}_2} \leq R$, for sufficiently small T.

To this end, we have to prove that, for any $u \in \mathcal{B}^\varphi_R$,

$$\|\mathcal{L}u\|_{\mathcal{X}_2} = \left(\sup_{t\in[-h,T]} \mathbb{E}\|(\mathcal{L}u)(t)\|^2_{L^2} \right)^{\frac{1}{2}} \leq R. \tag{4.8}$$

For $t \in [-h,0]$, we have

$$\mathbb{E}\|(\mathcal{L}u)(t)\|^2_{L^2} = \mathbb{E}\|\varphi(t)\|^2_{L^2} \leq \sup_{t\in[-h,0]} \mathbb{E}\|\varphi(t)\|^2_{L^2}. \tag{4.9}$$

If $t \in (0,T]$, it follows

$$\begin{aligned} \mathbb{E}\|(\mathcal{L}u)(t)\|^2_{L^2} &\leq 3\mathbb{E}\|\mathbf{E}_\alpha(-t^\alpha A)\varphi(0)\|^2_{L^2} + 3\mathbb{E}\left\| \int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)F(s,u_s)ds \right\|^2_{L^2} \\ &\quad + 3\mathbb{E}\left\| \int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)G(s,u_s)dW(s) \right\|^2_{L^2} := \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3. \end{aligned} \tag{4.10}$$

We now estimate each term on the right hand side of (4.10). For $\mathcal{I}_1$, by Lemma 4.3(i), it is obvious that

$$\mathcal{I}_1 = 3\mathbb{E}\|\mathbf{E}_\alpha(-t^\alpha A)\varphi(0)\|^2_{L^2} \leq 3C\mathbb{E}\|\varphi(0)\|^2_{L^2} \leq 3C \sup_{t\in[-h,0]} \mathbb{E}\|\varphi(t)\|^2_{L^2}. \tag{4.11}$$

For $\mathcal{I}_2$, by Lemma 4.3(i), (H_1), the Cauchy-Schwarz inequality and Fubini's theorem, we obtain

$$\begin{aligned}
\mathcal{I}_2 &= 3\mathbb{E}\left\| \int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)F(s,u_s)ds \right\|_{L^2}^2 \\
&\le 3C\mathbb{E}\left(\int_0^t \|\mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)F(s,u_s)\|_{L^2} ds \right)^2 \\
&\le 6Ct\left(\int_0^t \mathbb{E}\|F(s,u_s)-F(s,0)\|_{L^2}^2 ds + \int_0^t \mathbb{E}\|F(s,0)\|_{L^2}^2 ds \right) \\
&\le 6CL_f t \int_{-h}^t \mathbb{E}\|u(s)\|_{L^2}^2 ds + 6Ct\int_0^t \mathbb{E}\|F(s,0)\|_{L^2}^2 ds \\
&\le 6CL_f t\left(\int_{-h}^0 \mathbb{E}\|\varphi(s)\|_{L^2}^2 ds + \int_0^t \mathbb{E}\|u(s)\|_{L^2}^2 ds \right) \\
&\quad + 6Ct\int_0^t \mathbb{E}\|F(s,0)\|_{L^2}^2 ds \\
&\le 6ChL_f t \sup_{t\in[-h,0]} \mathbb{E}\|\varphi(t)\|_{L^2}^2 + 6CL_f t\int_0^t \mathbb{E}\|u(s)\|_{L^2}^2 ds \\
&\quad + 6Ct^2 \sup_{s\in[0,t]} \mathbb{E}\|F(s,0)\|_{L^2}^2 \\
&\le 6ChL_f t \sup_{t\in[-h,0]} \mathbb{E}\|\varphi(t)\|_{L^2}^2 + 6Ct^2\left(L_f R^2 + \sup_{s\in[0,t]} \mathbb{E}\|F(s,0)\|_{L^2}^2 \right).
\end{aligned} \tag{4.12}$$

For $\mathcal{I}_3$, by Lemma 4.3(i), Itô's isometry and (H_2),

$$\begin{aligned}
\mathcal{I}_3 &= 3\mathbb{E}\left\| \int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)G(s,u_s)dW(s) \right\|_{L^2}^2 \\
&\le 3C\int_0^t \mathbb{E}\|G(s,u_s)\|_{L^2}^2 ds \\
&\le 6C\int_0^t \mathbb{E}\|G(s,u_s)-G(s,0)\|_{L^2}^2 ds + 6C\int_0^t \mathbb{E}\|G(s,0)\|_{L^2}^2 ds \\
&\le 6CL_g\left(\int_{-h}^0 \mathbb{E}\|\varphi(s)\|_{L^2}^2 ds + \int_0^t \mathbb{E}\|u(s)\|_{L^2}^2 ds \right) + 6C\int_0^t \mathbb{E}\|G(s,0)\|_{L^2}^2 ds \\
&\le 6ChL_g \sup_{t\in[-h,0]} \mathbb{E}\|\varphi(t)\|_{L^2}^2 + 6Ct\Bigg(L_g \sup_{s\in[0,t]} \mathbb{E}\|u(s)\|_{L^2}^2 \\
&\quad + \sup_{s\in[0,t]} \mathbb{E}\|G(s,0)\|_{L^2}^2 \Bigg) \\
&\le 6ChL_g \sup_{t\in[-h,0]} \mathbb{E}\|\varphi(t)\|_{L^2}^2 + 6Ct\left(L_g R^2 + \sup_{s\in[0,t]} \mathbb{E}\|G(s,0)\|_{L^2}^2 \right).
\end{aligned} \tag{4.13}$$

Substituting (4.11)-(4.13) into (4.10), combining with (4.9), it is obvious that

$$\begin{aligned}\mathbb{E}\|(\mathcal{L}u)(t)\|_{L^2}^2 \leq 3((C+1)+2ChL_f t+2ChL_g) &\sup_{s\in[-h,0]}\mathbb{E}\|\varphi(s)\|_{L^2}^2\\ &+6Ct^2\left(L_f R^2+\sup_{s\in[0,t]}\mathbb{E}\|F(s,0)\|_{L^2}^2\right)\\ &+6Ct\left(L_g R^2+\sup_{s\in[0,t]}\mathbb{E}\|G(s,0)\|_{L^2}^2\right).\end{aligned}$$

Consequently, thanks to the choice of R, we can choose T small enough such that

$$\begin{aligned}\|\mathcal{L}u\|_{\mathcal{X}_2} = \left(\sup_{t\in[-h,T]}\mathbb{E}\|(\mathcal{L}u)(t)\|_{L^2}^2\right)^{\frac{1}{2}} &\leq \Bigg(3((C+1)+2ChL_f T+2ChL_g)\\ \times \sup_{t\in[-h,0]}\mathbb{E}\|\varphi(t)\|_{L^2}^2 &+6CT^2\left(L_f R^2+\sup_{t\in[0,T]}\mathbb{E}\|F(t,0)\|_{L^2}^2\right)\\ +6CT\left(L_g R^2+\sup_{t\in[0,T]}\mathbb{E}\|G(t,0)\|_{L^2}^2\right)\Bigg)^{\frac{1}{2}} &\leq R.\end{aligned} \tag{4.14}$$

Assertion 3: Operator $\mathcal{L}:\mathcal{B}_R^{\varphi}\to\mathcal{B}_R^{\varphi}$ is a contraction.

