Global stability and H theorem in lattice models with nonconservative interactions

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In kinetic theory, a system is usually described by its one-particle distribution function $f(\boldsymbol{r},\boldsymbol{v},t)$, such that $f(\boldsymbol{r},\boldsymbol{v},t)d\boldsymbol{r}\,d\boldsymbol{v}$ is the fraction of particles with positions and velocities in the intervals $(\boldsymbol{r},\boldsymbol{r}+d\boldsymbol{r})$ and $(\boldsymbol{v},\boldsymbol{v}+d\boldsymbol{v})$, respectively. Therein, global stability and the possible existence of an associated Lyapunov function or H theorem are open problems when nonconservative interactions are present, as in granular fluids. Here, we address this issue in the framework of a lattice model for granularlike velocity fields. For a quite general driving mechanism, including both boundary and bulk driving, we show that the steady state reached by the system in the long-time limit is globally stable. This is done by proving analytically that a certain H functional is nonincreasing in the long-time limit. Moreover, for a quite general energy injection mechanism, we are able to demonstrate that the proposed H functional is nonincreasing for all times. Also, we put forward a proof that clearly illustrates why the "classical" Boltzmann functional $H_B[f] = \int d\boldsymbol{r}\,d\boldsymbol{v}\,f(\boldsymbol{r},\boldsymbol{v},t)\ln f(\boldsymbol{r},\boldsymbol{v},t)$ is inadequate for systems with nonconservative interactions. This is done not only for the simplified kinetic description that holds in the lattice models analyzed here but also for a general kinetic equation, like Boltzmann's or Enskog's.

I. INTRODUCTION

In thermodynamics and statistical mechanics, global stability of the equilibrium state is usually proven by introducing a Lyapunov functional [1]. This Lyapunov functional of the probability distribution function (PDF) has the following three properties: (i) it is bounded from below, (ii) it monotonically decreases with time, and (iii) its time derivative equals 0 only when the PDF is the equilibrium one. Therefore, in the long-time limit, the Lyapunov functional must tend to a finite value and thus its time derivative vanishes. As a consequence, any PDF, corresponding to an arbitrary initial preparation, tends to the equilibrium PDF: the equilibrium state is irreversibly approached and said to be globally stable.

The first example of such a Lyapunov functional is the renowned Boltzmann H functional. In the Boltzmann description, the nonequilibrium behavior of a dilute gas is completely encoded in the one-particle velocity distribution function $f(\boldsymbol{r},\boldsymbol{v},t)$. By introducing the Stosszahlansatz or molecular chaos hypothesis, Boltzmann derived a closed nonlinear integrodifferential equation for $f(\boldsymbol{r},\boldsymbol{v},t)$ governing its time evolution [2]. Also, for a spatially homogeneous state, he showed that the functional $H_B[f] = \int d\boldsymbol{v}\,f(\boldsymbol{v},t)\ln f(\boldsymbol{v},t)$ has the three properties of a Lyapunov functional. This H theorem shows that all solutions of the Boltzmann equation tend in the long-time limit to the Maxwell velocity distribution and irreversibility naturally stems from a molecular picture [3,4]. Interestingly, a key point for deriving the H theorem is the reversibility of the underlying microscopic dynamics. In an inhomogeneous situation, one has to consider the spatial dependence of the one-particle distribution function $f(\boldsymbol{r},\boldsymbol{v},t)$, and the above functional must be generalized to

$$H_B[f] = \int d\boldsymbol{r}\,d\boldsymbol{v}\,f(\boldsymbol{r},\boldsymbol{v},t)\ln f(\boldsymbol{r},\boldsymbol{v},t). \tag{1}$$

Provided that the walls of the gas container are smooth, in the sense that there is no energy transport through them, it can be also shown that this is a nonincreasing Lyapunov functional [5].

Another example of a Lyapunov functional can be found in the realm of Markovian stochastic processes. Therein, the stochastic process $X(t)$ is completely determined by the conditional probability density $P_{1|1}(X,t|X_0,t_0)$ of finding the system in state X at time t, given that it was in state X_0 at time t_0, and the probability density $P(X,t)$ of finding the system in state X at time t [6]. Both probability densities satisfy the same evolution equation, called the master equation, but with different initial conditions: one always has that $P_{1|1}(X,t_0|X_0,t_0) = \delta(X - X_0)$, whereas $P(X,t_0) = P_{\text{ini}}(X)$, with $P_{\text{ini}}(X)$ corresponding to the (arbitrary) initial preparation. When the stochastic process is irreducible or ergodic, that is, every state can be reached from any other state by a chain of transitions with nonzero probability, there is only one stationary solution of the master equation. In physical systems, this steady solution must correspond to the equilibrium-statistical-mechanics distribution $P_{\text{eq}}(X)$. What is more, a Lyapunov functional can be constructed as

$$\mathcal{H}[P] = \int dX\,P_{\text{eq}}(X)\,g\!\left[\frac{P(X,t)}{P_{\text{eq}}(X)}\right], \tag{2}$$

where $g(x)$ is any positive-definite convex function ($g''(x) \geqslant 0$). It must be stressed that the proof of this H theorem for master equations rely only on the ergodicity of the underlying microscopic dynamics: it is not necessary to assume that detailed balance, which is connected with the microscopic reversibility, holds [6].

The most usual choice for g is $g(x) = x\ln x - x + 1$, which leads to

$$\mathcal{H}[P] = \int dX\,P(X,t)\,\ln\!\left[\frac{P(X,t)}{P_{\text{eq}}(X)}\right]. \tag{3}$$

The physical reason behind this choice is the "extensiveness" of $\mathcal{H}[P]$: if the system at hand comprises two independent

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subsystems A and B, so that $dX \equiv dX_A dX_B$ and $P(X) = P_A(X_A)P_B(X_B)$, one has that $\mathcal{H}[P] = \mathcal{H}_A[P_A] + \mathcal{H}_B[P_B]$. It is to consider $-\mathcal{H}$ as a nonequilibrium entropy S that this extensiveness is desirable: in this way, the nonincreasing behavior of $\mathcal{H}$ leads to a nondecreasing time evolution of S. Moreover, $\mathcal{H}[P]$ remains invariant upon a change of variables $Y = f(X)$, as emphasized in Refs. [7,8].

