

# Fully Analytical Circuit-like Approach for the TE Scattering by Narrow-Slit Printed Gratings

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**Abstract**—A fully analytical solution for the scattering of a uniform TE polarized plane wave by an infinite periodic printed array of narrow slits is provided in this contribution. The model is based on some simplifications made on the formulas arising from a rigorous full-wave formulation. The present approach gives the topology of the transverse equivalent network as well as closed-form expressions for all its components, and overall very accurate results.

## I. INTRODUCTION

The study of the scattering by periodic gratings has been a topic of intense research since long [1]–[3]. The specific case of metallic gratings has received considerable attention, being a notably interesting contribution to the modeling of this phenomenon the proposal of equivalent transmission-line circuits that account in a simple manner for the behavior of the reflected and transmitted fields. In the very general case, only very complicated networks can be obtained [4]. However, in some cases, it is possible to find simple equivalent circuits that model very accurately the original structure [5], [6]. In this paper we present a transverse equivalent network to model the scattering of an obliquely incident TE-polarized plane wave impinging on a periodic narrow-slit grating printed on dielectric slabs. Closed-form expressions are obtained for all the circuit parameters. Surprisingly good accuracy is verified by comparison with method-of-moments results.

## II. FORMULATION

Consider the situation shown in Fig.1, where a TE-polarized oblique incident plane wave impinges on a periodic grating. The tangential electric field at both sides of the screen can be written in terms of the following Floquet-series expansion:

$$E(y) = \left\{ 1 + R + \sum_{n=1}^{\infty} [E_n e^{-jk_n y} + E_{-n} e^{jk_n y}] \right\} e^{-jk_t y} \quad (1)$$

where the amplitude of the incident field has been normalized to unity,  $R$  is the complex reflection coefficient, and

$$k_n = \frac{2\pi n}{p} \quad (2)$$

$$k_t = k_0 \sin \theta_{\text{inc}}. \quad (3)$$

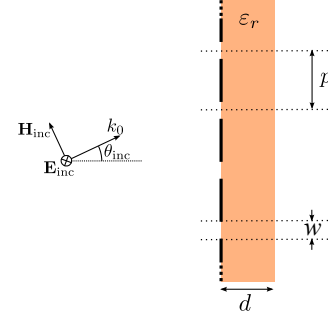


Fig. 1. A TE-polarized incident plane wave impinges obliquely on a periodic slit grating printed on a dielectric slab.

The magnetic field at both sides of the screen is given by

$$H^{\text{left}}(y) = \left\{ Y_0^{(1)}(1 - R) - \sum_{n=1}^{\infty} [Y_n^{(1)} E_n e^{-jk_n y} + Y_{-n}^{(1)} E_{-n} e^{jk_n y}] \right\} e^{-jk_t y} \quad (4)$$

$$H^{\text{right}}(y) = \left\{ Y_0^{(\text{in})}(1 + R) + \sum_{n=1}^{\infty} [Y_n^{(\text{in})} E_n e^{-jk_n y} + Y_{-n}^{(\text{in})} E_{-n} e^{jk_n y}] \right\} e^{-jk_t y} \quad (5)$$

where  $Y_n^{(\varepsilon_r)}$  is the wave admittance of the  $n$ -th spatial harmonic inside a dielectric medium with relative permittivity  $\varepsilon_r$ , given by

$$Y_n^{(\varepsilon_r)} = \eta_0^{-1} \frac{\beta_n^{(\varepsilon_r)}}{k_0} \quad (6)$$

with  $\eta_0 = \sqrt{\mu_0/\varepsilon_0}$  being the free-space impedance and

$$\beta_{\pm n}^{(\varepsilon_r)} = \sqrt{\varepsilon_r k_0^2 - (k_t \pm k_n)^2} \quad (7)$$