To this end, for any u, $v\in\mathcal{B}_R^{\varphi}$, it follows that

$$\|\mathcal{L}u-\mathcal{L}v\|_{\mathcal{X}_2} := \left(\sup_{t\in[-h,T]}\mathbb{E}\|(\mathcal{L}u)(t)-(\mathcal{L}v)(t)\|_{L^2}^2\right)^{\frac{1}{2}}. \tag{4.15}$$

For $t\in[-h,0]$, one has $(\mathcal{L}u)(t)=(\mathcal{L}v)(t)=\varphi(t)$. Thus, it is sufficient to consider the case $t\in[0,T]$. Observe that

$$\begin{aligned}\mathbb{E}\,\|(\mathcal{L}u)(t)-(\mathcal{L}v)(t)\|_{L^2}^2 &\leq 2\mathbb{E}\left\|\int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^{\alpha}A)(F(s,u_s)-F(s,v_s))ds\right\|_{L^2}^2\\ &+2\mathbb{E}\left\|\int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^{\alpha}A)(G(s,u_s)-G(s,v_s))dW(s)\right\|_{L^2}^2\\ &:= \mathcal{J}_1+\mathcal{J}_2.\end{aligned} \tag{4.16}$$

For $\mathcal{J}_1$, by Lemma 4.3(i), (H_2), the Cauchy-Schwarz inequality and Fubini's theorem, we obtain

$$\begin{aligned}\mathcal{J}_1 &= 2\mathbb{E}\left\|\int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^{\alpha}A)(F(s,u_s)-F(s,v_s))ds\right\|_{L^2}^2\\ &\leq 2C\mathbb{E}\left(\int_0^t \|\mathbf{E}_{\alpha,\alpha}(-(t-s)^{\alpha}A)(F(s,u_s)-F(s,v_s))\|_{L^2}ds\right)^2\\ &\leq 2CL_f t\int_{-h}^t \mathbb{E}\|u(s)-v(s)\|_{L^2}^2 ds\\ &\leq 2CL_f t^2\sup_{s\in[0,t]}\mathbb{E}\|u(s)-v(s)\|_{L^2}^2.\end{aligned} \tag{4.17}$$

For $\mathcal{J}_2$, by Lemma 4.3(i), (H_2) and Itô's isometry, one has

$$\begin{aligned}\mathcal{J}_2 &= 2\mathbb{E}\left\| \int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)(G(s,u_s)-G(s,v_s))dW(s)\right\|_{L^2}^2 \\ &\leq 2CL_g \int_{-h}^t \mathbb{E}\|u(s)-v(s)\|_{L^2}^2 ds \\ &= 2CL_g \int_0^t \mathbb{E}\|u(s)-v(s)\|_{L^2}^2 ds \\ &\leq 2CL_g t \sup_{s\in[0,t]} \mathbb{E}\|u(s)-v(s)\|_{L^2}^2.\end{aligned} \tag{4.18}$$

Hence, substituting (4.17)-(4.18) into (4.16), it follows that

$$\begin{aligned}\|\mathcal{L}u-\mathcal{L}v\|_{\mathcal{X}_2} &\leq \left(2CT(L_fT+L_g)\sup_{t\in[0,T]}\mathbb{E}\|u(t)-v(t)\|_{L^2}^2\right)^{\frac{1}{2}} \\ &:= \mathcal{M}\|u(t)-v(t)\|_{\mathcal{X}_2},\end{aligned}$$

where

$$\mathcal{M}^2 = 2CT(L_fT+L_g).$$

Therefore, we can choose T small enough such that $0<\mathcal{M}<1$, in other words, we can choose T small enough such that operator $\mathcal{L}$ maps $\mathcal{B}_R^\varphi$ into itself, and it is a contraction as well. The Banach fixed-point theory yields that operator $\mathcal{L}$ possesses a fixed point in $\mathcal{B}_R^\varphi$. Namely, problem (4.16) has a unique local mild solution on $[-h,T]$, and the proof of this theorem is completed. □

Proposition 4.9. *Under the assumptions of Theorem 4.8, the mild solution to problem (4.6) is continuous with respect to the initial data* $\varphi\in C([-h,0];L^2(\Omega;L_\sigma^2))$. *In particular, if* $u(t)$, $w(t)$ *are the corresponding mild solutions on the interval* $[-h,T]$, *to the initial data* ϕ *and* ψ, *then the following estimate holds*

$$\|u-w\|_{\mathcal{X}_2} \leq 3\|\phi-\psi\|_{C([-h,0];L^2(\Omega;L_\sigma^2))}\exp(3C(L_ft+L_g)t), \qquad \forall t\in[0,T].$$

Proof. The result of this proposition is proved by the similar arguments to those concerning the uniqueness of next theorem, so we omit the details here. □

In the following lines, a theorem will be considered to prove the global existence and uniqueness of mild solution to problem (4.6).

Theorem 4.10. *Assume the hypotheses of Theorem 4.8 hold. Then for every initial value* $\varphi\in C([-h,0];L^2(\Omega;L_\sigma^2))$, *the initial value problem (4.6) has a unique mild solution defined globally in the sense of Definition 4.5.*

Proof. Initially, we assume that there exist two solutions, u and v on $[0,T_1]$ and $[0,T_2]$, respectively to problem (4.6). Next let us prove that $u=v$ on $[-h,T_1\wedge T_2]$. It is clear that $u(t)=v(t)=\varphi(t)$ on $[-h,0]$, so we only need to prove that $u(t)=v(t)$ for any $t\in[0,T_1\wedge T_2]$. Notice that

$$\|u-v\|_{\mathcal{X}_2}^2 := \sup_{t\in[-h,T_1\wedge T_2]}\mathbb{E}\|u(t)-v(t)\|_{L^2}^2. \tag{4.19}$$

On the one hand, it has

$$
\begin{aligned}
\mathbb{E}\|u(t)-v(t)\|_{L^2}^2 \le{}& 2\mathbb{E}\left\|\int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)(F(s,u_s)-F(s,v_s))ds\right\|_{L^2}^2 \\
&+ 2\mathbb{E}\left\|\int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)(G(s,u_s)-G(s,v_s))dW(s)\right\|_{L^2}^2 \\
:={}& I_1 + I_2.
\end{aligned}
\tag{4.20}
$$

For I_1, by Lemma 4.3(i), (H_1) and the Cauchy-Schwarz inequality, it follows that

$$
\begin{aligned}
I_1 &\le 2C\mathbb{E}\left(\int_0^t \|\mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)(F(s,u_s)-F(s,v_s))\|_{L^2}ds\right)^2 \\
&\le 2C\mathbb{E}\left(\int_0^t \|F(s,u_s)-F(s,v_s)\|_{L^2}ds\right)^2 \\
&\le 2CL_f t\int_0^t \mathbb{E}\|u(s)-v(s)\|_{L^2}^2 ds \\
&\le 2CL_f t\int_0^t \sup_{\sigma\in[0,s]} \mathbb{E}\|u(\sigma)-v(\sigma)\|_{L^2}^2 ds.
\end{aligned}
\tag{4.21}
$$

