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Theory of frequency and phase synchronization in a rocked bistable stochastic system

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We investigate the role of noise in the phenomenon of stochastic synchronization of switching events in a rocked, overdamped bistable potential driven by white Gaussian noise, the archetype description of stochastic resonance. We present an approach to the stochastic counting process of noise-induced switching events: starting from the Markovian dynamics of the nonstationary, continuous particle dynamics, one finds upon contraction onto two states a non-Markovian renewal dynamics. A proper definition of an output discrete phase is given, and the time rate of change of its noise average determines the corresponding output frequency. The phenomenon of noise-assisted phase synchronization is investigated in terms of an effective, instantaneous phase diffusion. The theory is applied to rectangular-shaped rocking signals versus increasing input-noise strengths. In this case, for an appropriate choice of the parameter values, the system exhibits a noise-induced frequency locking accompanied by a very pronounced suppression of the phase diffusion of the output signal. Precise numerical simulations corroborate very favorably our analytical results. The novel theoretical findings are also compared with prior ones.

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I. INTRODUCTION

The theme of synchronization has widespread applications, covering a plenitude of phenomena [1–4]. Some characteristic examples are the entrainment of a system by external, time-dependent forcing, or the generalization of the synchronization concept for systems that exhibit a chaotic dynamics [5], lag synchronization [6], and also phase synchronization [7]. Synchronization phenomena play not only a key role for diverse technological applications, but increasingly as well for the description, the control, and even for the therapy of selected medical disorders [8].

Due to the interaction with a surrounding environment or with internal degrees of freedom, noise is present in many physical systems. This being so, its role cannot be ignored when investigating synchronization phenomena. In recent years, it has turned out that noise can actually play a constructive role in many physical situations. In particular, noise can boost the transduction of information by means of the phenomenon of stochastic resonance [9] in an ample number of metastable physical and biological systems [10,11]. Furthermore, noise enables Brownian motors to do work against external load forces [12], or to induce phase transitions far away from thermal equilibrium [13,14].

Our focus here is on the role of phase synchronization in stochastic overdamped systems driven by white Gaussian noise. In these cases, the velocity of the dynamics is not a measurable quantity because the stochastic trajectories are neither differentiable nor of finite variation, see, e.g., Ref. [15]. A recently proposed method for measuring the average phase velocity or frequency which is based on the generalization of a Rice rate formula for threshold crossings is consequently not a suitable method [16]. An alternative approach is based on the so-called "Hilbert phase" dynamics, as pioneered by Gabor [17] for *deterministic* systems. In the present work, we shall take a closer look at the synchronization phenomenon in a periodically driven bistable system. Then it is advantageous to introduce a discrete phase dynamics, as recently proposed by Schimansky-Geier and collaborators [4,14,18,19]. In order to extract this discrete phase dynamics from the underlying continuous process, we shall consider the stochastic counting process of the noise-induced switches between the two potential minima. It turns out that this counting process is in fact a nonstationary renewal process [20].

The outline of the present work is as follows: First, we introduce a dichotomic process associated to the original stochastic process by filtering out the fluctuations around the potential minima. This dichotomic process possesses a clear interpretation in terms of a discrete phase. By contrast to the underlying stochastic process, this two-state process, however, is no longer Markovian. Subsequently, in Sec. III, we analyze in detail the statistical properties of the random switching times associated to the dichotomic process. The one-time statistical properties of the discrete phase are then studied in Sec. IV. Based on these results, exact analytical expressions for the instantaneous output frequency and the phase diffusion are derived. Approximate expressions, valid in the weak-noise limit and for a slow external driving, are then obtained. Finally, our analytical findings will be applied to the case of a symmetric bistable potential driven by a periodic rectangular input signal. To corroborate our analytical results, we compare them with those obtained from a numerical simulation of the original stochastic process.

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II. DESCRIPTION OF THE MODEL AND DEFINITION OF THE DISCRETE PHASE

To start, we consider a stochastic dynamics characterized by a single degree of freedom $x(t)$, whose dynamics (in dimensionless units) is described by the stochastic differential equation

$$\dot{x}(t) = -U'(x(t),t) + \xi(t), \tag{1}$$

where $\xi(t)$ is a Gaussian white noise of zero mean with autocorrelation $\langle \xi(t)\xi(s)\rangle = 2D\delta(t-s)$, and $U'(x,t)$ is the derivative with respect to x of the bistable quartic potential,

$$U(x,t) = \frac{x^4}{4} - \frac{x^2}{2} - F(t)x, \tag{2}$$

$F(t)$ representing a periodic forcing with period T. Our focus is on subthreshold signals; more precisely, we will assume that, for any instant of time, $|F(t)| < A_{\text{th}} = 2/\sqrt{27}$, where A_{th} is the static threshold value (the dynamical threshold value always exceeds this adiabatic threshold A_{th}). In this case, the potential possesses two minima at $q_{-1}(t) < 0$ and $q_{+1}(t) > 0$, and a maximum at $q_M(t)$. Introducing the function $\nu(t) = \arccos[F(t)/A_{\text{th}}]$, with arccos y being the principal value of the arc cosine of y (i.e., the value in the interval $[0,\pi]$), and

$$\eta_n(t) = \frac{2}{\sqrt{3}} \cos\left[\frac{\nu(t) + 2\pi n}{3}\right], \tag{3}$$

then $\eta_0(t)$ yields the location of the minimum to the right of the barrier [i.e., $q_{+1}(t) = \eta_0(t)$], $\eta_1(t)$ yields the location of the minimum to the left of the barrier [i.e., $q_{-1}(t) = \eta_1(t)$], and $\eta_2(t)$ yields the location of the maximum [i.e., $q_M(t) = \eta_2(t)$]. From now on, we will assume that at an initial instant of time t_0 the system is placed at one of the minima of the potential $q_{\alpha_0}(t_0)$, with $\alpha_0 = +1$ or -1. The long-time behavior of the quantities of interest can be obtained by taking the limit as $t_0 \to -\infty$ at the end of the calculations. Henceforth, we will make explicit the dependence of all the quantities on the initial preparation by the superscript α_0, t_0; see also the discussion after Eq. (26). Thus, for instance, we will write $x^{\alpha_0,t_0}(t)$ instead of $x(t)$, meaning that $x^{\alpha_0,t_0}(t_0) = q_{\alpha_0}(t_0)$.