Although the Boltzmann equation is not a master equation, we may wonder why the expressions for H_B in Eq. (1) and $\mathcal{H}[P]$ in Eq. (3) are different. Specifically, we may wonder, Why not write

$$H[f] = \int d\boldsymbol{r}\, d\boldsymbol{v}\, f(\boldsymbol{r},\boldsymbol{v},t) \ln\left[\frac{f(\boldsymbol{r},\boldsymbol{v},t)}{f_{\mathrm{eq}}(\boldsymbol{v})}\right] \qquad (4)$$

instead of $H_B[f]$? Up to now, we have been implicitly considering the "classic" problem with elastic collisions between particles, in which the system eventually reaches thermodynamic equilibrium. Therein, the answer is trivial: since $\ln f_{\mathrm{eq}}(\boldsymbol{v})$ is a sum of constants of motion, $H[f] - H_B[f]$ is constant and both are utterly equivalent.

Whether or not there exists an extensive H functional is an important question in nonequilibrium statistical physics. If the answer were positive, it would make it possible to define a nonequilibrium entropy $-H$ that monotonically increases for all times, extending the Clausius inequality. In general, the system at hand does not reach equilibrium but a nonequilibrium steady state. Thus, the equilibrium distribution f_{eq} in H has to be substituted with the stationary one f_{st}. In this context, the field of granular fluids is a benchmark for intrinsically out-of-equilibrium, dissipative systems: the microscopic dynamics is not time reversible because collisions between particles are inelastic, but a nonequilibrium steady state can be attained if some driving mechanism injects energy into the system.

In granular fluids, the functionals $H[f]$ and $H_B[f]$ are no longer equivalent, since $\ln f_{\mathrm{st}}$ is not a sum of constants of motion. Indeed, for granular gases described by the inelastic Boltzmann equation [9,10], there are some results that hint at H_B not being a Lyapunov functional. Within the first Sonine approximation, it has been proven that the time derivative of H_B does not have a definite sign in the linear approximation around the steady state [11]. Moreover, Marconi *et al.* have numerically shown that H_B is nonmonotonic and even steadily increases from certain initial conditions [7]. They have also put forward some numerical evidence (further reinforced by García de Soria *et al.* [8]) in favor of H being a "good" Lyapunov functional. Notwithstanding, only spatially homogeneous situations, in which the $\boldsymbol{r}$ dependence of f and thus the integration over $\boldsymbol{r}$ may be dropped, are analyzed in Refs. [7,8].

Some years ago, a simplified model for a granularlike velocity field was introduced to study correlations in granular gases [12]. Very recently, a variant of this model on a one-dimensional (1D) lattice has been proposed to mimic the velocity component along the shear direction [13], and both its hydrodynamic limit and finite-size effects have been analyzed [13–15]. This model has been shown to retain a relevant part of the granular phenomenology: the shear instability of the homogeneous cooling state, the existence

of boundary-driven steady states such as the Couette and uniform shear flow (USF) states, the renormalization of the cooling rate due to fluctuations close to the shear instability, etc. Other properties thereof, when it is driven by a mechanism resembling collisions with a randomly moving inelastic wall, have been studied in Ref. [16]. At the N-particle level, the dynamics of the system is governed by a master equation, which is analogous to the Kac equation [17], which leads to a "kinetic" equation at the one-particle level, which is analogous to the Boltzmann equation. In the latter, the collision term, although being simpler than that in the Boltzmann equation, remains a nonlinear integrodifferential one [14].

It must be recalled that an analytical proof of either global stability or the H theorem is currently unavailable at the level of the kinetic description for granular gases. This is true even for simple collision terms, such as those corresponding to hard spheres or the cruder Maxwell particle model (where the collision rate is considered to be velocity independent), which are considered in the pioneering work in Refs. [7,8]. Therefore, it seems worthwhile to investigate this subject in simplified models, for which analytical calculations are more feasible.

Our main goal here is to investigate the global stability and the possibly associated H theorem in the above class of lattice models. Unlike the approach in Refs. [7,8], we do not restrict ourselves to spatially homogeneous situations but consider the whole space and velocity dependence of the one-particle PDF $f(\boldsymbol{r},\boldsymbol{v},t)$. Specifically, we introduce a general energy injection mechanism, in which the system may be driven both through the boundaries and in the bulk. We show that, under quite general conditions, the steady state is globally stable: independently of the initial preparation, the system always ends up in the steady state. Interestingly, it is not necessary to have an H theorem to prove this: it suffices to show that H is decreasing in the long-time limit, not for all times. In this sense, the situation is analogous to the proof of the tendency towards the equilibrium curve in systems whose dynamics is governed by master equations with time-dependent transition rates [18–24].

Our proof of global stability also enables us to show the inadequacy of Boltzmann's H_B as a candidate for a Lyapunov functional in inelastic systems. This is done not only for the simplified models considered in the paper, but for a general collision term that does not conserve energy in collisions. Therefore, this result also applies to the inelastic Boltzmann or Enskog equations used for granular fluids. The main idea is that the sign of dH_B/dt can be reversed by a suitable choice of the initial PDF and, thus, cannot have a definite sign. In this respect, our result generalizes that in Ref. [11], which was derived within the first Sonine approximation of the inelastic Boltzmann equation, to an arbitrary collision kernel with nonconservative interactions.

Having proved global stability by showing that H is a nonincreasing functional for long times, a natural question remains. Is H a Lyapunov function, that is, a nonincreasing functional for all times? There does not seem to be a unique proof, valid for any driving mechanism, even within our simplified model. Nevertheless, we have been able to derive a specific proof for a quite general driving mechanism, which includes as limiting cases both the sheared system, in

which the steady state is the USF state, and the uniformly heated system, heated by means of the so-called stochastic thermostat [25–33]. The proof is based on a suitable expansion of the one-particle PDF in Hermite polynomials, which is a generalization of the usual Sonine expansion of kinetic theory.