For I_2, by Lemma 4.3(i), (H_2) and Itô's isometry, we derive

$$
\begin{aligned}
I_2 &\le 2\int_0^t \mathbb{E}\|\mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)(G(s,u_s)-G(s,v_s))\|_{L^2}^2 ds \\
&\le 2CL_g\int_0^t \mathbb{E}\|u(s)-v(s)\|_{L^2}^2 ds \\
&\le 2CL_g\int_0^t \sup_{\sigma\in[0,s]} \mathbb{E}\|u(\sigma)-v(\sigma)\|_{L^2}^2 ds.
\end{aligned}
\tag{4.22}
$$

Substituting (4.21)-(4.22) to (4.20), it yields

$$
\mathbb{E}\|u(t)-v(t)\|_{L^2}^2 \le 2C\left(L_f t + L_g\right)\int_0^t \sup_{\sigma\in[0,s]} \mathbb{E}\|u(\sigma)-v(\sigma)\|_{L^2}^2 ds.
$$

Denote by $\mathcal{M}_1 = 2C(L_f(T_1\wedge T_2)+L_g)$, we have

$$
\sup_{t\in[-h,T_1\wedge T_2]} \mathbb{E}\|u(t)-v(t)\|_{L^2}^2 \le \mathcal{M}_1\int_0^{T_1\wedge T_2}\left(\sup_{\sigma\in[-h,t]}\|u(\sigma)-v(\sigma)\|_{L^2}^2\right)dt,
$$

the Gronwall Lemma implies that

$$
\|u-v\|_{\mathcal{X}_2} = 0.
$$

Therefore, $u=v$ on $[-h, T_1\wedge T_2]$ for every initial function $\varphi(t)$.

Now we prove that for each given $T > 0$, the mild solution u to problem (4.6) is bounded with $\mathcal{X}_2$ norm. Taking into account Lemma 4.3(i), (H_1)-(H_2), Itô's isometry, the Cauchy-Schwarz inequality and Fubini's theorem, we have

$$\begin{aligned}
\mathbb{E}\|u(t)\|_{L^2}^2 \le 3C \sup_{t\in[-h,0]} \mathbb{E}\|\varphi(t)\|_{L^2}^2 &+ 6C\mathbb{E}\bigg((L_f t + L_g)\int_{-h}^{0} \|\varphi(s)\|_{L^2}^2 ds\bigg) \\
&+ 6C\mathbb{E}\bigg((L_f t + L_g)\int_0^t \|u(s)\|_{L^2}^2 ds\bigg) \\
&+ \int_0^t \left(\|F(s,0)\|_{L^2}^2 ds + \|G(s,0)\|_{L^2}^2\right) ds \\
&\le 6C(1 + L_f th + L_g h) \sup_{t\in[-h,0]} \mathbb{E}\|\varphi(t)\|_{L^2}^2 \\
&+ 6Ct^2 \sup_{s\in[0,t]} \mathbb{E}\|F(s,0)\|_{L^2}^2 + 6Ct \sup_{s\in[0,t]} \mathbb{E}\|G(s,0)\|_{L^2}^2 \\
&+ 6C(L_f t + L_g)\int_0^t \sup_{\sigma\in[0,s]} \mathbb{E}\|u(\sigma)\|_{L^2}^2 ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sup_{t\in[-h,T]} \mathbb{E}\|u(t)\|_{L^2}^2 \le 6((C+1) + CL_f Th + CL_g h) &\sup_{t\in[-h,0]} \mathbb{E}\|\varphi(t)\|_{L^2}^2 \\
&+ 6CT^2 \sup_{t\in[0,T]} \mathbb{E}\|F(t,0)\|_{L^2}^2 + 6CT \sup_{t\in[0,T]} \mathbb{E}\|G(t,0)\|_{L^2}^2 \\
&+ 6C(L_f T + L_g)\int_0^T \sup_{\sigma\in[0,t]} \mathbb{E}\|u(\sigma)\|_{L^2}^2 dt \\
&:= A(\varphi, T, F, G) + \mathcal{M}_2 \int_0^T \sup_{\sigma\in[-h,t]} \mathbb{E}\|u(\sigma)\|_{L^2}^2 dt,
\end{aligned}$$

where we have used the notation that

$$\begin{aligned}
A(\varphi, T, F, G) : = 6((C+1) + CL_f Th + CL_g h) &\sup_{t\in[-h,0]} \mathbb{E}\|\varphi(t)\|_{L^2}^2 \\
&+ 6CT^2 \sup_{t\in[0,T]} \mathbb{E}\|F(t,0)\|_{L^2}^2 + 6CT \sup_{t\in[0,T]} \mathbb{E}\|G(t,0)\|_{L^2}^2,
\end{aligned}$$

and

$$\mathcal{M}_2 := 6C(L_f T + L_g).$$

Applying the Gronwall lemma, for any fixed $T > 0$ and all $t \in [0, T]$, we obtain

$$\|u\|_{\mathcal{X}_2}^2 \le A(\varphi, T, F, G)\exp(\mathcal{M}_2 T).$$

Because of the arbitrariness of T, together with the conclusion of uniqueness of u on $[-h, T]$, it is straightforward that the mild solution u to problem (4.6) is defined globally. The proof of this theorem is complete. □

4.3 Well-posedness results with unbounded delay

Let us consider the well-posedness of mild solution to the following stochastic time fractional 2D-Stokes equation with unbounded delay:

$$\begin{cases} D_t^\alpha u = -Au + F(t, u_t) + G(t, u_t)\frac{dW(t)}{dt}, & t > 0, \\ u(t) = \varphi(t), & t \in (-\infty, 0]. \end{cases} \tag{4.23}$$

Before going a step further to prove the main results, we first introduce a suitable space motivated by our unbounded delay. Let $\mathbb{H}$ be a separable Hilbert space, then the space $\mathcal{C}_X$ on $\mathbb{H}$ is defined as

$$\mathcal{C}_X(\mathbb{H}) = \{\varphi \in C((-\infty, 0]; \mathbb{H}) : \lim_{\theta \to -\infty} \varphi(\theta) \text{ exists in } \mathbb{H}\},$$

which is a Banach space equipped with the norm

$$\|\varphi\|_{\mathcal{C}_X} = \sup_{\theta \in (-\infty, 0]} \|\varphi(\theta)\|_{\mathbb{H}}.$$

Let us denote $\mathbb{R}_+ = [0, \infty)$ and enumerate now the assumptions on the delay terms F and G. Assume that F, $G : [0, \infty) \times \mathcal{C}_X(L^2(\Omega; L^2_\sigma)) \to L^2(\Omega; L^2_\sigma)$, then

(H_3) For any $\xi \in \mathcal{C}_X(L^2(\Omega; L^2_\sigma))$, the mappings $[0, \infty) \ni t \to F(t, \xi) \in L^2(\Omega; L^2_\sigma)$ and $[0, \infty) \ni t \to G(t, \xi) \in L^2(\Omega; L^2_\sigma)$ are measurable.

(H_4) $F(\cdot, 0) = 0, \quad G(\cdot, 0) = 0$ (for simplicity).

(H_5) There exist two constants L'_f and L'_g, such that for all $t \in [0, \infty)$, and for all ξ, $\eta \in \mathcal{C}_X(L^2(\Omega; L^2_\sigma))$,

$$\|F(t, \xi) - F(t, \eta)\|_{L^2(\Omega; L^2_\sigma)} \le L'_f \|\xi - \eta\|_{\mathcal{C}_X(L^2(\Omega; L^2_\sigma))},$$

$$\|G(t, \xi) - G(t, \eta)\|_{L^2(\Omega; L^2_\sigma)} \le L'_g \|\xi - \eta\|_{\mathcal{C}_X(L^2(\Omega; L^2_\sigma))}.$$

At this point, some remarks are in order.