To analyze the synchronization phenomenon in this stochastic bistable system, it is convenient to introduce a discrete phase associated to the continuous stochastic process $x^{a_0,t_0}(t)$. In order to do so, first we will proceed to filter out the fluctuations around the minima of the stochastic process $x^{\alpha_0,t_0}(t)$ to obtain a two-state stochastic process $\chi^{\alpha_0,t_0}(t)$ which only takes the values $+1$ or -1. The procedure used is as follows: At the initial instant of time t_0 we set $\chi^{\alpha_0,t_0}(t_0) = \alpha_0$. A switch of state from $\pm\alpha_0$ to $\mp\alpha_0$ occurs whenever the system, having started in one of the minima, reaches the other minimum for the first time. The instant of time at which the nth switch of state takes place is a random variable which will be denoted by $\mathcal{T}_n^{\alpha_0,t_0}$, with $n = 1,2,\ldots$. Formally, these random variables can be defined recursively as

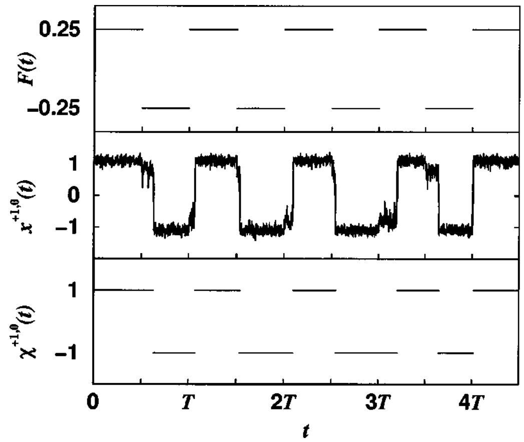

FIG. 1. Illustration of the procedure used to define the stochastic process $\chi^{\alpha_0,t_0}(t)$ in the particular case of a rectangular signal [see Eq. (65) and text below] with amplitude $A = 0.25$ and frequency $\Omega = 2\pi/T = 0.01$, and a noise strength $D = 0.02$. The initial instant of time has been chosen to be $t_0 = 0$, and the system has been initially placed at $q_{+1}(0)$, so that $\alpha_0 = +1$. In the upper panel, we have sketched the rectangular periodic signal $F(t)$, whereas in the middle and lower panels, we have depicted a random trajectory of the stochastic process $x^{+1,0}(t)$ and the corresponding realization of the process $\chi^{+1,0}(t)$, respectively.

$$\mathcal{T}_n^{\alpha_0,t_0} = \min\Big[t : t > \mathcal{T}_{n-1}^{\alpha_0,t_0} \quad \text{and} \quad x^{\alpha_0,t_0}(t) = q_{\alpha_n}(t)\Big], \tag{4}$$

where $\mathcal{T}_0^{\alpha_0,t_0} = t_0$ and $\alpha_n = (-1)^n \alpha_0$. Thus, if we introduce the stochastic process

$$N^{\alpha_0,t_0}(t) = \max\Big[n : \mathcal{T}_n^{\alpha_0,t_0} \leqslant t\Big], \tag{5}$$

which counts the number of switches of state in the interval $(t_0,t]$, then the two-state stochastic process $\chi^{\alpha_0,t_0}(t)$ can be expressed as

$$\chi^{\alpha_0,t_0}(t) = \alpha_0 \cos\Big[\pi N^{\alpha_0,t_0}(t)\Big]. \tag{6}$$

By analogy with the case of a sinusoidal signal, we will define the discrete phase $\varphi^{\alpha_0,t_0}(t)$ associated with $x^{\alpha_0,t_0}(t)$ as the stochastic process

$$\varphi^{\alpha_0,t_0}(t) = \pi N^{\alpha_0,t_0}(t). \tag{7}$$

In Fig. 1, we illustrate the procedure just described in the particular case of a rectangular signal [see Eq. (65) and text below] with amplitude $A = 0.25$ and frequency $\omega = 2\pi/T = 0.01$, and a noise strength $D = 0.02$. The initial instant of time has been chosen to be $t_0 = 0$, and the system has been initially placed at $q_{+1}(0)$, so that $\alpha_0 = +1$. In the upper panel, we have sketched the rectangular periodic signal $F(t)$, whereas in the middle and lower panels, we have depicted a random trajectory of the stochastic process $x^{+1,0}(t)$ and the corresponding realization of the process $\chi^{+1,0}(t)$, respectively.

III. STATISTICAL CHARACTERIZATION OF THE SWITCHING TIMES $\mathcal{T}_n^{\alpha_0,t_0}$

According to Eqs. (5) and (7), the statistical properties of the discrete phase $\varphi^{\alpha_0,t_0}(t)$ are closely related to those of the random switching times $\mathcal{T}_n^{\alpha_0,t_0}$. The aim of this section is to provide a detailed description of the statistical characterization of these random variables previous to the analysis of $\varphi^{\alpha_0,t_0}(t)$, which will be postponed to the next section. The connection between the statistical properties of the switching times and the original stochastic process $x^{\alpha_0,t_0}(t)$ will be also analyzed in this section.

Following the approach presented in Ref. [20], the random variable $\mathcal{T}_n^{\alpha_0,t_0}$ can be characterized statistically by its probability *density* function

$$g_n^{\alpha_0,t_0}(t) = \lim_{\Delta t\to 0^+}\frac{\mathrm{Prob}[t<\mathcal{T}_n^{\alpha_0,t_0}\leqslant t+\Delta t]}{\Delta t}. \tag{8}$$

Besides this probability distribution function, it is also convenient to introduce the (cumulative) distribution function

$$G_n^{\alpha_0,t_0}(t) = \mathrm{Prob}[\mathcal{T}_n^{\alpha_0,t_0}\leqslant t] = \int_{t_0}^{t} dt'\, g_n^{\alpha_0,t_0}(t'), \tag{9}$$

as well as its complementary,

$$\mathcal{G}_n^{\alpha_0,t_0}(t) = \mathrm{Prob}[\mathcal{T}_n^{\alpha_0,t_0}>t] = 1-G_n^{\alpha_0,t_0}(t). \tag{10}$$

In the particular case $n=1$, these functions can be directly determined from the solution of the Fokker-Planck equation (FPE)

$$\frac{\partial}{\partial t}P(x,t) = \frac{\partial}{\partial x}\left[D\frac{\partial}{\partial x}+U'(x,t)\right]P(x,t), \tag{11}$$

with initial condition

$$P(x,t_0) = \delta\big(x-q_{\alpha_0}(t_0)\big), \tag{12}$$

and absorbing boundary condition at $q_{-\alpha_0}(t)$, i.e.,

$$P[q_{-\alpha_0}(t),t]=0 \quad \text{for all} \quad t\geqslant t_0. \tag{13}$$

Denoting by $P^{\alpha_0,t_0}(x,t)$ the solution of the above problem, it follows from the definition of $\mathcal{G}_1^{\alpha_0,t_0}(t)$ in Eq. (10), with $n=1$, that

$$\mathcal{G}_1^{\alpha_0,t_0}(t) = \delta_{\alpha_0,+1}\int_{q_{-1}(t)}^{+\infty} dx\, P^{+1,t_0}(x,t) + \delta_{\alpha_0,-1}\int_{-\infty}^{q_{+1}(t)} dx\, P^{-1,t_0}(x,t), \tag{14}$$

for $t\geqslant t_0$, and $\mathcal{G}_1^{\alpha_0,t_0}(t)=1$ for $t<t_0$. The function $\mathcal{G}_1^{\alpha_0,t_0}(t)$ is the conditional survival probability of the discrete state α_0. The function $G_1^{\alpha_0,t_0}(t)$ is then given by Eq. (10) with $n=1$, whereas $g_1^{\alpha_0,t_0}(t)=-\dot{\mathcal{G}}_1^{\alpha_0,t_0}(t)$ is the corresponding conditional residence time distribution (RTD). The dot indicates the derivative with respect to time t. The knowledge of either the conditional survival probabilities $\mathcal{G}_1^{\alpha_0,t_0}(t)$, or (equivalently) the conditional RTDs $g_1^{\alpha_0,t_0}(t)$, is sufficient to specify a driven two-state non-Markovian renewal process [21]. These functions can be found from the underlying continuous-state Markovian dynamics by solving Eqs. (11)–(14).