The paper is organized as follows. In Sec. II, we briefly introduce the model, its dynamics, and the continuum limit. Section III is devoted to the proof of the global stability of the nonequilibrium steady states, for a general energy injection mechanism. The inadequacy of Boltzmann's H_B as a Lyapunov functional for inelastic systems is discussed in Sec. IV. Later, in Sec. V, we consider some concrete physical situations in our model, which include the sheared and the uniformly heated systems. Therein, we show that $H[f]$ is a monotonically decreasing Lyapunov functional. Finally, Sec. VI gives the main conclusions of the paper. Some technical details, which are omitted in the text, are given in the Appendixes.

II. THE MODEL: DYNAMICS AND CONTINUUM LIMIT

Here, we present the general class of models that was introduced in Ref. [13], focusing on the continuum description obtained in the large-system-size limit [14]. Specifically, our system is defined on a 1D lattice: at each lattice site l, there is a particle with velocity v_l. Thus, at a given time τ, the configuration of the system is completely determined by $\boldsymbol{v} \equiv \{v_1, \ldots, v_N\}$. The dynamics proceeds through inelastic nearest-neighbor binary collisions: each pair $(l, l+1)$ collides inelastically with a characteristic rate ω^{-1}, independently of their relative velocity (the so-called Maxwell-molecule model [34]) and the state of the other pairs. We introduce the operator $\hat{b}_l$ that transforms the precollisional velocities into the postcollisional ones,

$$\hat{b}_l v_l = v_l - \frac{1+\alpha}{2}(v_l - v_{l+1}), \quad (5a)$$

$$\hat{b}_l v_{l+1} = v_{l+1} + \frac{1+\alpha}{2}(v_l - v_{l+1}), \quad (5b)$$

where α is the normal restitution coefficient, with $0 < \alpha \leqslant 1$.

In addition to collisions, the system is heated by a stochastic force that is modeled by a white noise, the so-called *stochastic thermostat* [25–33]. Specifically, for a short time interval, the change in the velocity due to the heating is given by

$$\Delta v_i(\tau)|_{\text{noise}} \equiv v_i(\tau + \Delta\tau) - v_i(\tau)|_{\text{noise}}$$
$$= \left(\xi_i(\tau) - \frac{1}{N} \sum_{j=1}^{N} \xi_j(\tau) \right) \Delta\tau, \quad (6)$$

where $\xi_i(t)$ are Gaussian white noises, verifying

$$\langle \xi_i(\tau) \rangle_{\text{noise}} = 0, \quad \langle \xi_i(\tau)\xi_j(\tau') \rangle_{\text{noise}} = \chi \delta_{ij} \delta(\tau - \tau') \quad (7)$$

for $i, j = 1, \ldots, N$. Above, χ is the amplitude of the noise, and $\langle \cdots \rangle_{\text{noise}}$ denotes the average over the different realizations of the noise. Note that this version of the stochastic thermostat conserves the total momentum, a necessary condition to have a steady state [30,33].

We define $P_N(\boldsymbol{v}, \tau)$ as the probability density of finding the system in state $\boldsymbol{v}$ at time τ. The stochastic process $\boldsymbol{v}(\tau)$ is Markovian and the equation governing the time evolution of $P_N(\boldsymbol{v}, \tau)$ has two contributions. First, we have a master equation contribution stemming from collisions [14,15],

$$\partial_\tau P_N(\boldsymbol{v}, \tau)|_{\text{coll}} = \omega \sum_{l=1}^{N} \left[\frac{P_N(\hat{b}_l^{-1}\boldsymbol{v}, \tau)}{\alpha} - P_N(\boldsymbol{v}, \tau) \right], \quad (8)$$

in which the operator $\hat{b}_l^{-1}$ is the inverse of $\hat{b}_l$; that is, it changes the postcollisional velocities into the precollisional ones when the colliding pair is $(l, l+1)$. Second, there is a Fokker-Planck contribution stemming from the stochastic forcing [7,8,35]:

$$\partial_\tau P_N(\boldsymbol{v}, \tau)|_{\text{noise}} = \frac{\chi}{2} \sum_{i,j=1}^{N} \left(\delta_{ij} - \frac{1}{N} \right) \frac{\partial^2}{\partial v_i \partial v_j} P_N(\boldsymbol{v}, \tau). \quad (9)$$

The time evolution of $P_N(\boldsymbol{v}, \tau)$ is obtained by combining Eqs. (8) and (9), that is,

$$\partial_\tau P_N(\boldsymbol{v}, \tau) = \partial_\tau P_N(\boldsymbol{v}, \tau)|_{\text{coll}} + \partial_\tau P_N(\boldsymbol{v}, \tau)\big|_{\text{noise}}. \quad (10)$$

In this work, we focus on the evolution of quantities that can be written in terms of the one-particle distribution function, namely,

$$P_1(v; l, \tau) = \int d\boldsymbol{v} \, P_N(\boldsymbol{v}, \tau) \delta(v_l - v). \quad (11)$$

All the one-site velocity moments can be calculated from P_1,

$$\langle v_l^n(\tau) \rangle \equiv \int_{-\infty}^{+\infty} dv \, v^n P_1(v; l, \tau). \quad (12)$$

The first two moments give the hydrodynamic fields: the average velocity $u_l(\tau)$ and granular temperature $T_l(\tau)$ [36], which are defined by the relations

$$u_l(\tau) \equiv \langle v_l \rangle, \quad T_l(\tau) \equiv \langle v_l^2(\tau) \rangle - u_l^2(\tau). \quad (13)$$

Here, we do not write the evolution equations on the lattice for either P_1 or the hydrodynamic fields (u and T), since they are not necessary for our present purposes. The unforced case ($\chi = 0$) can be found in Ref. [14]. However, we would like to stress that the evolution equation for P_1 is not closed, since the collision term involves the two-particle distribution function $P_2(v, v'; l, l+1, \tau)$. As usual in kinetic theory, one can write a closed equation for P_1 after introducing the molecular chaos assumption, that is, $P_2(v, v'; l, l+1, \tau) = P_1(v; l, \tau)P_1(v'; l+1, \tau) + O(N^{-1})$. In other words, one assumes that the correlations at different sites are of the order of N^{-1} and thus negligible in the large-system-size limit.