Remark 4.11. *i) Notice that in this unbounded delay case, assumptions (H_4) and (H_5) imposed on the delay terms are simply Lipschitz continuity while in the bounded delay case we need to impose (H_1) and (H_2) which are some kind of integral Lipschitz condition. The main reason is that in the current situation, we can use the estimate* $\sup_{\theta \le 0} \|u_t(\theta)\|_{\mathbb{H}} \le \sup_{\theta \le 0} \|u_s(\theta)\|_{\mathbb{H}}$*, if $s > t$, while in the bounded delay case this is not true. This will make our computations different in both cases. Also, this is why we will include the complete details in this section.*

ii) It is quite usual when dealing with unbounded delay differential equations, to adopt a different space for the initial data ([83]), namely,

$$C^\gamma(\mathbb{H}) = \{\varphi \in C((-\infty, 0]; \mathbb{H}) : \sup_{\theta \in (-\infty, 0]} e^{\gamma\theta} \|\varphi(\theta)\|_{\mathbb{H}} < +\infty\}.$$

However, if we consider this space, then hypotheses (H_4) *and* (H_5) *are not fulfilled when the delay in* F *or* G *is a variable delay one. For instance,* $F(t, u_t) = F_0(u(t-\rho(t)))$*, where* ρ *is a measurable function taking nonnegative values and* $F_0 : \mathbb{R}^2 \to \mathbb{R}^2$ *is a Lipschitz function. Therefore, this new space, although it is a bit more restrictive than the usual one, allows us to consider more general delay terms in the functional formulation.*

At this point, we are in position to prove our main results on well-posedness of mild solution to problem (4.23).

Theorem 4.12. *Let* $\alpha \in (0,1)$*,* F *and* G *satisfy assumptions* (H_3)*-*(H_5)*. Then for each initial function* $\varphi \in C((-\infty,0];(L^2(\Omega;L^2_\sigma)))$*, such that* $\varphi(t,\cdot)$ *is a* $\mathcal{F}_0$*-measurable random variable for all* $t \le 0$*, problem (4.23) admits a unique mild solution* u *in the sense of Definition 4.5 on* $(-\infty,T]$*, for* $T > 0$ *small enough.*

Proof. To start off, let us pick an initial function $\varphi(t) \in C((-\infty,0];L^2(\Omega;L^2_\sigma))$ such that $\|\varphi\|_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}$ is small enough compared with R, namely, we choose R such that

$$3(C+1)\|\varphi\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} \le \frac{R^2}{3}.$$

Define the following space $\mathcal{V}_R^\varphi$ with $\alpha \in (0,1)$, $R > 0$

$$\mathcal{V}_R^\varphi = \Big\{ u \in C((-\infty,0];L^2(\Omega;L^2_\sigma)) : \ u(t) = \varphi(t) \text{ for } t \in (-\infty,0],$$
$$\text{and } u_t \in \mathcal{C}_X(L^2(\Omega;L^2_\sigma)) \text{ for } t \ge 0, \text{ satisfying } \|u_t\|_{\mathcal{C}_X} \le R. \Big\}$$

As a preparation for handling the main result, with the choice of an initial value $\varphi(t) \in C((-\infty,0];L^2(\Omega;L^2_\sigma))$, let us define the operator $\mathcal{K}$ on $\mathcal{V}_R^\varphi$ as follows,

$$(\mathcal{K}u)(t) = \begin{cases} \varphi(t), & t \in (-\infty,0], \\ \mathbf{E}_\alpha(-t^\alpha A)\varphi(0) + \displaystyle\int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)F(s,u_s)ds & \\ \quad + \displaystyle\int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)G(s,u_s)dW(s), & t \in [0,T], \ \mathbb{P}\text{-}a.s. \end{cases} \tag{4.24}$$

Assertion 1: $\mathcal{K}u \in C((-\infty,T];L^2(\Omega;L^2_\sigma))$, for all $u \in C((-\infty,0];L^2(\Omega;L^2_\sigma))$.

Observe that, if $t \in (-\infty,0]$, then $(\mathcal{K}u)(t) = \varphi(t)$. Therefore, we only need to check the continuity of $\mathcal{K}u$ on $[0,T]$. For any t_1, $t_2 \in [0,T]$, $\delta > 0$ small enough with $0 < |t_2 - t_1| < \delta$. By slightly modifying the proof in [22, Lemma 11], with the help of the analyticity of Mittag-Leffler operators in time (see Lemma 4.2 (ii)), the result holds immediately.

Assertion 2: $\|(\mathcal{K}u)_t\|_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} \le R$, for all $t \in [0,T]$ with sufficiently small T.

For every $u \in \mathcal{V}_R^\varphi$, we have to show that

$$\|(\mathcal{K}u)_t\|_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} := \left(\sup_{\theta\in(-\infty,0]} \mathbb{E}\|(\mathcal{K}u)(t+\theta)\|^2_{L^2} \right)^{\frac{1}{2}} \le R.$$

For $t \in (-\infty, 0]$, we have

$$\mathbb{E}\|(\mathcal{K}u)(t)\|_{L^2}^2 = \mathbb{E}\|\varphi(t)\|_{L^2}^2 \leq \sup_{t\in(-\infty,0]} \mathbb{E}\|\varphi(t)\|_{L^2}^2. \tag{4.25}$$

If $t+\theta \in (0,T]$, then it follows that

$$\begin{aligned}\mathbb{E}\|(\mathcal{K}u)(t)\|_{L^2}^2 \leq\ & 3\mathbb{E}\|\mathbf{E}_\alpha(-t^\alpha A)\varphi(0)\|_{L^2}^2 + 3\mathbb{E}\left\|\int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)F(s,u_s)ds\right\|_{L^2}^2 \\ & + 3\mathbb{E}\left\|\int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)G(s,u_s)dW(s)\right\|_{L^2}^2 \\ & := \mathcal{I}^1 + \mathcal{I}^2 + \mathcal{I}^3.\end{aligned} \tag{4.26}$$

We will do estimates one by one. For $\mathcal{I}^1$, by Lemma 4.3(i), it is obvious that

$$\mathcal{I}^1 = 3\mathbb{E}\|\mathbf{E}_\alpha(-t^\alpha A)\varphi(0)\|_{L^2}^2 \leq 3C\mathbb{E}\|\varphi(0)\|_{L^2}^2 \leq 3C \sup_{t\in(-\infty,0]} \mathbb{E}\|\varphi(t)\|_{L^2}^2. \tag{4.27}$$

For $\mathcal{I}^2$, by Lemma 4.3(i), (H_1), the Cauchy-Schwarz inequality and Fubini's theorem, we obtain

$$\begin{aligned}\mathcal{I}^2 =& 3\mathbb{E}\left\|\int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)F(s,u_s)ds\right\|_{L^2}^2 \\ &\leq 3\mathbb{E}\left(\int_0^t \|\mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)F(s,u_s)\|_{L^2}ds\right)^2 \\ &\leq 3Ct\int_0^t \mathbb{E}\|F(s,u_s)-F(s,0)\|_{L^2}^2 ds \\ &\leq 3CL'_f t\int_0^t \|u_s\|_{\mathcal{C}_X(L^2(\Omega,L^2_\sigma))}^2 ds \\ &\leq 3CL'_f t^2\|u_t\|_{\mathcal{C}_X(L^2(\Omega,L^2_\sigma))}^2 \leq 3CL'_f t^2R^2.\end{aligned} \tag{4.28}$$