For $n>1$, the functions $\mathcal{G}_n^{\alpha_0,t_0}(t)$, $G_n^{\alpha_0,t_0}(t)$, and $g_n^{\alpha_0,t_0}(t)$ can be obtained iteratively from the ones corresponding to the case $n=1$ by making use of three integral equations. To obtain the first integral equation, let us consider the consistency condition

$$\mathcal{G}_{n+1}^{\alpha_0,t_0}(t) = \int_{-\infty}^{+\infty} dt'\, \mathrm{Prob}[\mathcal{T}_{n+1}^{\alpha_0,t_0}>t|\mathcal{T}_n^{\alpha_0,t_0}=t']g_n^{\alpha_0,t_0}(t'). \tag{15}$$

By use of the definition of the switching times in Eq. (4), as well as the Markovian character of the original stochastic process $x^{\alpha_0,t_0}(t)$, it is straightforward to verify that

$$\mathrm{Prob}[\mathcal{T}_{n+1}^{\alpha_0,t_0}>t|\mathcal{T}_n^{\alpha_0,t_0}=t'] = \mathcal{G}_1^{\alpha_n,t'}(t). \tag{16}$$

Inserting the above expression into Eq. (15) and taking into account that $\mathcal{G}_1^{\alpha_n,t'}(t)=1$ for $t<t'$ and $g_n^{\alpha_0,t_0}(t)=0$ for $t<t_0$, we obtain

$$\mathcal{G}_{n+1}^{\alpha_0,t_0}(t) = \mathcal{G}_n^{\alpha_0,t_0}(t) + \int_{t_0}^{t} dt'\,\mathcal{G}_1^{\alpha_n,t'}(t)g_n^{\alpha_0,t_0}(t') \tag{17}$$

for $n\geqslant 1$. The interpretation of this result is straightforward: The probability that the $(n+1)$th switch of state occurs after the time t is equal to the probability that the nth switch occurs after that instant of time plus the probability that the nth switch has happened at any time t' before t with the next switch taking place after t. Similar interpretations hold for the integral equations; i.e.,

$$G_{n+1}^{\alpha_0,t_0}(t) = \int_{t_0}^{t} dt'\, G_1^{\alpha_n,t'}(t)g_n^{\alpha_0,t_0}(t'), \tag{18}$$

$$g_{n+1}^{\alpha_0,t_0}(t) = \int_{t_0}^{t} dt'\, g_1^{\alpha_n,t'}(t)g_n^{\alpha_0,t_0}(t'), \tag{19}$$

which are obtained from Eq. (17) by using Eq. (10) and $g_n^{\alpha_0,t_0}(t)=-\dot{\mathcal{G}}_n^{\alpha_0,t_0}(t)$, respectively.

A formal solution for $g_n^{\alpha_0,t_0}(t_n)$ is obtained by solving iteratively the integral equation (19). The result is

$$g_n^{\alpha_0,t_0}(t_n) = \int_{t_0}^{t_n} dt_{n-1}\cdots\int_{t_0}^{t_2} dt_1\prod_{j=0}^{n-1} g_1^{\alpha_j,t_j}(t_{j+1}) \tag{20}$$

for $n\geqslant 2$. Thus, the probability distribution corresponding to the first switch of state, $g_1^{\alpha_0,t_0}(t)$, which can be obtained from the solution of the FPE (11) with Eqs. (12) and (13), determines completely the statistical properties of the rest of the switching times.

IV. ONE-TIME STATISTICAL PROPERTIES OF THE DISCRETE PHASE: THE OUTPUT FREQUENCY AND THE PHASE DISPERSION

The one-time statistical properties of the discrete phase $\varphi^{\alpha_0,t_0}(t)$ can be evaluated by making use of the probability distribution of the number of switches of state

$$\rho_n^{\alpha_0,t_0}(t) = \mathrm{Prob}\big[N^{\alpha_0,t_0}(t)=n\big], \tag{21}$$

with $n=0, 1, 2, \ldots$. From the definition of $N^{\alpha_0,t_0}(t)$ in Eq. (5) it follows that

$$\mathrm{Prob}\left[N^{\alpha_0,t_0}(t) \geqslant n\right] = G_n^{\alpha_0,t_0}(t), \tag{22}$$

with $G_0^{\alpha_0,t_0}(t)=1$. Consequently, the probability distribution of the number of switches of state and its derivative with respect to t can be expressed, respectively, as

$$\rho_n^{\alpha_0,t_0}(t) = G_n^{\alpha_0,t_0}(t) - G_{n+1}^{\alpha_0,t_0}(t) = \mathcal{G}_{n+1}^{\alpha_0,t_0}(t) - \mathcal{G}_n^{\alpha_0,t_0}(t), \tag{23}$$

and

$$\dot{\rho}_n^{\alpha_0,t_0}(t) = g_n^{\alpha_0,t_0}(t) - g_{n+1}^{\alpha_0,t_0}(t), \tag{24}$$

with $\mathcal{G}_0^{\alpha_0,t_0}(t)=g_0^{\alpha_0,t_0}(t)=0$. The average of an arbitrary one-time function of $N^{\alpha_0,t_0}(t)$, $K[N^{\alpha_0,t_0}(t)]$, is obviously given by

$$\left\langle K\left[N^{\alpha_0,t_0}(t)\right]\right\rangle = \sum_{n=0}^{\infty} K(n)\rho_n^{\alpha_0,t_0}(t). \tag{25}$$

Equations (23) and (24) can be written in a more transparent form by introducing the probability of an almost immediate switch of state after n switches,

$$\Gamma_n^{\alpha_0,t_0}(t) = \lim_{\Delta t\to 0^+} \frac{\mathrm{Prob}[t < \mathcal{T}_{n+1}^{\alpha_0,t_0} \leqslant t+\Delta t | N^{\alpha_0,t_0}(t) = n]}{\Delta t}. \tag{26}$$

Note that if the process $\chi^{\alpha_0,t_0}(t)$ were Markovian, these probabilities could only depend on the state α_n. The explicit dependence on the number of jumps n and on the initial preparation α_0 at time t_0 is a consequence of the non-Markovian character of the process. Another fingerprint of the non-Markovian nature of the dichotomic process is the fact that these probabilities depend on the time t even in the absence of the external driving. In order to clarify this point, let us consider, e.g., the particular case $n=0$. Then, while initially, right after the particle has been prepared at one of the minima, the distribution function $P^{\alpha_0,t_0}(x,t)$ is still very sharply peaked around $q_{\alpha_0}(t_0)$, it becomes smeared out around the minimum after the intrawell relaxation time. Consequently, the probability of an immediate switch will be different before and after this relaxation time, even without an external driving.