The continuum limit of the model is introduced for a large system size $N \gg 1$, in which we expect the average velocity u_l and temperature T_l to be smooth functions of space and time. This is expressed mathematically by defining "hydrodynamic" continuous space and time variables by $x = l/N$ and $t = \omega\tau/N^2$, respectively [14]. Note that $0 \leqslant x \leqslant 1$ and $t \geqslant 0$. In the continuum limit, the one-particle distribution function also becomes a smooth function of x and τ, $P_1(v; x, t) \equiv P_1(v; l = Nx, \tau = N^2t/\omega)$.

From now on, we use the usual notation in kinetic theory, $f(x, v, t) \equiv P_1(v; x, t)$. The physical picture is straightforward: $f(x, v, t) \, dx \, dv$ gives the fraction of the total number of particles

with positions between x and $x + dx$ and velocities between v and $v + dv$. We have that $\int_{-\infty}^{+\infty} dv\, f(x,v,t) = 1$ for all x and t, since there is no mass transport in the system. The time evolution of f is governed by the nonlinear integrodifferential (pseudo-Boltzmann) equation [14]

$$\partial_t f = \partial_x^2 f + \frac{\nu}{2}\partial_v\{[v - u(x,t)]f\} + \frac{\xi}{2}\partial_v^2 f, \quad (14)$$

where $u(x,t)$ is the local average velocity, ν is the macroscopic dissipation coefficient and ξ is the macroscopic noise strength, which are, respectively, given by

$$\nu = (1 - \alpha^2)N^2, \quad \xi = \frac{\chi N^2}{\omega}. \quad (15)$$

This shows that the microscopic noise strength χ must scale as N^{-2} in order to have a finite contribution in the continuum limit. Of course, for $\xi = 0$, we recover the kinetic equation for the case in which there is no stochastic forcing; see Ref. [14]. The N scaling of the macroscopic dissipation coefficient ν is similar to that found in the dissipative version of the Kipnis-Marchioro-Presutti model [37–41].

The average velocity $u(x,t)$ and granular temperature $T(x,t)$ are the continuum limits of u_l and T_l defined in Eq. (13),

$$u(x,t) \equiv \langle v \rangle(x,t), \quad T(x,t) \equiv \langle v^2 \rangle(x,t) - u^2(x,t), \quad (16)$$

where the velocity moments are given by $\langle v^n \rangle(x,t) = \int dv\, v^n f(x,v,t)$. From the kinetic equation for $f(x,v,t)$, one can derive the evolution equations of u and T,

$$\partial_t u = \partial_{xx} u, \quad (17a)$$

$$\partial_t T = -\nu T + \partial_x^2 T + 2(\partial_x u)^2 + \xi. \quad (17b)$$

On the one hand, Eq. (17a) is a diffusion equation for the average velocity, which expresses the conservation of the total momentum. On the other hand, the temperature equation, (17b), contains a purely dissipative term $-\nu T$ that stems from the inelasticity of collisions and always contributes to "cooling" the system, a diffusive term $\partial_x^2 T$, a viscous heating term $2(\partial_x u)^2$, and, finally, the term corresponding to the uniform heating ξ. Of course, either the kinetic equation for f or the average equations for (u,T) must be complemented with suitable boundary conditions in each physical situation.

A. Nonequilibrium steady states and boundary conditions

We are interested in driven cases, in which there is an input of energy that balances (on average) the energy loss in collisions, so that the system eventually reaches a steady state. These nonequilibrium steady states (NESSs) are described by the corresponding stationary solutions $f_{st}(x,v)$ of the kinetic equation, which verify

$$0 = \partial_x^2 f_{st} + \frac{\nu}{2}\partial_v\{[v - u_{st}(x)]f_{st}\} + \frac{\xi}{2}\partial_v^2 f_{st}, \quad (18)$$

where $u_{st}(x) = \int dv\, v f_{st}(x,v)$ is the stationary average velocity profile. To be concrete, we consider two cases: a system that is (a) sheared and (b) uniformly heated.

First, let us consider a sheared system: there is no stochastic forcing, $\xi = 0$, and the driving is introduced by imposing a velocity difference ("shear") between the left and the right

edges of the system. At the level of the hydrodynamic description, the corresponding boundary conditions are

$$u(1,t) = u(0,t) + a, \quad u'(1,t) = u'(0,t), \quad (19a)$$

$$T(1,t) = T(0,t), \quad T'(1,t) = T'(0,t), \quad (19b)$$

which are said to be of the Lees-Edwards type [42]. We have used a prime to denote a spatial derivative. The imposed shear allows the viscous heating term, which is proportional to $(\partial_x u)^2$, to compensate for the energy dissipation term, $-\nu T$. The boundary conditions for the one-particle distribution function read

$$f(1,v,t) = f(0,v-a,t), \quad f'(1,v,t) = f'(0,v-a,t), \quad (20)$$

from which Eq. (19) directly follows. Equation (20) has a simple physical interpretation: particles that leave the system through its right edge at velocity v are reinserted through its left edge at velocity $v - a$.

The steady state for the sheared system is known as the USF state, which has a linear velocity profile and a homogeneous temperature:

$$u_{st}(x) = a\left(x - \frac{1}{2}\right), \quad T_{st} = \frac{2a^2}{\nu}. \quad (21)$$

For our simplified model, the stationary PDF is Gaussian,

$$f_{st}(x,v) = (2\pi T_{st})^{-1/2} \exp\left[-\frac{(v - u_{st}(x))^2}{2T_{st}}\right]. \quad (22)$$

An extensive investigation of the sheared system, at the level of the average hydrodynamic equations, can be found in Ref. [14].

Second, we address the uniformly heated system, in which there is no shear, $a = 0$, but there is stochastic forcing, $\xi \neq 0$. In this case, we have the usual periodic boundary conditions. In particular, for the PDF we have

$$f(1,v,t) = f(0,v,t), \quad f'(1,v,t) = f'(0,v,t). \quad (23)$$

At steady state, the system is homogeneous: there is no average velocity and the temperature is uniform,

$$u_{st}(x) = 0, \quad T_{st} = \frac{\xi}{\nu}. \quad (24)$$

The corresponding stationary PDF is also Gaussian,

$$f_{st}(\not{x},v) = (2\pi T_{st})^{-1/2} \exp\left[-\frac{v^2}{2T_{st}}\right]. \quad (25)$$

With this "stochastic thermostat" forcing, the system remains homogeneous for all times if it is initially so, as is also the case for an inelastic gas of hard particles described by the inelastic Boltzmann equation [25].