For $\mathcal{I}^3$, by Lemma 4.3(i), Itô's isometry and (H_2),

$$\begin{aligned}\mathcal{I}^3 =& 3\mathbb{E}\left\|\int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)G(s,u_s)dW(s)\right\|_{L^2}^2 \\ &\leq 3C\int_0^t \mathbb{E}\|G(s,u_s)-G(s,0)\|_{L^2}^2 ds \\ &\leq 3CL'_g\int_0^t \|u_s\|_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}^2 ds \\ &\leq 3CL'_g t\|u_t\|_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} \leq 3CL'_g tR^2.\end{aligned} \tag{4.29}$$

Replacing (4.27)-(4.29) into (4.26), combining with (4.25), it is obvious that

$$\mathbb{E}\|(\mathcal{K}u)_t\|_{L^2}^2 \leq 3\bigg((C+1)\sup_{t\in(-\infty,0]} \mathbb{E}\|\varphi(t)\|_{L^2}^2 + Ct^2L'_fR^2 + CtL'_gR^2\bigg).$$

Consequently, due to the choice of R, we can choose T small enough such that

$$\begin{aligned}\|(\mathcal{K}u)_t\|_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} &= \left(\sup_{\theta\in(-\infty,0]} \mathbb{E}\|(\mathcal{K}u)(t+\theta)\|^2_{L^2}\right)^{\frac{1}{2}} \\ &\le 3\bigg((C+1)\|\varphi(t)\|_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} + CT^2L'_fR^2 + CTL'_gR^2\bigg)^{\frac{1}{2}} \\ &\le R.\end{aligned}$$

Assertion 3: Operator $\mathcal{K}:\mathcal{V}^\varphi_R\to\mathcal{V}^\varphi_R$ is a contraction.

To this end, for any u, $v\in\mathcal{V}^\varphi_R$, it follows that

$$\|(\mathcal{K}u)_t-(\mathcal{K}v)_t\|_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} := \left(\sup_{\theta\in(-\infty,0]} \mathbb{E}\|(\mathcal{K}u)(t+\theta)-(\mathcal{K}v)(t+\theta)\|^2_{L^2}\right)^{\frac{1}{2}}. \tag{4.30}$$

For $t\in(-\infty,0]$, one has $(\mathcal{K}u)(t)=(\mathcal{K}v)(t)=\varphi(t)$. Thus, we only need to consider the case $t\in[0,T]$. Observe that

$$\begin{aligned}\mathbb{E}\,\|(\mathcal{K}u)(t)-(\mathcal{K}v)(t)\|^2_{L^2} &\le 2\mathbb{E}\bigg\|\int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)(F(s,u_s)-F(s,v_s))ds\bigg\|^2_{L^2} \\ &\quad + 2\mathbb{E}\bigg\|\int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)(G(s,u_s)-G(s,v_s))dW(s)\bigg\|^2_{L^2} \\ &:= \mathcal{J}^1+\mathcal{J}^2.\end{aligned} \tag{4.31}$$

For $\mathcal{J}^1$, by Lemma 4.3(i), (H_2), the Cauchy-Schwarz inequality and Fubini's theorem, we obtain

$$\begin{aligned}\mathcal{J}^1 &= 2\mathbb{E}\bigg\|\int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)(F(s,u_s)-F(s,v_s))ds\bigg\|^2_{L^2} \\ &\le 2\mathbb{E}\left(\int_0^t \|\mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)(F(s,u_s)-F(s,v_s))\|_{L^2}ds\right)^2 \\ &\le 2CL'_ft\int_0^t \|u_s-v_s\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}ds \\ &\le 2CL'_ft^2\|u_t-v_t\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}.\end{aligned} \tag{4.32}$$

For $\mathcal{J}^2$, by Lemma 4.3(i), (H_2) and Itô's isometry, one has

$$\begin{aligned}\mathcal{J}^2 &= 2\mathbb{E}\bigg\|\int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)(G(s,u_s)-G(s,v_s))dW(s)\bigg\|^2_{L^2} \\ &\le 2CL'_g\int_0^t \|u_s-v_s\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}ds \\ &\le 2CL'_gt\|u_t-v_t\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}.\end{aligned} \tag{4.33}$$

Hence, substituting (4.31)-(4.33) into (4.30), it follows that

$$\|(\mathcal{K}u)_t - (\mathcal{K}v)_t\|_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} \leq \left(2C(L'_f T^2 + L'_g T)\|u_t - v_t\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}\right)^{\frac{1}{2}}$$
$$:= \mathcal{W}\|u(t) - v(t)\|_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))},$$

where

$$\mathcal{W}^2 = 2C(L'_f T^2 + L'_g T).$$

Therefore, we can choose T small enough such that $0 < \mathcal{W} < 1$, which means that, the operator $\mathcal{K}$ maps $\mathcal{V}_R^\varphi$ into itself, also it is a contraction. The Banach fixed-point theorem yields that operator $\mathcal{K}$ has a fixed point in $\mathcal{V}_R^\varphi$. Namely, problem (4.23) has a unique local mild solution on $(-\infty, T]$. The proof of this theorem is completed. □

Proposition 4.13. *Under the assumptions of Theorem 4.12, the mild solution to problem (4.23) is continuous with respect to the initial data $\varphi \in C((-\infty,0];L^2(\Omega;L^2_\sigma))$. In particular, if $u(t)$, $w(t)$ are the corresponding mild solutions, on the interval $(-\infty,T]$, to the initial data ϕ and ψ, then the following estimate holds*

$$\|u_t - w_t\|_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} \leq 3C\|\phi - \psi\|_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}\exp(3C(L'_f t + L'_g)t), \qquad \forall t \in [0,T].$$

Proof. The result of this theorem is proved by the similar arguments to those concerning the uniqueness of next theorem, so we omit the details. □

The following result is concerned with the existence and uniqueness of the global mild solution to problem (4.23).

Theorem 4.14. *Assume the hypotheses of Theorem 4.12 hold. Then for every initial value $\varphi \in C((-\infty,0];L^2(\Omega;L^2_\sigma))$, the initial value problem (4.23) has a unique mild solution defined globally in the sense of Definition 4.5.*

Proof. Although the proof of this theorem follows the same lines as the case of bounded delay in Section 3, but with differences in the estimates, we prefer to include it here because the proof of Proposition 4.13 is similar to the uniqueness below.