Multiplying and dividing the right-hand side of the above expression by $\rho_n^{\alpha_0,t_0}(t)$ and taking into account that

$$\lim_{\Delta t\to 0^+} \frac{\mathrm{Prob}\left[t < \mathcal{T}_{n+1}^{\alpha_0,t_0} \leqslant t+\Delta t \text{ and } N^{\alpha_0,t_0}(t) = n\right]}{\Delta t} = g_{n+1}^{\alpha_0,t_0}(t), \tag{27}$$

it is readily seen that

$$\Gamma_n^{\alpha_0,t_0}(t) = \frac{g_{n+1}^{\alpha_0,t_0}(t)}{\rho_n^{\alpha_0,t_0}(t)}. \tag{28}$$

Then, it follows from Eqs. (17), (23), and (28) that

$$\rho_n^{\alpha_0,t_0}(t) = \int_{t_0}^{t} dt' \,\mathcal{G}_1^{\alpha_n,t'}(t)\Gamma_{n-1}^{\alpha_0,t_0}(t')\rho_{n-1}^{\alpha_0,t_0}(t') \tag{29}$$

for $n\geqslant 1$, with $\rho_0^{\alpha_0,t_0}(t)=\mathcal{G}_1^{\alpha_0,t_0}(t)$. Analogously, Eq. (24) leads to the following hierarchy of differential equations:

$$\dot{\rho}_n^{\alpha_0,t_0}(t) = \Gamma_{n-1}^{\alpha_0,t_0}(t)\rho_{n-1}^{\alpha_0,t_0}(t) - \Gamma_n^{\alpha_0,t_0}(t)\rho_n^{\alpha_0,t_0}(t) \tag{30}$$

for $n\geqslant 1$, and

$$\dot{\rho}_0^{\alpha_0,t_0}(t) = -\Gamma_0^{\alpha_0,t_0}(t)\rho_0^{\alpha_0,t_0}(t), \tag{31}$$

which must be solved with the initial condition $\rho_n^{\alpha_0,t_0}(t_0) = \delta_{n,0}$.

We will also introduce the conditional probability for $\chi^{\alpha_0,t_0}(t)$ to take the value $\beta=\pm 1$ at time t, provided that it took the value α_0 with probability 1 at the initial instant of time t_0,

$$p_\beta^{\alpha_0,t_0}(t) = \mathrm{Prob}\left[\chi^{\alpha_0,t_0}(t) = \beta\right]. \tag{32}$$

Noting that after an even number of switches of state the system ends up in the same state as it was initially, whereas for an odd number of switches the system ends up in the other state, it is clear that the events $\{\chi^{\alpha_0,t_0}(t)=\alpha_0\}$ and $\{\chi^{\alpha_0,t_0}(t)=-\alpha_0\}$ are, respectively, equivalent to the events $\{N^{\alpha_0,t_0}(t) \text{ is even}\}$ and $\{N^{\alpha_0,t_0}(t) \text{ is odd}\}$, and consequently

$$p_\beta^{\alpha_0,t_0}(t) = \delta_{\alpha_0,\beta}\sum_{n=0}^{\infty} \rho_{2n}^{\alpha_0,t_0}(t) + \delta_{-\alpha_0,\beta}\sum_{n=0}^{\infty} \rho_{2n+1}^{\alpha_0,t_0}(t). \tag{33}$$

Besides the probability distribution of the number of switches of state, later we will also use the probability distribution of the number of switches of state conditioned to the value of $\chi^{\alpha_0,t_0}(t)$,

$$\rho_n^{\alpha_0,t_0}(t|\beta) = \mathrm{Prob}[N^{\alpha_0,t_0}(t) = n | \chi^{\alpha_0,t_0}(t) = \beta]. \tag{34}$$

Multiplying and dividing the right-hand side of the above expression by $p_\beta^{\alpha_0,t_0}(t)$ and taking into account that $\mathrm{Prob}[N^{\alpha_0,t_0}(t)=n \text{ and } \chi^{\alpha_0,t_0}(t)=\beta]=\rho_n^{\alpha_0,t_0}(t)\,\delta_{\alpha_n,\beta}$, it results that

$$\rho_n^{\alpha_0,t_0}(t|\beta) = \frac{\rho_n^{\alpha_0,t_0}(t)}{p_\beta^{\alpha_0,t_0}(t)}\,\delta_{\alpha_n,\beta}. \tag{35}$$

The average of an arbitrary one-time function of $N^{\alpha_0,t_0}(t)$, $K[N^{\alpha_0,t_0}(t)]$, conditioned to the event $\{\chi^{\alpha_0,t_0}(t)=\beta\}$ will be denoted by $\langle K[N^{\alpha_0,t_0}(t)]\rangle_\beta$, with $\beta=+1$ or -1, and it is given by

$$\begin{aligned}\left\langle K[N^{\alpha_0,t_0}(t)]\right\rangle_\beta &= \sum_{n=0}^{\infty} K(n)\rho_n^{\alpha_0,t_0}(t|\beta)\\ &= \delta_{\alpha_0,\beta}\sum_{n=0}^{\infty} K(2n)\frac{\rho_{2n}^{\alpha_0,t_0}(t)}{p_{\alpha_0}^{\alpha_0,t_0}(t)}\\ &\quad + \delta_{-\alpha_0,\beta}\sum_{n=0}^{\infty} K(2n+1)\frac{\rho_{2n+1}^{\alpha_0,t_0}(t)}{p_{-\alpha_0}^{\alpha_0,t_0}(t)}.\end{aligned} \tag{36}$$

From the above expression and Eq. (25), it follows that

$$\begin{aligned}\left\langle K[N^{\alpha_0,t_0}(t)]\right\rangle &= \left\langle K[N^{\alpha_0,t_0}(t)]\right\rangle_{+1} p_{+1}^{\alpha_0,t_0}(t)\\ &\quad + \left\langle K[N^{\alpha_0,t_0}(t)]\right\rangle_{-1} p_{-1}^{\alpha_0,t_0}(t).\end{aligned} \tag{37}$$

Another interesting quantity which will be useful later is the probability of an almost immediate switch from state β, defined as

$$\gamma_{\beta}^{\alpha_0,t_0}(t)=\lim_{\Delta t\to 0^+}\frac{\mathrm{Prob}\Big[t<\mathcal{T}^{\alpha_0,t_0}_{N^{\alpha_0,t_0}(t)+1}\leqslant t+\Delta t|\chi^{\alpha_0,t_0}(t)=\beta\Big]}{\Delta t}. \tag{38}$$

Notice that in the Markovian limit, these probabilities are independent of the initial preparation α_0 at time t_0 and cannot be distinguished from the probabilities defined in Eq. (26). Multiplying and dividing the right-hand side of the above expression by $p_{\beta}^{\alpha_0,t_0}(t)$ and taking into account the equivalence of the event $\{\chi^{\alpha_0,t_0}(t)=\beta\}$ with the event $\{N^{\alpha_0,t_0}(t)\text{ is even}\}\delta_{\alpha_0,\beta}+\{N^{\alpha_0,t_0}(t)\text{ is odd}\}\delta_{-\alpha_0,\beta}$, as well as Eqs. (26) and (36), it is easy to see that

$$\gamma_{\beta}^{\alpha_0,t_0}(t)=\Big\langle\Gamma^{\alpha_0,t_0}_{N^{\alpha_0,t_0}(t)}(t)\Big\rangle_{\beta}. \tag{39}$$