III. GLOBAL STABILITY

In this section, we analyze the global stability of the nonequilibrium stationary solutions of the kinetic equation, (14), submitted to quite a general class of boundary conditions. Following the discussion in Sec. I, we define the H functional as

$$H[f] = \int dx\, dv f(x,v,t) \ln\left[\frac{f(x,v,t)}{f_{st}(x,v)}\right]. \quad (26)$$

Let us consider the time evolution of $H[f]$. It is directly obtained that

$$\frac{dH}{dt} = \int dx\,dv\,\partial_t f \ln\left(\frac{f}{f_{st}}\right) = \int dx\,dv\,\mathcal{L}f \ln\left(\frac{f}{f_{st}}\right),$$
(27)

where $\mathcal{L}$ stands for the nonlinear evolution operator on the right-hand side (r.h.s.) of the kinetic equation, (14); that is, $\partial_t f = \mathcal{L}f$. Now we note the following property: if we define $\Delta f = f - f_{st}$ to be the deviation of the PDF from the steady state, the linear terms in the deviation vanishes, since both factors in the integrand of (27) are equal to 0 for $f = f_{st}$. This is a desirable property: were it not true, the sign of dH/dt could be reversed for initial conditions close enough to the steady state by simply reversing the initial value of Δf. Thus, the existence of an H theorem would be utterly impossible (see also the next section).

Then we can write

$$\frac{dH}{dt} = \int dx\,dv\,\mathcal{L}f \ln\left(\frac{f}{f_{st}}\right) - \int dx\,dv\,\mathcal{L}f_{st}\frac{f - f_{st}}{f_{st}}.$$
(28)

Now, the idea is to split the operator $\mathcal{L}$ into the three contributions on the r.h.s. of Eq. (14): first, the diffusive one; second, the one proportional to ν, which is intrinsically dissipative; and third, the one proportional to the noise strength ξ—$\mathcal{L}_{\text{diff}}$, $\mathcal{L}_{\text{inel}}$, and $\mathcal{L}_{\text{noise}}$, respectively. Accordingly, we have that the time derivative of H has three contributions,

$$\frac{dH}{dt} = \left.\frac{dH}{dt}\right|_{\text{diff}} + \left.\frac{dH}{dt}\right|_{\text{inel}} + \left.\frac{dH}{dt}\right|_{\text{noise}},$$
(29)

obtained by inserting into Eq. (28) the relevant part of the evolution operator $\mathcal{L}$. Note that, although $\mathcal{L}f_{st} = 0$, in general $\mathcal{L}_{\text{diff}}f_{st} \neq 0$, $\mathcal{L}_{\text{inel}}f_{st} \neq 0$, and $\mathcal{L}_{\text{noise}}f_{st} \neq 0$.

After some tedious but easy algebra, a summary of which is given in Appendix A, the following expressions are derived. First, for the diffusive term,

$$\left.\frac{dH}{dt}\right|_{\text{diff}} = -\int dx\,dv\,f(\partial_x \ln f - \partial_x \ln f_{st})^2 \leqslant 0.$$
(30)

Second, for the inelastic term, proportional to ν,

$$\left.\frac{dH}{dt}\right|_{\text{inel}} = -\frac{\nu}{2}\int dx(u - u_{st})\int dv\,f\,\partial_v \ln f_{st}.$$
(31)

Finally, the noise term, proportional to ξ, reads

$$\left.\frac{dH}{dt}\right|_{\text{noise}} = -\frac{\xi}{2}\int dx\,dv\,f(\partial_v \ln f - \partial_v \ln f_{st})^2 \leqslant 0.$$
(32)

These results, and the following throughout this section, are valid for a quite general set of boundary conditions, leading to the cancellation of all the boundary terms arising after integrating by parts, as detailed in Appendix A. This set includes but is not limited to the Lees-Edwards and periodic boundary conditions corresponding to the sheared and uniformly heated situations, respectively. For instance, they also apply to the Couette state, in which the system is driven by keeping its two edges at two (in general, different) fixed temperatures T_L and T_R.

The inelastic term $dH/dt|_{\text{inel}}$ in Eq. (31) does not have a definite sign in general. Therefore, it is the inelastic term that prevents us from proving H to be a nonincreasing function of time. It must be stressed that the diffusive, inelastic, and noise contributions to dH/dt in Eqs. (30) and (31) come exclusively from the diffusive, noise, and inelastic contributions in the kinetic equation, respectively, only once the linear terms have been subtracted as done in Eq. (28): see Appendix A for details.

Despite the above discussion, global stability of the steady state can be established without proving an H theorem. The key point is the following: the long-time limit of dH/dt is nonpositive and thus H has a finite limit, since it is bounded from below. Therefore, dH/dt tends to 0 in the long-time limit and it can be shown that this is only the case if $f(x,v,\infty) \equiv \lim_{t\to\infty} f(x,v,t) = f_{st}(x,v)$.

The average velocity $u(x,t)$ satisfies a diffusive equation, (17a), and thus it irreversibly tends to the steady profile corresponding to the given boundary conditions in the long-time limit. Therefore, $u(x,\infty) \equiv \lim_{t\to\infty} u(x,t) = u_{st}(x)$, and taking into account Eq. (31),

$$\lim_{t\to\infty}\left.\frac{dH}{dt}\right|_{\text{inel}} = 0 \Rightarrow \lim_{t\to\infty}\frac{dH}{dt} \leqslant 0.$$
(33)

Since $H[f]$ is bounded from below, the only possibility is

$$\lim_{t\to\infty}\frac{dH}{dt} = 0,$$
(34)

and all the contributions to dH/dt in Eqs. (30)–(32) vanish in the long-time limit. The vanishing of Eq. (30) imposes that $f(x,v,\infty) = f_{st}(x,v)\phi(v)$, where $\phi(v)$ is an arbitrary function of v. For $\xi \neq 0$, Eq. (32) implies that $\phi(v)$ must be a constant, independent of v, and normalization yields $\phi(v) = 1$. For $\xi = 0$, Eq. (32) identically vanishes but it can also be shown that $\phi(v) = 1$ by using the kinetic equation in the limit as $t \to \infty$. Therefore, for arbitrary ξ, including $\xi = 0$, we have that

$$f(x,v,\infty) = f_{st}(x,v).$$
(35)

This completes the proof. The steady distribution $f_{st}(x,v)$ is globally stable: each time evolution $f(x,v,t)$ (corresponding to a given initial condition) tends to it in the long-time limit.