Assume that there exist two solutions, u and v on $[0,T_1]$ and $[0,T_2]$, respectively to problem (4.23). Next let us prove that $u = v$ on $(-\infty, T_1 \wedge T_2]$. It is notable that $u(t) = v(t) = \varphi(t)$ on $(-\infty,0]$, so we only need to prove that $u(t) = v(t)$ for any $t \in [0, T_1 \wedge T_2]$. Observe that

$$\|u - v\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} := \sup_{t\in(-\infty,T_1\wedge T_2]} \mathbb{E}\|u(t) - v(t)\|^2_{L^2}. \tag{4.34}$$

On the one hand, it has

$$\begin{aligned}\mathbb{E}\|u(t) - v(t)\|^2_{L^2} \leq\ & 2\mathbb{E}\left\|\int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)(F(s,u_s) - F(s,v_s))ds\right\|^2_{L^2}\\ &+ 2\mathbb{E}\left\|\int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)(G(s,u_s) - G(s,v_s))dW(s)\right\|^2_{L^2}\\ :=\ & I^1 + I^2.\end{aligned} \tag{4.35}$$

For I^1, by Lemma 4.3(i), (H_1) and the Cauchy-Schwarz inequality, it follows that

$$\begin{aligned} I^1 &\leq 2\mathbb{E}\left(\int_0^t \|\mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)(F(s,u_s)-F(s,v_s))\|_{L^2}ds\right)^2 \\ &\leq 2C\mathbb{E}\left(\int_0^t \|F(s,u_s)-F(s,v_s)\|_{L^2}ds\right)^2 \\ &\leq 2CL_f't\int_0^t \|u_s-v_s\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}ds. \end{aligned} \tag{4.36}$$

For I^2, by Lemma 4.3(i), (H_2) and Itô's isometry, we derive

$$\begin{aligned} I^2 &\leq 2\int_0^t \mathbb{E}\|\mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)(G(s,u_s)-G(s,v_s))\|^2_{L^2}ds \\ &\leq 2CL_g'\int_0^t \|u_s-v_s\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}ds. \end{aligned} \tag{4.37}$$

Substituting (4.36)-(4.37) to (4.35), it yields

$$\mathbb{E}\|u(t)-v(t)\|^2_{L^2} \leq 2C\left(L_f't+L_g'\right)\int_0^t \|u_s-v_s\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}ds.$$

Denote by $\mathcal{W}_1 = 2C(L_f'(T_1\wedge T_2)+L_g')$, we have

$$\|u-v\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} \leq \mathcal{W}_1\int_0^{T_1\wedge T_2} \|u_s-v_s\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}dt.$$

The Gronwall Lemma implies that

$$\|u-v\|_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} = 0.$$

Therefore, $u=v$ on $(-\infty, T_1\wedge T_2]$ for every initial function $\varphi(t)$.

Now we prove that for each given $T>0$, the mild solution u to problem (4.23) is bounded with $\mathcal{C}_X(L^2(\Omega;L^2_\sigma))$ norm. Taking into account Lemma 4.3(i), (H_1)-(H_2), Itô's isometry, the Cauchy-Schwarz inequality and Fubini's theorem, we have

$$\begin{aligned} \mathbb{E}\|u(t)\|^2_{L^2} &\leq 3\mathbb{E}\,\|\mathbf{E}_\alpha(-t^\alpha A)\varphi(0)\|^2_{L^2} + 3\mathbb{E}\left\|\int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)F(s,u_s)ds\right\|^2_{L^2} \\ &\quad + 3\mathbb{E}\left\|\int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)G(s,u_s)dW(s)\right\|^2_{L^2} \\ &\leq 3C\|\varphi\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} + 3CL_f't\int_0^t \|u_s\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}ds \\ &\quad + 3CL_g'\int_0^t \|u_s\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}ds \\ &\leq 3C\|\varphi\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} + 3C(L_f't+L_g')\int_0^t \|u_s\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}ds. \end{aligned}$$

Applying the Gronwall lemma, for any fixed $T > 0$ and all $t \in [0, T]$,

$$\|u_t\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} \leq (3C + 1)\|\varphi\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} \exp(3C(L'_f T + L'_g)T).$$

Because of the arbitrariness of T, together with the conclusion of uniqueness of u on $(-\infty, T]$, it is straightforward that the mild solution u to problem (4.23) is defined globally. It finishes the proof of this theorem. □

Remark 4.15. *The well-posedness results to problems (4.6) and (4.23) can be modified to the case that the driven process is an additive fractional Brownian motion, which is L^2_σ-value. Of course we need to redefine $G(t, \cdot)$ and impose certain assumptions similar to (H_2)-(H_5), see [85], for example.*

Remark 4.16. *Although we have performed our analysis for the stochastic time fractional 2D-Stokes delay differential equations, the results of sections 3 and 4 still hold true when the phase spaces are extended to $C([-h, 0]; L^2(\Omega, L^N_\sigma))$ and $C((-\infty, 0]; L^2(\Omega, L^N_\sigma))$ respectively, where $N > 2$ ([22]).*

We end this chapter with the following conclusion. In this chapter we have considered a quite general time-fractional stochastic Stokes model with finite and infinite delay and multiplicative Brownian motion. As we said, this is only a first approach to our goal concerning the case of stochastic time fractional delay Navier-Stokes with multiplicative noise. But, to that end, a new technique has to be designed because the fixed point theorem used in our proofs is not appropriate to handle the nonlinear term: the appearance of expectation in the norm does not allow us to bound that term in an appropriate way as it is done in the deterministic case, specially for the contraction property. Therefore, this is a challenging problem to be analyzed shortly. However, it is not surprising that the problem cannot be analyzed with this technique since, to the best of our knowledge, even the non-fractional time derivative system has not been solved for the multiplicative noise case, but only for the additive one.

Bibliography

[1] A. A. Alikhanov, A priori estimates for solutions of boundary value problems for equations of fractional order, *Differ. Equ.*, **46** (2010), 660-666.

[2] M. Allen, L. Caffarelli and A. Vasseur, A parabolic problem with a fractional time derivative, *Arch. Ration. Mech. Anal.*, **221**(2016), 603-630.

[3] D. Araya and C. Lizama, Almost automorphic mild solutions to fractional differential equations, *Nonlinear Anal. TMA*, **69** (2008), 3692-3705.

[4] P. Bates, H. Lisei and K. Lu, Attractors for stochastic lattice dynamical systems, *Stoch. Dyn.*, **6** (2006), 1-21.

[5] P. Bates, K. Lu and B. Wang, Attractors for lattice dynamical systems,*Int. J. Bifurcation Chaos*, **11** (2001), 143-153.

[6] D. Bahuguna, R. Sakthivel and A. Chadha, Asymptotic stability of fractional impulsive neutral stochastic partial integro-differential equations with infinite delay, *Stoch. Anal. Appl.*, **35** (2017), 63-88.

[7] M. Benchohra, J. Henderson and S. Ntouyas, emphImpulsive Differential Equations and Inclusions, Hindawi Publishing Corporation, 2006.

[8] M. Benchohra, J. Henderson, S. K. Ntouyas and A. Ouahad, Existence results for fractional order functional differential equations with infinite delay, *J. Math. Anal. Appl.*, **338** (2008), 1340-1350.

[9] T. Blouhi, T. Caraballo and A. Ouahab, Existence and stability results for semilinear systems of impulsive stochastic differential equations with fractional Brownian motion, *Stoch. Anal. Appl.*, **34** (2016), 792-834.

[10] E. M. Bonotto, M. C. Bortolan, T. Caraballo and R. Collegari, Attractors for impulsive non-autonomous dynamical systems and their relations, *J. Differential Equations*, **262** (2017), 3524-3550.