Differentiating Eq. (33) with respect to t and taking into account Eqs. (30), (36), and (39), it is straightforward to obtain that

$$\dot{p}_{\beta}^{\alpha_0,t_0}(t)=-\gamma_{\beta}^{\alpha_0,t_0}(t)p_{\beta}^{\alpha_0,t_0}(t)+\gamma_{-\beta}^{\alpha_0,t_0}(t)p_{-\beta}^{\alpha_0,t_0}(t) \tag{40}$$

for $\beta=1$ and -1. Equation (40) is a non-Markovian master equation for the conditional probabilities $p_{\beta}^{\alpha_0,t_0}(t)$. It is of the time-convolutionless form [22,23]. The rate parameters $\gamma_{\beta}^{\alpha_0,t_0}(t)$ entering this equation are time-dependent quantities even in the absence of time-dependent driving. Such a time dependence reflects primarily a nonexponential distribution of the residence times of the renewal two-state non-Markovian process [20,24], which results from the projection of a continuous-state Markovian stochastic dynamics onto the two discrete states $\beta=\pm 1$. A time-dependent driving introduces an additional time dependence into $\gamma_{\beta}^{\alpha_0,t_0}(t)$ which is present also in the *driven* Markovian case. In this latter case, $\gamma_{\beta}^{\alpha_0,t_0}(t)$ becomes a time-dependent rate and this rate depends neither on α_0 nor on t_0 (see below). Two other forms are possible to describe the evolution of conditional probabilities $p_{\beta}^{\alpha_0,t_0}(t)$. One is given by the generalized master equations (GMEs) with the memory kernels expressed via the corresponding RTDs. In the driven case, the kernels of corresponding GMEs will become functionals of the driving and will depend on both time arguments. Alternatively, integral equations for the conditional probabilities $p_{\beta}^{\alpha_0,t_0}(t)$ can be derived for the driven two-state renewal process in terms of (conditional) RTDs $g_1^{\alpha_0,t_0}(t)$ [21]. Such integral equations present a generalization of the integral renewal equations of Ref. [20] to the driven case. We apply in this work a *time-convolutionless* description of non-Markovian dynamics [22,23] to the synchronization problem.

After these rather formal considerations, we shall now apply these results to the evaluation of two important quantities in the study of the synchronization phenomenon: The instantaneous output frequency and phase diffusion.

A. The instantaneous output frequency

The instantaneous output frequency is defined as the derivative with respect to t of the averaged discrete phase [18], i.e.,

$$\Omega_{\text{out}}^{\alpha_0,t_0}(t)=\frac{\partial}{\partial t}\Big\langle\varphi^{\alpha_0,t_0}(t)\Big\rangle=\pi\frac{\partial}{\partial t}\Big\langle N^{\alpha_0,t_0}(t)\Big\rangle. \tag{41}$$

Multiplying Eq. (30) by n, summing up the series $\Sigma_{n=1}^{\infty}n\dot{\rho}_n^{\alpha_0,t_0}(t)$, and taking into account Eq. (25), it is easy to obtain that

$$\frac{\partial}{\partial t}\langle N^{\alpha_0,t_0}(t)\rangle=\Big\langle\Gamma^{\alpha_0,t_0}_{N^{\alpha_0,t_0}(t)}(t)\Big\rangle. \tag{42}$$

Consequently, from Eqs. (37), (39), and (41), it results

$$\Omega_{\text{out}}^{\alpha_0,t_0}(t)=\pi\big[\gamma_{+1}^{\alpha_0,t_0}(t)p_{+1}^{\alpha_0,t_0}(t)+\gamma_{-1}^{\alpha_0,t_0}(t)p_{-1}^{\alpha_0,t_0}(t)\big]. \tag{43}$$

This finding for the averaged frequency of the discrete phase dynamics constitutes a first main finding of this work.

B. The instantaneous phase diffusion

Let us now proceed to the evaluation of the instantaneous phase diffusion $D_{\text{out}}^{\alpha_0,t_0}(t)$, which can be defined as [18]

$$\begin{aligned}D_{\text{out}}^{\alpha_0,t_0}(t)&=\frac{\partial}{\partial t}\Big\{\Big\langle\big[\varphi^{\alpha_0,t_0}(t)\big]^2\Big\rangle-\Big\langle\varphi^{\alpha_0,t_0}(t)\Big\rangle^2\Big\}\\&=\pi^2\frac{\partial}{\partial t}\Big\{\Big\langle\big[N^{\alpha_0,t_0}(t)\big]^2\Big\rangle-\Big\langle N^{\alpha_0,t_0}(t)\Big\rangle^2\Big\}.\end{aligned} \tag{44}$$

Multiplying Eq. (30) by n^2, summing up the series $\Sigma_{n=1}^{\infty}n^2\dot{\rho}_n^{\alpha_0,t_0}(t)$, and taking into account Eq. (25), it is straightforward to see that

$$\frac{\partial}{\partial t}\Big\langle\big[N^{\alpha_0,t_0}(t)\big]^2\Big\rangle=2\Big\langle N^{\alpha_0,t_0}(t)\Gamma^{\alpha_0,t_0}_{N^{\alpha_0,t_0}(t)}(t)\Big\rangle+\Big\langle\Gamma^{\alpha_0,t_0}_{N^{\alpha_0,t_0}(t)}(t)\Big\rangle. \tag{45}$$

Replacing the above expression into Eq. (44) and taking into account Eqs. (41) and (42), it results that

$$\begin{aligned}D_{\text{out}}^{\alpha_0,t_0}(t)=\pi\Omega_{\text{out}}^{\alpha_0,t_0}(t)+2\pi^2\Big[&\Big\langle N^{\alpha_0,t_0}(t)\Gamma^{\alpha_0,t_0}_{N^{\alpha_0,t_0}(t)}(t)\Big\rangle\\&-\langle N^{\alpha_0,t_0}(t)\rangle\Big\langle\Gamma^{\alpha_0,t_0}_{N^{\alpha_0,t_0}(t)}(t)\Big\rangle\Big].\end{aligned} \tag{46}$$

Equation (46) can be expressed in a more convenient form by writing all the averages $\langle\cdots\rangle$ in terms of the conditional averages $\langle\cdots\rangle_{\beta}$, according to Eq. (37). Then, after some simplifications one obtains

$$\begin{aligned}D_{\text{out}}^{\alpha_0,t_0}(t)=&\pi\Omega_{\text{out}}^{\alpha_0,t_0}(t)+2\pi^2\Delta\gamma^{\alpha_0,t_0}(t)\Psi^{\alpha_0,t_0}(t)\\&+2\pi^2\sum_{\beta=\pm1}C_{\beta}^{\alpha_0,t_0}(t)p_{\beta}^{\alpha_0,t_0}(t),\end{aligned} \tag{47}$$

where

$$\Delta\gamma^{\alpha_0,t_0}(t)=\gamma_{+1}^{\alpha_0,t_0}(t)-\gamma_{-1}^{\alpha_0,t_0}(t), \tag{48}$$

$$\Psi^{\alpha_0,t_0}(t)=\big[\langle N^{\alpha_0,t_0}(t)\rangle_{+1}-\langle N^{\alpha_0,t_0}(t)\rangle_{-1}\big]p_{+1}^{\alpha_0,t_0}(t)p_{-1}^{\alpha_0,t_0}(t), \tag{49}$$

and we have introduced the conditional covariance

$$\begin{aligned}C_{\beta}^{\alpha_0,t_0}(t)=\Big\langle&\big[N^{\alpha_0,t_0}(t)-\langle N^{\alpha_0,t_0}(t)\rangle_{\beta}\big]\\&\times\big[\Gamma^{\alpha_0,t_0}_{N^{\alpha_0,t_0}(t)}(t)-\Big\langle\Gamma^{\alpha_0,t_0}_{N^{\alpha_0,t_0}(t)}(t)\Big\rangle_{\beta}\big]\Big\rangle_{\beta}.\end{aligned} \tag{50}$$