the wavenumber along  $z$ . In (4) we actually have free space,  $\varepsilon_r = 1$ , and therefore  $Y_n^{(\varepsilon_r)} \equiv Y_n^{(1)}$ . The  $Y_n^{(\text{in})}$  admittances in (5) represent the input wave admittances to the dielectric slab seen from the right hand side of the screen, and can therefore be written as

$$Y_n^{(\text{in})} = Y_n^{(\varepsilon_r)} \frac{Y_n^{(1)} + jY_n^{(\varepsilon_r)} \tan\left(\beta_n^{(\varepsilon_r)} d\right)}{Y_n^{(\varepsilon_r)} + jY_n^{(1)} \tan\left(\beta_n^{(\varepsilon_r)} d\right)}. \quad (8)$$

Now we make the assumption that the tangential electric field in the aperture (slit) can be expressed as

$$E_a(y) = A f(y) \quad (9)$$

where  $f(y)$  is a normalized field profile. The  $E_n$  coefficients can be given in terms of the aperture field using the following identity:

$$\int_{-p/2}^{p/2} E(y) e^{j(k_t \pm k_n)y} dy = \int_{-w/2}^{w/2} E_a(y) e^{j(k_t \pm k_n)y} dy. \quad (10)$$

Thus, introducing (1) in (10) we obtain

$$(1 + R)p = A f_0 w \quad (11)$$

$$E_n p = A w f_n \quad (12)$$

with

$$f_{\pm n} = \frac{1}{w} \int_{-w/2}^{w/2} f(x) e^{j(k_t \pm k_n)y} dy, \quad (13)$$

and therefore

$$A = \frac{1 + R}{f_0} \frac{p}{w} \quad (14)$$

$$E_n = (1 + R) \frac{f_n}{f_0}. \quad (15)$$

Since the magnetic field is continuous through the slit, we can write

$$H^{\text{left}}(y) = H^{\text{right}}(y) \quad \text{for } |y| < w/2. \quad (16)$$

Following a Galerking-like testing scheme, we can project (16) over the aperture field profile to obtain the following equation:

$$\int_{-w/2}^{w/2} f(y) [H^{\text{left}}(y) - H^{\text{right}}(y)] dy = 0. \quad (17)$$

Substituting (4) and (5) into (17) and taking into account (15), we obtain

$$R = \frac{Y_0^{(1)} - Y_0^{(\text{in})} - f_0^{-2} \sum_{n \neq 0} (Y_n^{(1)} + Y_n^{(\text{in})}) f_n^2}{Y_0^{(1)} + Y_0^{(\text{in})} + f_0^{-2} \sum_{n \neq 0} (Y_n^{(1)} + Y_n^{(\text{in})}) f_n^2} \quad (18)$$

where the sum extends over all non-zero integers (positive and negative harmonics). Looking at (18) we can see that the effect of the slit grating can be accounted for by the following equivalent admittance:

$$Y_{\text{eq}} = f_0^{-2} \sum_{n \neq 0} (Y_n^{(1)} + Y_n^{(\text{in})}) f_n^2. \quad (19)$$

For narrow slits (in practice for  $w/\lambda_0 \lesssim 0.2$ ), the following field profile is expected to give a good approximation of the aperture field:

$$f(y) = \frac{4}{\pi} \sqrt{1 - \left(\frac{2y}{w}\right)^2}. \quad (20)$$

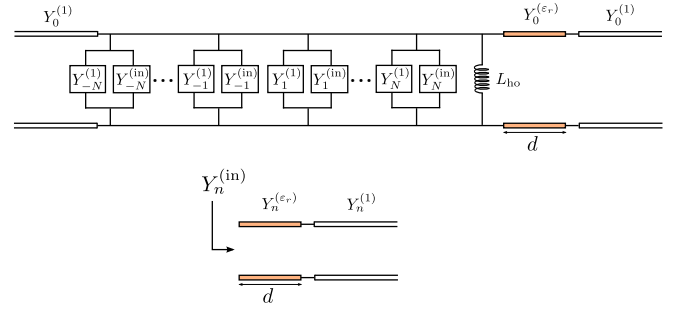


Fig. 2. Equivalent network obtained for the scattering of a TE-polarized incident plane wave impinging obliquely on a narrow-slit periodic grating.