[11] E. M. Bonotto, J. G. Mesquita and R. P. Silva, Global mild solutions for a nonautonomous 2D Navier-Stokes equations with impulses at variable times, *J. Math. Fluid Mech.*, **20** (2018), 801-818.

[12] A. Boudaoui, T. Caraballo and A. Ouahab, Impulsive stochastic functional differential inclusions driven by a fractional Brownian motion with infinite delay, *Math. Methods Appl. Sci.*, **39** (2016), 1435-1451.

[13] T. Caraballo, M. J. Garrido-Atienza and T. Taniguchi, The existence and exponential behavior of solutions to stochastic delay evolution equations with a fractional Brownian motion, *Nonlinear Anal.*, **74** (2011), 3671-3684.

[14] T. Caraballo, F. Morillas and J. Valero, Random attractors for stochastic lattice systems with non-Lipschitz nonlinearity, *J. Difference Equ. Appl.*, **17** (2011), 161-184.

[15] T. Caraballo, F. Morillas and J. Valero, Attractors of stochastic lattice dynamical systems with a multiplicative noise and non-Lipschitz nonlinearities, *J. Differential Equations*, **253** (2012), 667-693.

[16] T. Caraballo, F. Morillas and J. Valero, On differential equations with delay in Banach spaces and sttractors for retarded lattice dynamical systems, *Discrete Contin. Dyn. Syst.*, **34** (2014), 51-77.

[17] T. Caraballo and J. Real, Navier-Stokes equations with delays, *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.*, **457** (2014), 2441-2453.

[18] T. Caraballo and X. Y. Han, A survey on Navier-Stokes models with delays: existence, uniqueness and asymptotic behavior of solutions, *Discrete Contin. Dyn. Syst. Ser. S*, **8** (2015), 1079-1101.

[19] S. Carmi, L. Turgeman and E. Barkai, On distributions of functionals of anomalous diffusion paths, *J. Stat. Phys.*, **141** (2010), 1071-1092.

[20] M. H. Chen and W. H. Deng, Discretized fractional substantial calculus, *ESAIM Math. Model. Numer. ANal.*, **49** (2015), 373-394.

[21] P. M. Carvalho-Neto, *Fractional differential equations: a novel study of local and global solutions in Banach spaces*, PhD thesis, Universidade de São Paulo, São Carlos, 2013.

[22] P. M. Carvalho-Neto and G. Planas, Mild solutions to the time fractional Navier-Stokes equations in $\mathbb{R}^N$, *J. Differential Equations*, **259** (2015), 2948-2980.

[23] A. Chauhan and J. Dabas, Local and global existence of mild solution to an impulsive fractional functional integro-differential equations with nonlocal condition, *Commun. Nonlinear Sci. Numer. Simul.*, **19** (2014), 821-829.

[24] Z. Q. Chen, K. H. Kim, and P. Kim, Fractional time stochastic partial differnetial equations, *Stochastic Process. Appl.*, **125** (2015), 1470-1499.

[25] J. Cui and L. Yan, Existence result for fractional neutral stochastic integro-differential equations with infinite delay, *J. Phys. A*, **44** (2011), 335201.

[26] R. F. Curtain and P. L. Falb, Stochastic differential equations in Hilbert space, *J. Differential Equations*, **10** (1971), 412-430.

[27] M. Herrera-Cobos, T. Caraballo and P. Marín-Rubio, *Long-time behavivor of nonlocal partial differential equations*, PhD-thesis, Universidad de Sevilla, 2016.

[28] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, Cambridge, 1992.

[29] L. Debbi, Well-posedness of the multidimensional fractional stochastic Navier-Stokes equations on the torus and on bounded domains, *J. Math. Fluid Mech.*, **18** (2016), 25-69.

[30] J. Dabas and A. Chauhan, Existence and uniqueness of mild solutions for an impulsive neutral fractional integro-differential equation with infinite delay, *Math. Comput. Modelling*, **57** (2013), 754-763.

[31] W. H. Deng, M. H. Chen and E. Barkai, Numercial algorithms for the forward and backward fractional Feynman-Kac equations, *J. Sci. Comput.*, **62** (2015), 718-746.

[32] T. S. Doan, M. Rasmussen and P. E. Kloeden, The mean-square dichotomy spectrum and a bifurcation to a mean-square attractors, *Discrete Contin. Dyn. Syst. Ser. B*, **20** (2015), 875-887.

[33] M. M. EI-Borai, K. EI-said EI-Nadi and H. A. Fouad, On some fractional stochastic delay differential equations, *Comput. Math. Appl.*, **59** (2010), 1165-1170.

[34] Z. B. Fan, Characterization of compactness for resolvents and its applications, *Appl. Math. Comput.*, **232** (2014), 60-67.

[35] R. Friedrich, F. Jenko, A. Baule and S. Eule, Anomalous diffusion of inertial, weakly damped particles, *Phys. Rev. Lett.*, **96** (2006), 230601.

[36] A. Friedman and B. Hu, Bifurcation from stability to instability for a free boundary problem modeling tumor growth by Stokes equation, *J. Math. Anal. Appl.*, **327** (2007), 643-664.

[37] A. Friedman and B. Hu, Bifurcation for a free boundary problem modeling tumor growth by Stokes equation, *SIAM J. Math. Anal.*, **39** (2007), 174-194.

[38] B. H. Haak and P. C. Kunstmann, On Kato's method for Navier-Stokes equations, *J. Math. Fluid Mech.*, **11** (2009), 492-535.

[39] X. Y. Han, Random attractors for stochastic sine-Gordon lattice systems with multiplicative white noise, *J. Math. Anal. Appl.*, **376** (2011), 481-493.

[40] X. Y. Han, Random attractors for second order stochastic lattice dynamical sysems with multiplicative noise in weighted spaces, *Stoch. Dyn.*, **12** (2012), 1150024.

[41] X. Y. Han, Asymptotic behaviors for second order stochastic lattice dynamical systems on $\mathbb{Z}^k$ in weighted spaces, *J. Math. Anal. Appl.*, **397** (2012), 242-254.

[42] X. Y. Han, W. Shen and S. Zhou, Random attractors for stochastic lattice dynamical systems in weighted spaces, *J. Differential Equations*, **250** (2011), 1235-1266.

[43] W. R. Hao, J. D. Hauenstein, B. Hu, T. McCoy and A. J. Sommese, Computing steady-state solutions for a free boundary problem modeling tumor growth by Stokes equation, *J. Comput. Appl. Math.*, **237** (2013), 326-334.

[44] D. Herry, *Geometric Theory of Semilinear Parabolic Partial Differential Equations*, Springer-Verlag, Berlin, 1989.

[45] I. Podlubny, *Fractional differential equations*, Academic Press, San Diego, 1999.

[46] T. Kato, Strong L^p-solutions of the Navier-Stokes equation in R^m, with applications to weak solution, *Math. Z.*, **187** (1984), 471-480.

[47] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier Science B.V, Amsterdam, 2006.

[48] P. E. Kloeden and J. Valero, The Kneser property of the weak solutions of the three dimensional Navier-Stokes equations, *Discrete Contin. Dyn. Syst.*, **28** (2010), 161-179.

[49] P. E. Kloeden and T. Lorenz, Mean-square random dynamical systems, *J. Differential Equations*, **253** (2012), 1422-1438.