Making use of Eqs. (30) and (40) and after some lengthy calculations, it is possible to prove that $\Psi^{\alpha_0,t_0}(t)$ satisfies the differential equation

$$\dot{\Psi}^{\alpha_0,t_0}(t) = -\gamma^{\alpha_0,t_0}(t)\Psi^{\alpha_0,t_0}(t) - \sum_{\beta=\pm 1} \beta[C_\beta^{\alpha_0,t_0}(t) + \gamma_\beta^{\alpha_0,t_0}(t)p_\beta^{\alpha_0,t_0}(t)]p_\beta^{\alpha_0,t_0}(t), \tag{51}$$

where

$$\gamma^{\alpha_0,t_0}(t) = \gamma_{+1}^{\alpha_0,t_0}(t) + \gamma_{-1}^{\alpha_0,t_0}(t). \tag{52}$$

Equation (51) can be formally solved taking into account that, as it follows from the definition (49), $\Psi^{\alpha_0,t_0}(t_0)=0$. The result is

$$\Psi^{\alpha_0,t_0}(t) = -\sum_{\beta=\pm 1} \beta \int_{t_0}^{t} dt' \left[C_\beta^{\alpha_0,t_0}(t') + \gamma_\beta^{\alpha_0,t_0}(t')p_\beta^{\alpha_0,t_0}(t')\right] \times p_\beta^{\alpha_0,t_0}(t')e^{-\int_{t'}^{t}dt''\gamma^{\alpha_0,t_0}(t'')}. \tag{53}$$

Replacing the above expression into Eq. (47), one obtains

$$D_{\text{out}}^{\alpha_0,t_0}(t) = \pi\,\Omega_{\text{out}}^{\alpha_0,t_0}(t) + Q^{\alpha_0,t_0}(t) - 2\pi^2\Delta\gamma^{\alpha_0,t_0}(t) \times \sum_{\beta=\pm 1} \beta \int_{t_0}^{t} dt'\,\gamma_\beta^{\alpha_0,t_0}(t')\left[p_\beta^{\alpha_0,t_0}(t')\right]^2 e^{-\int_{t'}^{t}dt''\gamma^{\alpha_0,t_0}(t'')}, \tag{54}$$

where all the dependence on the conditional covariance $C_\beta^{\alpha_0,t_0}(t)$ has been included in the function

$$Q^{\alpha_0,t_0}(t) = 2\pi^2 \sum_{\beta=\pm 1} C_\beta^{\alpha_0,t_0}(t)p_\beta^{\alpha_0,t_0}(t) - 2\pi^2\Delta\gamma^{\alpha_0,t_0}(t) \times \sum_{\beta=\pm 1} \beta \int_{t_0}^{t} dt'\, C_\beta^{\alpha_0,t_0}(t')p_\beta^{\alpha_0,t_0}(t')e^{-\int_{t'}^{t}dt''\gamma^{\alpha_0,t_0}(t'')}. \tag{55}$$

The expression (54) for the instantaneous phase diffusion presents a second main result of this work.

C. The weak-noise and low-frequency limit

Throughout the following, we will assume that the noise strength D is sufficiently small so that the intrawell relaxation time scale is negligible compared with the time scale associated to the interwell transitions and, as well, the driving time scale T. In this case, for $t-t_0$ much larger than the characteristic intrawell relaxation time, the probability of an almost immediate switch of state after 0 switches, $\Gamma_0^{\beta,t_0}(t)$, can be approximated by the Kramers rate of escape [25] from the state β at time t, i.e.,

$$\Gamma_0^{\beta,t_0}(t) \approx \gamma_\beta^{\mathrm{K}}(t) = \frac{\omega_\beta(t)\ \omega_M(t)}{2\pi} \times \exp\left\{-\frac{U[q_M(t),t] - U[q_\beta(t),t]}{D}\right\}, \tag{56}$$

where $\omega_\beta(t)=\sqrt{U''[q_\beta(t),t]}=\sqrt{3[q_\beta(t)]^2-1}$ and $\omega_M(t) =\sqrt{|U''[q_M(t),t]|}=\sqrt{1-3[q_M(t)]^2}$. Furthermore, from Eqs. (17), (19), (23), and (28) it follows that, within this approximation, we also have that $\Gamma_n^{\beta,t_0}(t)\approx\gamma_\beta^{\mathrm{K}}(t)$ for $n\geqslant 1$ and, consequently, $\gamma_\beta^{\alpha_0,t_0}(t)\approx\gamma_\beta^{\mathrm{K}}(t)$ and $C_\beta^{\alpha_0,t_0}(t)\approx 0$. In this case, the conditional survival propabilities and the residence time distributions in state α_0 read

$$\mathcal{G}_1^{\alpha_0,t_0}(t) = \exp\left[-\int_{t_0}^{t}\gamma_{\alpha_0}^{K}(t')dt'\right] \tag{57}$$

and

$$g_1^{\alpha_0,t_0}(t) = \gamma_{\alpha_0}^{K}(t)\exp\left[-\int_{t_0}^{t}\gamma_{\alpha_0}^{K}(t')dt'\right], \tag{58}$$

respectively. This corresponds to a two-state Markovian process with rates $\gamma_{\pm 1}^{K}(t)$. In this Markovian limit for the reduced, two-state dynamics, the instantaneous output frequency and phase diffusion become

$$\Omega_{\text{out}}^{\alpha_0,t_0}(t) = \pi\left[\gamma_{+1}^{\mathrm{K}}(t)p_{+1}^{\alpha_0,t_0}(t) + \gamma_{-1}^{\mathrm{K}}(t)p_{-1}^{\alpha_0,t_0}(t)\right] \tag{59}$$

and

$$D_{\text{out}}^{\alpha_0,t_0}(t) = \pi\Omega_{\text{out}}^{\alpha_0,t_0}(t) - 2\pi^2\Delta\gamma^{\mathrm{K}}(t)\sum_{\beta=\pm 1}\beta\int_{t_0}^{t}dt'\,\gamma_\beta^{\mathrm{K}}(t')\left[p_\beta^{\alpha_0,t_0}(t')\right]^2 \times e^{-\int_{t'}^{t}dt''\gamma^{\mathrm{K}}(t'')}, \tag{60}$$

respectively. Here, $\gamma^{\mathrm{K}}(t)=\gamma_{+1}^{\mathrm{K}}(t)+\gamma_{-1}^{\mathrm{K}}(t)$, $\Delta\gamma^{\mathrm{K}}(t)=\gamma_{+1}^{\mathrm{K}}(t)-\gamma_{-1}^{\mathrm{K}}(t)$, and $p_\beta^{\alpha_0,t_0}(t)$ is obtained by solving the master equation

$$\dot{p}_\beta^{\alpha_0,t_0}(t) = -\gamma_\beta^{\mathrm{K}}(t)p_\beta^{\alpha_0,t_0}(t) + \gamma_{-\beta}^{\mathrm{K}}(t)p_{-\beta}^{\alpha_0,t_0}(t), \tag{61}$$

with initial condition $p_\beta^{\alpha_0,t_0}(t_0)=\delta_{\alpha_0,\beta}$.