This profile implies that the coefficients  $f_{\pm n}$  are given by

$$f_{\pm n} = 2 \frac{J_1\left((k_t \pm k_n) \frac{w}{2}\right)}{(k_t \pm k_n) \frac{w}{2}} \quad (21)$$

where  $J_1(\cdot)$  is the Bessel function of the first kind and order 1. For higher-order harmonics that are well below cutoff, we can make the following approximations:

$$k_t \pm k_n \approx \pm k_n \quad (22)$$

$$\beta_n^{(\epsilon_r)} \approx -jk_n \quad (23)$$

$$Y_n^{(\epsilon_r)} \approx \frac{1}{j\omega} \frac{ck_n}{\eta_0} \quad (24)$$

with  $c$  being the speed of light in vacuum. In (24) we observe that the admittances account for frequency-independent inductances given by  $L_n = \eta_0/(ck_n)$ . Note that the  $L_n$  inductances do not depend on the permittivity. The input admittances  $Y_n^{(\text{in})}$  can as well be approximated by

$$Y_n^{(\text{in})} \approx Y_n^{(\epsilon_r)} \approx \frac{1}{j\omega L_n} \quad (25)$$

and, therefore, the equivalent admittance defined in (19) can be written as

$$Y_{\text{eq}} \approx \sum_{\substack{n=-N \\ n \neq 0}}^N A_n (Y_n^{(1)} + Y_n^{(\text{in})}) + \frac{1}{j\omega L_{\text{ho}}} \quad (26)$$

where  $A_n = (f_n/f_0)^2$  [ $f_0$  and  $f_{\pm n}$  are given by (21)],  $N$  is the number of positive or negative harmonics (the larger of the two) that are propagative or almost propagative inside the dielectric, and

$$\frac{1}{L_{\text{ho}}} = \frac{8c}{w\eta_0} \left[ \frac{J_1(k_t w/2)}{k_t w/2} \right]^{-2} \sum_{n=N+1}^{\infty} \frac{J_1^2(k_n w/2)}{k_n w/2}. \quad (27)$$

The above series does not depend on frequency and therefore it has to be computed only once.

Taking now into account the final form of  $Y_{\text{eq}}$  in (26), we observe that the narrow-slit periodic grating can be modelled by the equivalent circuit depicted in Fig.2, where all the elements are known in closed form.

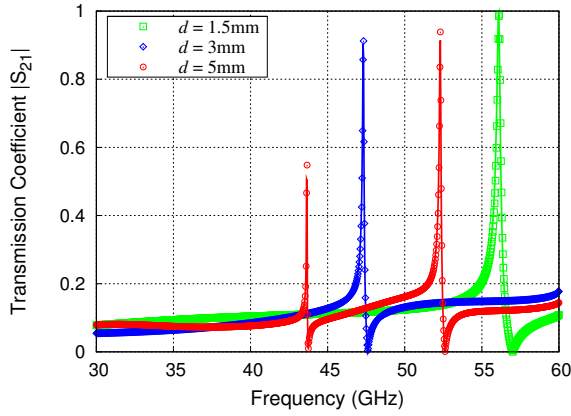


Fig. 3. Full-wave MoM (solid lines) and analytical (symbols) results for a printed grating with  $p = 5$  mm,  $w = 1$  mm, and  $\epsilon_r = 2.2$ .  $\theta_{\text{inc}} = 0$  and  $N = 2$  in Eq.(26).