IV. INADEQUACY OF H_B AS A LYAPUNOV FUNCTIONAL FOR NONCONSERVATIVE SYSTEMS

Here we show that Boltzmann's $H_B[f]$ cannot be used to build a Lyapunov functional for intrinsically dissipative systems, in agreement with the numerical results of Marconi *et al.* [7]. We prove this not only for the simplified models considered here, but for a general kinetic equation in which energy is not conserved in collisions, such as the inelastic Boltzmann or Enskog equations. To keep the notation simple, we still write $\partial_t f = \mathcal{L}f$, but now $\mathcal{L}$ stands for the evolution operator in the considered kinetic description, which is nonlinear in general.

First, we restrict ourselves to homogeneous situations and thus drop the integral over x,

$$H_B[f] = \int dv f \ln f, \tag{36a}$$

$$\frac{dH_B}{dt} = \int dv \, \partial_t f \ln f = \int dv \, \mathcal{L}f \, \ln f. \tag{36b}$$

Also, we consider a system that is initially close to the steady state, such that we can expand everything in powers of $\Delta f = f - f_{\text{st}}$. Then

$$\mathcal{L}f \equiv \mathcal{L}(f_{\text{st}} + \Delta f) = \underbrace{\mathcal{L}f_{\text{st}}}_{0} + \mathcal{L}_{\text{lin}}\Delta f + O(\Delta f)^2, \tag{37}$$

in which $\mathcal{L}_{\text{lin}}$ is the linearized evolution operator. Neglecting $O(\Delta f)^2$ terms, the linear approximation arises:

$$\left.\frac{dH_B}{dt}\right|_{\text{lin}} = \int dv \, (\mathcal{L}_{\text{lin}}\Delta f) \ln f_{\text{st}} = \left.\frac{d}{dt}\langle \ln f_{\text{st}}\rangle\right|_{\text{lin}}. \tag{38}$$

On the one hand, the linear contribution vanishes in the elastic case: $\ln f_{\text{st}}$ is a sum of constants of motion, which are unchanged by the linearized kinetic operator. Thus H_B can be a candidate for a Lyapunov functional. On the other hand, only the mass and linear momentum are conserved for nonconservative interactions. Thus, no longer is $\ln f_{\text{st}}$ the sum of conserved quantities, and

$$\left.\frac{dH_B}{dt}\right|_{\text{lin}} \neq 0. \tag{39}$$

Therefore, by changing the initial sign of $\Delta f = f - f_{\text{st}}$, which can always be done, the initial sign of dH_B/dt is reversed and H_B cannot be a Lyapunov functional.

In Fig. 1, we show the evolution of H_B in our kinetic model. We consider a uniformly heated system, so that the system remain homogeneous for all times, as described in Sec. II A. Two initial conditions are considered, corresponding to Gaussian distributions with zero average velocity but nonsteady values of the temperature, specifically $1.1\,T_{\text{st}}$ and $0.9\,T_{\text{st}}$. We can see how, in agreement with our discussion, not only is one of the functionals increasing, but also it can be obtained as the mirror image of the decreasing one through the stationary value. Technical details about the simulation are provided in Appendix B.

Taking into account the specific (Gaussian) shape of the steady PDF for the uniformly heated system, as given by Eq. (25), the time derivative of H_B in Eq. (38) reduces to

$$\frac{dH_B}{dt} = -\frac{1}{2T_{\text{st}}}\frac{d\langle v^2\rangle}{dt}. \tag{40}$$

Since the plots in Fig. 1 correspond to evolutions of the system for which $u(x,t) \equiv 0$ for all times, therein $\langle v^2\rangle = T$ and, consistently, the H_B curve corresponding to an initial value of the temperature that is higher (lower) than the steady one monotonically increases (decreases).

This is consistent with the situation found in Ref. [11], in which the uniformly heated granular gas described by the inelastic Boltzmann equation was investigated within the first Sonine approximation. Therein, the entropy production was shown to have linear terms in the deviations of the temperature
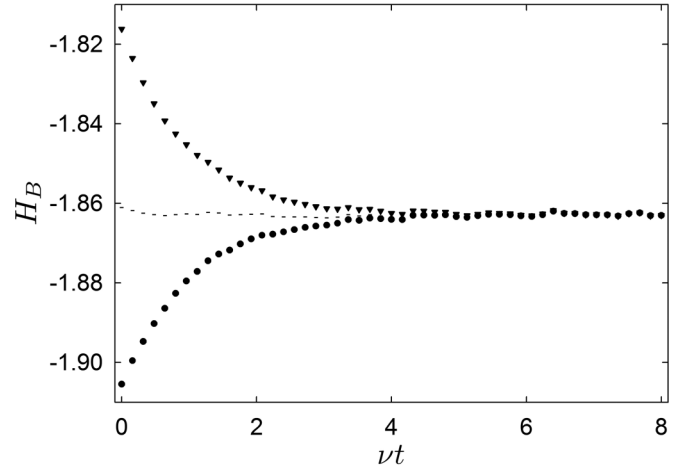

FIG. 1. Evolution of the functional H_B for two initial conditions in a uniformly heated system. Both simulations start from the stationary Gaussian shape but with a homogeneous temperature slightly shifted from the stationary one: (i) $T(t = 0) = 1.1\,T_{\text{st}}$ (circles) and (ii) $T(t = 0) = 0.9\,T_{\text{st}}$ (triangles). As predicted by the linear approximation, both functionals are symmetric with respect to the stationary value, which is schematically illustrated by plotting their mean value (dashed line). A system with $N = 330$ sites has been considered, with $\nu = 20$ and $\xi = 50$. Plots correspond to averages over 3000 trajectories.

and the excess kurtosis. Also, our result is consistent with the numerical results in Ref. [7] for several collision models. The above argument also proves why H_B is not nonincreasing for an elastic system immersed in a heat bath at a temperature different from the initial temperature of the gas, as also observed in Ref. [7]. Although $\ln f_{\text{st}}$ is conserved in collisions, the evolution operator includes a term coming from the interaction with the bath that does not conserve the kinetic energy, and again, $dH/dt|_{\text{lin}} \neq 0$, making it impossible for H_B to be a Lyapunov functional.