In order to obtain expressions independent of the initial preparation, it is necessary to take the limit $t_0\to-\infty$ of Eqs. (59) and (60). In this limit, it can be shown that the functions $\Omega_{\text{out}}(t)=\lim_{t_0\to-\infty}\Omega_{\text{out}}^{\alpha_0,t_0}(t)$ and $D_{\text{out}}(t)=\lim_{t_0\to-\infty}D_{\text{out}}^{\alpha_0,t_0}(t)$ are periodic functions of the time t. Then, one can perform a cycle average and define the averaged output frequency

$$\Omega_{\text{out}} = \frac{1}{T}\int_0^T dt\,\Omega_{\text{out}}(t) = \frac{\pi}{T}\int_0^T dt\left[\gamma_{+1}^{\mathrm{K}}(t)p_{+1}(t) + \gamma_{-1}^{\mathrm{K}}(t)p_{-1}(t)\right], \tag{62}$$

where $p_\beta(t)$ is the periodic long-time solution, $p_\beta(t)=\lim_{t_0\to-\infty}p_\beta^{\alpha_0,t_0}(t)$, of Eq. (61). After some lengthy calculations, it is also possible to show from Eq. (60) that the averaged phase diffusion is given by

$$D_{\text{out}} = \frac{1}{T}\int_0^T dt D_{\text{out}}(t) = \pi\Omega_{\text{out}} - \frac{\pi^2}{T}\text{csch}\left(\frac{\overline{\gamma}^{\text{K}}T}{2}\right) \times \sum_{\beta=\pm 1}\beta\int_0^T dt\int_0^T dt'\Delta\gamma^{\text{K}}(t)\gamma_\beta^{\text{K}}(t')[p_\beta(t')]^2 \times \exp\left[\text{sgn}(t-t')\frac{\overline{\gamma}^{\text{K}}T}{2} - \int_{t'}^{t} dt''\gamma^{\text{K}}(t'')\right], \tag{63}$$

with

$$\overline{\gamma}^{\text{K}} = \frac{1}{T}\int_0^T dt\gamma^{\text{K}}(t). \tag{64}$$

In the next section we will consider the case of a rectangular input signal. In this case, explicit analytical evaluations of the integrals in Eqs. (62) and (63) can be carried out.

V. PERIODIC RECTANGULAR INPUT SIGNAL

As an example of the use of Eqs. (62) and (63) which is amenable to analytical treatment, we will consider the case of the periodic rectangular driving force

$$F(t) = (-1)^{n(t)}A, \tag{65}$$

where $n(t)=\lfloor 2t/T\rfloor$, $\lfloor z\rfloor$ being the floor function of z, i.e., the greatest integer less than or equal to z. In other words, $F(t)=A$ $[F(t)=-A]$ if $t\in[nT/2,(n+1)T/2]$ with n even (odd). Because the potential fulfills the symmetry property $U(x,t+T/2)=U(-x,t)$, we have $q_M(t)=(-1)^{n(t)}q_M(0)$, and

$$q_\beta(t) = \beta\frac{\Delta q(0)}{2} - (-1)^{n(t)}\frac{q_M(0)}{2}, \tag{66}$$

where $\Delta q(0)=q_{+1}(0)-q_{-1}(0)$. Here, we have taken into account Vieta's formula $q_{+1}(t)+q_{-1}(t)+q_M(t)=0$.

According to the above mentioned symmetry property of the potential, $\gamma_\beta^{\text{K}}(t)$ can be expressed in the form

$$\gamma_\beta^{\text{K}}(t) = \frac{\gamma}{2}[1-(-1)^{n(t)}\beta\Delta p^{\text{eq}}(0)], \tag{67}$$

where $\gamma=\gamma_{+1}^{\text{K}}(0)+\gamma_{-1}^{\text{K}}(0)$, and $\Delta p^{\text{eq}}(0)=p_{+1}^{\text{eq}}(0)-p_{-1}^{\text{eq}}(0)$, $p_\beta^{\text{eq}}(0)$ being the equilibrium population of the state β corresponding to the rates taken at time $t=0$, i.e., $p_\beta^{\text{eq}}(0)=[\delta_{\beta,-1}\gamma_{+1}^{\text{K}}(0)+\delta_{\beta,+1}\gamma_{-1}^{\text{K}}(0)]/\gamma$. Notice that for the rectangular input signal in Eq. (65), $\gamma^{\text{K}}(t)=\overline{\gamma}^{\text{K}}=\gamma$. We can also write

$$\Delta\gamma^{\text{K}}(t) = \gamma_{+1}^{\text{K}}(t) - \gamma_{-1}^{\text{K}}(t) = -(-1)^{n(t)}\gamma\Delta p^{\text{eq}}(0). \tag{68}$$

As shown in Ref. [26], the long-time probabilities $p_{\pm 1}(t)$ are given by

$$p_{-1}(t) = \tfrac{1}{2}[1-(-1)^{n(t)}\Delta p^{\text{eq}}(0)] + (-1)^{n(t)}\Delta p^{\text{eq}}(0)\,\frac{e^{-\gamma\{t-[n(t)T/2]\}}}{1+e^{-\gamma T/2}}, \tag{69}$$

and $p_{+1}(t)=1-p_{-1}(t)$.

Replacing the above expressions into Eqs. (62) and (63), one obtains after some lengthy simplifications that

$$\Omega_{\text{out}} = \frac{\pi\gamma}{2}\left\{1-[\Delta p^{\text{eq}}(0)]^2\left[1-\frac{4\tanh\left(\frac{\gamma T}{4}\right)}{\gamma T}\right]\right\}, \tag{70}$$

and

$$D_{\text{out}} = \pi\Omega_{\text{out}} - \frac{2\pi^2}{T}[\Delta p^{\text{eq}}(0)]^4\left[\tanh\left(\frac{\gamma T}{4}\right)\right]^3 - \frac{\pi^2}{2T}[\Delta p^{\text{eq}}(0)]^2\{1-[\Delta p^{\text{eq}}(0)]^2\}\left(12\tanh\left(\frac{\gamma T}{4}\right) - \gamma T\left\{1+2\left[\text{sech}\left(\frac{\gamma T}{4}\right)\right]^2\right\}\right). \tag{71}$$

In the next subsection, we will compare these analytical results for the averaged output frequency and phase diffusion with results obtained from a numerical solution of the stochastic differential equation (1).

Comparison with numerical results

Following the algorithm developed by Greenside and Helfand [27,28] (consult also the Appendix in Ref. [29]), we have integrated Eq. (1) for a large number of noise realizations, M, starting from one of the minima $q_{\alpha_0}(0)$. From the initial instant of time, which we set equal to zero, we start monitoring the switches of states and recording the instants of time at which those switches occur, according to Eq. (4). We will denote by $t_{n,i}^{\alpha_0,0}$ the instant of time of the nth switch of state in the ith trajectory. From the switching times $t_{n,i}^{\alpha_0,0}$, the realization of $N^{\alpha_0,0}(t)$ corresponding to the ith trajectory, $N_i^{\alpha_0,0}(t)$, can be easily calculated using Eq. (5), and the corresponding realization of the discrete phase, $\varphi_i^{\alpha_0,0}(t)$, by Eq. (7). The noise-averaged phase is then obtained by

$$\langle\varphi^{\alpha_0,0}(t)\rangle = \frac{1}{M}\sum_{i=1}^{M}\varphi_i^{\alpha_0,0}(t), \tag{72}$$

and the phase variance by

$$\upsilon^{\alpha_0,0}(t) = \langle[\varphi^{\alpha_0,0}(t)]^2\rangle - \langle\varphi^{\alpha_0,0}(t)\rangle^2 = \frac{1}{M}\sum_{i=1}^{M}[\varphi_i^{\alpha_0,0}(t)]^2 - \frac{1}{M^2}\left[\sum_{i=1}^{M}\varphi_i^{\alpha_0,0}(t)\right]^2. \tag{73}$$