### III. RESULTS

In this section we present some examples of application of the above discussed equivalent-circuit approach to characterize the scattering of a TE-polarized wave impinging on different narrow-slit periodic gratings. First, in Fig. 3 we analyze the case of a slit grating printed on a low-permittivity substrate ( $\epsilon_r = 2.2$ ) for various thicknesses of the substrate. In the figure we compare our fully analytical results (symbols) with those computed using a full-wave Method-of-Moment technique with multiple basis functions (solid lines). For the thinnest substrate case ( $d = 1.5$  mm) in Fig. 3, we can see that there is a peak of total transmission followed by a transmission null, and that this Fano-type resonance [3] is also very accurately captured by the analytical (circuit-model) results. For thicker slabs, there appear different Fano-type resonances, which are again in complete agreement with the fully analytical results.

In the second example shown in Fig. 4 we consider the case of a grating printed on a electrically thick dielectric slab (the relative permittivity of the substrate is now 10.2). In this case the transmission spectra shown in Fig. 4 is much more complicated than in the low-permittivity cases studied in Fig. 3. However, despite the complexity of the transmission spectra (multiple Fano-type resonances, maxima of partial transmission, etc), the agreement between the full-wave MoM results and the analytical ones is surprisingly good.

As a final example, we show in Fig. 5 the transmission spectra for the case of the thickest slab previously analyzed in Fig. 3, but now under oblique incidence and including frequencies in the grating-lobe regime (this regime starts at  $60/(1 + \sin \theta_{\text{inc}})$  GHz for the considered structure). Again, we can see an excellent agreement between the full-wave MoM results and our analytical data.

In summary, we have presented a fully analytical circuit-like approach to study the TE scattering of a plane wave impinging obliquely on a narrow-slit periodic grating printed on a dielectric slab. Our analytical results have proved to be very accurate when compared with full-wave numerical data, even for very complicated transmission spectra.

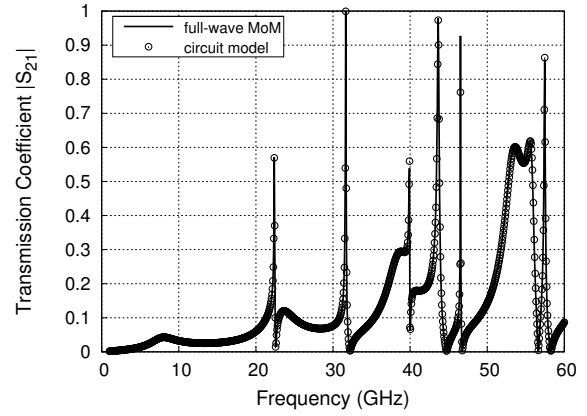


Fig. 4. Full-wave MoM and analytical results for a slit grating ( $p = 5$  mm,  $w = 1$  mm) printed on a electrically thick dielectric slab ( $h = 3$  mm and  $\epsilon_r = 10.2$ ).  $\theta_{\text{inc}} = 0$  and  $N = 4$  in Eq.(26).

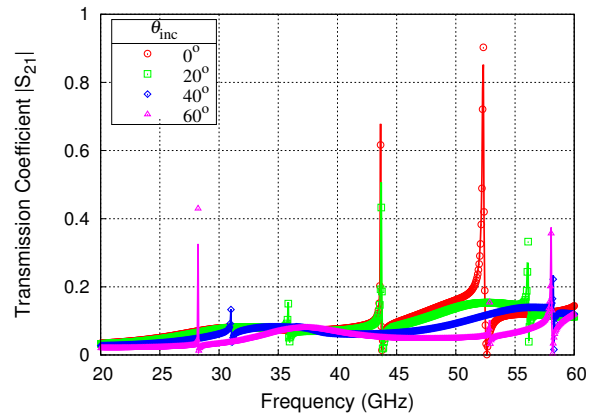


Fig. 5. Full-wave MoM (solid lines) and analytical results (symbols) for a slit grating printed on a dielectric slab under oblique incidence ( $p = 5$  mm,  $w = 1$  mm,  $d = 5$  mm,  $\epsilon_r = 2.2$  and  $N = 2$  in Eq.(26)).

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