[50] P. E. Kloeden and M. H. Yang, Forward attraction in nonautonomous difference equations, *J. Difference Equ. Appl.*, (2015), 1027-1039.

[51] B. Kirby, *Micro- and nanoscale fluid mechanics: transport in microfluidic devices*, Cambridge University Press, Cambridge, UK, 2010.

[52] V. Lakthivel and A. S. Vatsala, Basic theory of fractional differential equations, *nonlinear Anal. TMA*, **69** (2008), 2677-2682.

[53] V. Lakshmikantham, D. D. Bainov and P. S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, 1989.

[54] X. D. Li, M. Bohner and C. K. Wang, Impulsive differential equations: Periodic solutions and applications, *Automatica J. IFAC*, **52** (2015), 173-178.

[55] Y. J. Li and Y. J. Wang, Uniform asymptotic stability of solutions of fractional functional differential equations, *Abstr. Appl. Anal.*, **5** (2013), 685-696.

[56] Y. J. Li, Y. J. Wang and W. H. Deng, Galerkin finite element approximations for stochastic space-time fractional wave equations, *SIAM J. Numer. Anal.*, **55** (2017), 3173-3202.

[57] A. H. Lin, Y. Ren and N. M. Xia, On neutral impulsive stochastic integro-differential equations with infinite delays via fractional operators, *Math. Comput. Modelling*, **51** (2010), 413-424.

[58] Y. J. Li and Y. J. Wang, The existence and asymptotic behavior of solutions to fractional stochastic evolution equations with infinite delay, *J. Differential Equations*, **266** (2019), 3514-3558.

[59] W. Liu, M. Röckner and J. L. da Silva, Quasi-linear (stochastic) partial differential equations with time-fractional derivatives, *SIAM J. Math. Anal.*, **50** (2018), 2588-2607.

[60] F. Mainardi, On the initial value problem for the fractional diffusion-wave equation, *Ser. Adv. Math. Appl. Sci.*, **23** (1994), 246-251.

[61] X. R. Mao, *Stochastic differential equations and applications*, Horwood Publication, Chich- ester, 1997.

[62] P. Marín-Rubio, J. Real, and J. Valero, Pullback attractors for a two-dimensional Navier-Stokes model in an infinite delay case, *Nonlinear Anal.*, **74** (2011), 2012-2030.

[63] R. H. Martin, *Nonlinear Operators and Differential Equations in Banach Spaces*, Printed in the United States of America, 1976.

[64] V. D. Mil'man and A. D. Myskis, On the stability of motion in the prensence of impulses, *Sibirsk. Mat. Zh.*, **1** (1960), 233-237.

[65] Y. S. Mishura, *Stochastic Calculus for Fractional Brownian Motion and Related Processes*, Springer-verlag, 2008.

[66] B. Øksendal, *Stochastic differential equations*, Springer-Verlag, 1985.

[67] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag, New York, 1983.

[68] Y. Ren, X. Cheng and R. Sakthivel, Impulsive neutral stochastic integral-differential equations with infinite delay fBm, *Appl. Math. Comput.*, **247** (2014), 205-212.

[69] J. C. Robinson, *Infinite-dimensional dynamical systems: an introduction to dissipative parabolic PDEs and the theory of global attractors*, Cambridge University Press, 2001.

[70] R. Sakthivel, P. Revathi and Y. Ren, Existence of solutions for nonlinear fractional stochastic differenital equations, *Nonlinear Anal. TMA*, **81** (2013), 70-86.

[71] C. Schwab and R. Stevenson, Fractional space-time variational formulations of (Navier-) Stokes equations, *SIAM J. Math. Anal*, **49** (2017), 2442-2467.

[72] J. H. Shen and X. Z. Liu, Global existence results of impulsive differential equation, *J. Math. Anal. Appl.*, **314** (2006), 546-557.

[73] X. B. Shu, Y. Z. Lai and Y. M. Chen, The existence of mild solutions for impulsive fractional partial differential equations, *Nonlinear Anal.*, **74** (2011), 2003-2011.

[74] I. M. Sokolov and R. Metzler, Towards deterministic equations for Lévy walks: The fractional material derivative, *Phys. Rev. E.*, **67** (2003), 010101.

[75] H. Tang, On the pathwise solutions to the Camassa-Holm equation with multiplicative noise, *SIAM J. Math. Anal.*, **50** (2018), 1322-1366.

[76] T. Taniguchi, The existence and asymptotic behaviour of energy solutions to stochastic 2D functional Navier-Stokes equations driven by Lévy processes, *J. Math. Anal. Appl.*, **385** (2012), 634-654.

[77] S. Tindel, C. A. Tudor and F. Viens, Stochastic evolution equations with fractional Brownian motion, *Probab. Theory Related Fields*, **127** (2003), 186-204.

[78] R. N. Wang, D. H. Chen and T. J. Xiao, Abstract fractional Cauchy problems with almost sectorial operators, *J. Differential Equations*, **252** (2012), 202-235.

[79] Y. J. Wang, J. H Xu and P. E. Kloeden, Asymptotic behavior of stochastic lattice systems with a Caputo fractional time derivative, *Nonlinear Anal. TMA*, **135** (2016) 205-222.

[80] Y. J. Wang, F. S. Gao and P. E. Kloeden, On impulsive fractional functional differential equations with weakly continuous nonlinearity and asymptotic behavior of impulsive fractional delay lattice systems, *Electron. J. Differ. Equ.*, **285** (2017), 1-18.

[81] F. K. Wu and P. E. Kloeden, Mean-square random attractors of stochastic delay differential equations with random delay, *Discrete Contin. Dyn. Syst.*, **18** (2013), 1715-1734.

[82] P. F. Xu, C. B. Zeng and J. H. Huang, Well-posedness of the time-space fractional stochastic Navier-Stokes equations driven by fractional Brownian motion, *Math. Model. Nat. Phenom.*, **13** (2018), 11.

[83] J. H. Xu and T. Caraballo, Long time behavior of fractional impulsive stochastic differential equations with infinite delay. *Discrete Contin. Dyn. Syst. Ser. B*, **24** (2019), 2719-2743.

[84] J. H. Xu, Z. C. Zhang and T. Caraballo, Well-posedness and dynamics of impulsive fractional stochastic evolution equations with unbounded delay, *Commun. Nonlinear Sci. Numer. Simulat.*, **75** (2019), 121-139.

[85] J. H. Xu, Z. C. Zhang and T. Caraballo, Mild solutions to time fractional stochastic 2D-Stokes equations with bounded and unbounded delay, *J. Dyn. Differ. Equ.*, Online, Doi:org/10.1007/s10884-019-09809-3.

[86] S. Zhou and W. Shi, Attractors and dimension of dissipative lattice systems, *J. Differential Equations*, **224** (2006), 172-204.

[87] Y. Zhou and L. Peng, On the time-fractional Navier-Stokes equations, *Comput. Math. Appl.*, **73** (2017), 874–891.

[88] Y. Zhou and L. Peng, Weak solutions of the time-fractional Navier-Stokes equations and optimal control, *Comput. Math. Appl.*, **73**(2017), 1016-1027.

[89] G. A. Zou, G. Y. Lv and J. L. Wu, Stochastic Navier-Stokes equations with Caputo derivative driven by fractional noises, *J. Math. Anal. Appl.*, **461** (2018), 595-609.