After a sufficiently long number of periods L for the system to "forget" the initial preparation, the averaged output frequency is calculated from the expression

$$\Omega_{\text{out}} = \frac{\langle\varphi^{\alpha_0,0}[(L+1)T]\rangle - \langle\varphi^{\alpha_0,0}(LT)\rangle}{T}, \tag{74}$$

and the averaged phase diffusion from

$$D_{\text{out}} = \frac{\upsilon^{\alpha_0,0}[(L+1)T] - \upsilon^{\alpha_0,0}(LT)}{T}. \tag{75}$$

Figures 2–4 show the results of the numerical solution just described for a periodic rectangular input signal with

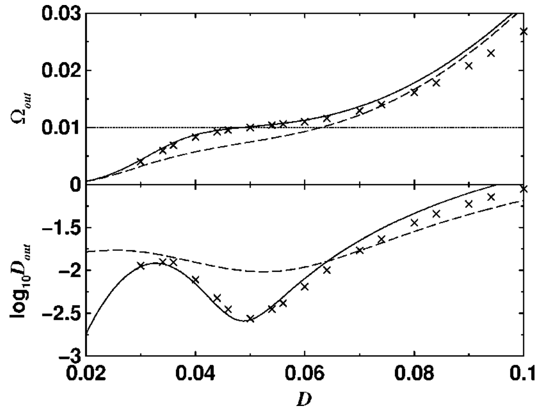

FIG. 2. Averaged output frequency (upper panel) and averaged phase diffusion (lower panel) as a function of the noise strength D for a periodic rectangular input signal [see Eq. (65) and text below] with amplitude $A=0.14$ and angular frequency $\Omega=2\pi/T=0.01$. Solid line: Analytical results obtained from Eq. (70) (upper panel) and Eq. (71) (lower panel), respectively. Dashed line: Theoretical result from Ref. [18]. Crosses: Precise numerical results. In the upper panel, a horizontal dotted line indicates the frequency of the input signal.

angular frequency $\Omega=2\pi/T=0.01$ and three different values of the driving amplitude: $A=0.14$ (see Fig. 2), $A=0.25$ (see Fig. 3), and $A=0.3$ (see Fig. 4). As reported previously in Ref. [18] and observed experimentally in Ref. [30], for high enough, but still subthreshold, driving amplitudes, the present system exhibits a noise-induced frequency locking, i.e., starting from a nonzero value of the noise strength D, the frequency of the output signal matches the frequency of the input signal, until, for strong noise, the output signal becomes desynchronized again. This effect is accompanied by a very pronounced suppression of the phase diffusion of the output signal, i.e., a noise-induced phase locking. For the relevant values of the noise strength D, our analytical estimates Eq. (70) and Eq. (71) agree very well with the results obtained from the numerical solution. Only for rather strong noise can a noticeable deviation be observed. In this regime, the Kramers rates (56) are no longer valid. We have also plotted the results of the previous work [18], using the rates (56). We note that while the improvement of our analytical

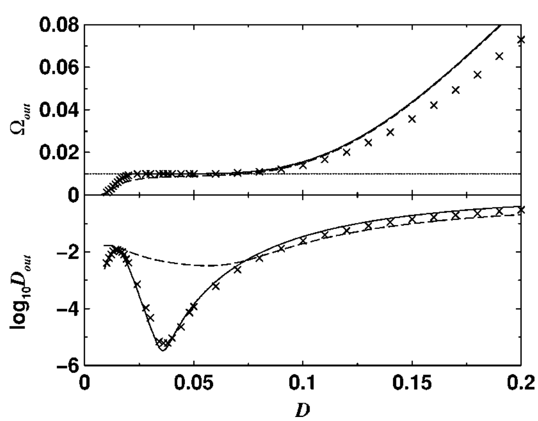

FIG. 3. Like Fig. 2 but for a driving amplitude $A=0.25$.

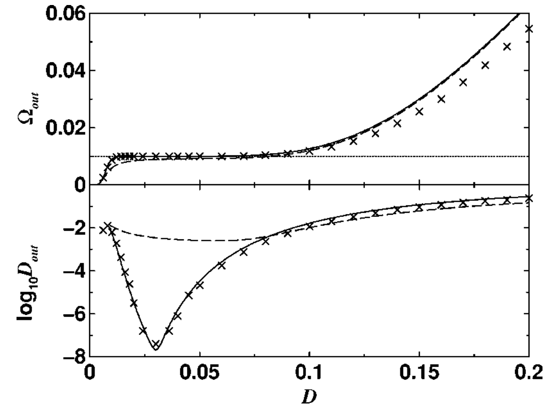

FIG. 4. Like Fig. 2 but for a driving amplitude $A=0.30$.

estimates for the frequency synchronization is moderate only, our prediction for the phase diffusion is strongly improved.

VI. CONCLUSIONS

With this work, we have investigated in detail the phenomenon of frequency and phase synchronization in bistable, periodically driven stochastic systems. This objective is not only of foremost interest for the well known phenomenon of stochastic resonance [9,10] and the topic of rocked Brownian motors [12], but also carries great potential for the study of driven stochastic neuronal dynamics and driven excitable sytems per se [31]. Our approach takes a new look at this prominent problem. Starting out from a driven, Markovian continuous dynamics, we derived in great detail the stochastic renewal dynamics of the noise-induced switching events. This contraction of the full Markovian dynamics in state space onto the discrete counting process of subsequent switches between the metastable states implies a non-Markovian dynamics for the switching times and the corresponding phase dynamics whose explicit time evolution depends on initial preparation effects. The resulting non-Markovian expressions still contain the full information of the driven dynamics in the relevant state space and thus are not readily accessible for analytical estimates.

In contrast, for weak noise and slow external driving, the dynamics of the underlying process simplifies considerably. Consequently, in the long time limit the phase dynamics now assumes again a Markovian nature. In this regime, we put forward new results for the phase diffusion and the frequency synchronization. In doing so, we have employed rectangular-shaped periodic driving signals. This choice entails two distinct advantages, namely (i) it allows a convenient analytic analysis of the corresponding synchronization quantities and (ii) its two-state character is also known to optimize the efficiency for the synchronization features. The same optimization feature holds true for related effects such as the achievements of optimizing the gain for stochastic resonance [32] or the enhancements of energy transduction in driven chemical reactions [33].

Our analysis is in the spirit of prior works [18,19]; our novel estimates, however, quantitatively supersede in accuracy those prior results, cf. the detailed comparison performed above. While the improvement for the frequency synchronization is moderate only, the novel estimates present a sizable improvement for the role of the phase diffusion. In the weak-noise regime and for slow external driving, the Markovian theory provides a very good agreement with numerical precise simulations. This being so, we are confident that the new insight gained into the complexity of non-Markovian, driven switching time dynamics together with its Markovian simplification obtained at weak noise and slow driving will prove useful for modeling and interpreting stochastic synchronization phenomena in driven metastable and excitable dynamics.

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