In spatially nonhomogeneous situations, the main difference is that an additional integral over x is present, in both H_B and, consequently, dH_B/dt. There is no reason to expect this integral over space to make $dH/dt|_{\text{lin}}$ vanish, since one still has that

$$\left.\frac{dH_B}{dt}\right|_{\text{lin}} = \left.\frac{d}{dt}\langle \ln f_{\text{st}}\rangle\right|_{\text{lin}}, \tag{41}$$

and in general, $\ln f_{\text{st}}$ is not a sum of constants of motion. In fact, again the sign of $dH_B/dt|_{\text{lin}}$ is reversed when $\Delta f \to -\Delta f$, similarly to the homogeneous case. We have numerically checked this prediction for the sheared system, with the resulting evolution of H_B being completely similar to that for the uniformly heated case in Fig. 1 and, thus, not shown here.

V. *H* THEOREM FOR SOME SPECIFIC NESSs

Here we prove that the functional $H[f]$ is monotonically decreasing for all times in some specific physical situations. Our proof applies to both the sheared and the uniformly heated systems described in Sec. II A. To be as general as possible, we consider a system that is both heated and sheared: $a \neq 0$ and $\xi \neq 0$. In this situation, the boundary conditions for the

PDF are given by Eq. (20), which lead to Eqs. (19a) and (19b) for the averages $u(x,t)$ and $T(x,t)$.

The steady solution of the hydrodynamic equations is

$$u_{st}(x) = a\left(x - \frac{1}{2}\right), \quad T_{st} = \frac{2a^2 + \xi}{\nu}. \quad (42)$$

On the one hand, the average velocity has a linear profile, similarly to the situation in the USF state. On the other hand, the temperature remains homogeneous but its steady value has two contributions, one coming from the shear and the other from the stochastic thermostat. The viscous heating $2(\partial_x u)^2$ and uniform heating ξ terms cancel the cooling term $-\nu T$ for all x. The stationary solution of the kinetic equation is readily found,

$$f_{st}(x,v) = (2\pi T_{st})^{-1/2} \exp\left[-\frac{(v - u_{st}(x))^2}{2T_{st}}\right], \quad (43)$$

that is, the Gaussian distribution corresponding to the hydrodynamic fields in Eq. (42). Of course, the USF state and NESS of the uniformly heated system in Sec. II A can be easily recovered as particular cases of Eq. (43): for $(a \neq 0, \xi = 0)$ and $(a = 0, \xi > 0)$, respectively.

We turn now to the question of the existence of an H theorem, that is, the existence of a nonequilibrium entropy ensuring the monotonic approach of the one-particle PDF to the steady state. Our starting point is the following expansion of the one-particle PDF in Hermite polynomials,

$$f(x,v,t) = \frac{1}{\sqrt{2\pi T(x,t)}} \exp\left[-\frac{[v - u(x,t)]^2}{2T(x,t)}\right]$$
$$\times \left[1 + \sum_{n=3}^{\infty} \gamma_n(x,t) H_n\left(\frac{v - u(x,t)}{\sqrt{T(x,t)}}\right)\right], \quad (44)$$

which is known as the Gram-Charlier series [43–46]. Therein, $u(x,t)$ and $T(x,t)$ are the (exact) average velocity and temperature stemming from the hydrodynamic equations for the considered distribution. The above expansion is suggested by the Gaussian shape of the stationary PDF in Eq. (43). Now we define

$$c = \frac{v - u(x,t)}{\sqrt{T(x,t)}}, \quad \tilde{f}(x,c,t) = \sqrt{T(x,t)}\, f(x,v,t). \quad (45)$$

From the orthogonality relation of the Hermite polynomials [47], it is readily obtained that

$$\gamma_n(x,t) = \frac{1}{n!}\int dc\, H_n(c)\,\tilde{f}(x,c,t). \quad (46)$$

Also, we could write γ_n as a combination of moments of the distribution.

Some comments on the Gram-Charlier expansion are pertinent. First, note that $n \geqslant 3$ in the sum: $\gamma_1 = \gamma_2 = 0$ because the zeroth-order Gaussian contribution exactly gives the first two moments $u(x,t)$ and $\langle v^2 \rangle(x,t) = u^2(x,t) + T(x,t)$. Second, if $f(x,v,t)$ were symmetric with respect to $v = u$, that is, $\langle (v - u)^{2n+1} \rangle = 0$ for all $n \in \mathbb{N}$, only even values of n would be present in the sum and one would end up with the usual expansion in Sonine-Laguerre polynomials of kinetic theory. Finally, it is worth stressing that series (44) converges for functions such that the tails of $\tilde{f}(x,c,t)$ approach 0 more rapidly than $e^{-c^2/4}$ for $c \to \pm\infty$ [46,48,49].

After a lengthy but straightforward calculation, which is summarized in Appendix C, it is shown that

$$\frac{dH}{dt} = A(t) + B(t), \quad \text{with both} \quad A(t), B(t) \leqslant 0. \quad (47)$$

The expressions for $A(t)$ and $B(t)$ are

$$A(t) = -\int dx\, T\left[\left(\frac{u'}{T} - \frac{u'_{st}}{T_{st}}\right)^2 + \frac{\xi}{2}\left(\frac{1}{T} - \frac{1}{T_{st}}\right)^2\right] \quad (48)$$

and

$$B(t) = -\frac{1}{\sqrt{2\pi}}\int dx\, dc\, \frac{e^{-c^2/2}}{1 + \sum_{n=3}^{\infty} \gamma_n H_n(c)}\left\{\frac{T'}{2T}H_2(c) + \sum_{n=3}^{\infty}\gamma_n' H_n(c) - \sum_{n=3}^{\infty}\frac{\gamma_n u'}{\sqrt{T}}n H_{n-1}(c) + \sum_{n=3}^{\infty}\frac{\gamma_n T'}{2T}[H_{n+2}(c) + n H_n(c)]\right\}^2$$
$$-\frac{\xi}{2\sqrt{2\pi}}\int dx\, dc\, \frac{e^{-c^2/2}}{1 + \sum_{n=3}^{\infty}\gamma_n H_n(c)}\frac{1}{T}\left[\sum_{n=3}^{\infty}\gamma_n n H_{n-1}(c)\right]^2. \quad (49)$$

We recall that a prime denotes a spatial derivative.

Therefore, $dH/dt \leqslant 0$ for all times and we have shown that the H theorem holds for the system that is both sheared and heated. Rigorously, our proof holds for those PDFs such that the above Hermite expansion converges. Also, note that the proof remains valid for the approach to any NESS whose PDF is a Gaussian with a homogeneous temperature, independently of the corresponding boundary conditions. In Sec. III, we have demonstrated that dH/dt only vanishes for $f(x,v,\infty) = f_{st}(x,v)$, but the same result can be rederived here in a different way. By imposing that both $A(t)$ and $B(t)$ vanish in the long-time limit and making use of the hydrodynamic equations for the averages, it can be shown that $u(x,\infty) = u_{st}(x)$, $T(x,\infty) = T_{st}$, and $\gamma_n(x,\infty) = 0$, $\forall n \geqslant 3$.

A. Numerical results for the USF state

Here we consider the sheared system, and we numerically check our theoretical predictions. Throughout this section, we use the values of the parameters $\nu = 20$, $a = 5$, and $\xi = 0$ (there is no stochastic forcing).

First, in Fig. 2, we show the evolution of the distribution and the H functional from a Gaussian initial condition with the steady velocity profile $u(x,0) = u_{st}(x)$ but a higher temperature, $T(t = 0) = 7 T_{st}$. In Fig. 2(a), we depict the velocity distribution at $x = 1/4$ for several times. All of them are Gaussian, which agrees with the theoretical prediction of the kinetic equation: when the initial average velocity profile coincides with the steady one and only the temperature is perturbed, an initially Gaussian PDF remains Gaussian for
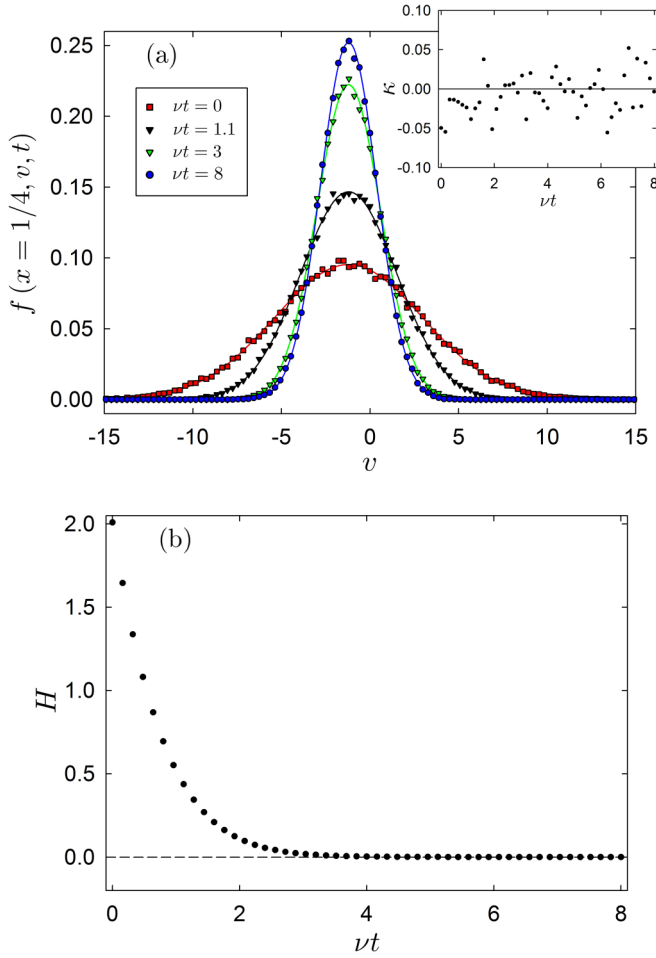
FIG. 2. Relaxation towards the USF state. The initial condition is Gaussian, centered in $u_{st}(x)$ and with variance $T = 7\,T_{st}$. (a) Velocity distribution function at $x = 1/4$ for four times. Inset: Evolution of the excess kurtosis. (b) Monotonic relaxation of the H functional, in clear agreement with the proven H theorem. Data are averaged over 6000 trajectories in a system with $N = 660$ sites, $\nu = 20$, and $a = 5$. Solid lines correspond to the (theoretical) Gaussian distributions for the plotted times, except the last time, for which it represents the theoretical steady distribution.

FIG. 3. The same plots as in Fig. 2, but starting from a different initial condition. Now, the initial PDF is a Gaussian centered in $u(x,0) = u_{st}(x) + 4.4\,\sin(2\pi x)$ and with variance $T(t = 0) = T_{st}$. (a) Solid lines correspond to the theoretical PDFs for the initial time and the steady state. (b) H decreases again monotonically towards its steady value, consistent with our theoretical prediction.

all times. Indeed, the inset shows how the excess kurtosis $\kappa = \langle [v - u(x)]^4 \rangle / \langle [v - u(x)]^2 \rangle^2 - 3$ only fluctuates around 0 at the considered position $x = 1/4$, consistently with the Gaussian shape. In Fig. 2(b), it is clearly shown that the H functional is monotonically decreasing with time.

Second, we study the relaxation to the USF state from another initial preparation, for which the average velocity profile $u(x,0)$ is different from the stationary one but $T(x,0) = T_{st}$. The numerical results are shown in Fig. 3, and for the sake of simplicity we again use an initial Gaussian distribution. Specifically, we use $u(x,0) = u_{st}(x) + 4.4\,\sin(2\pi x)$. Here, the departure from the Gaussian shape is evident, and thus we have not plotted the kurtosis. Consistent with our theoretical prediction, we again get a monotonous relaxation of H towards its null stationary value.

Finally, we consider situations for which the above-presented proof is not rigorously applicable. As stated before,
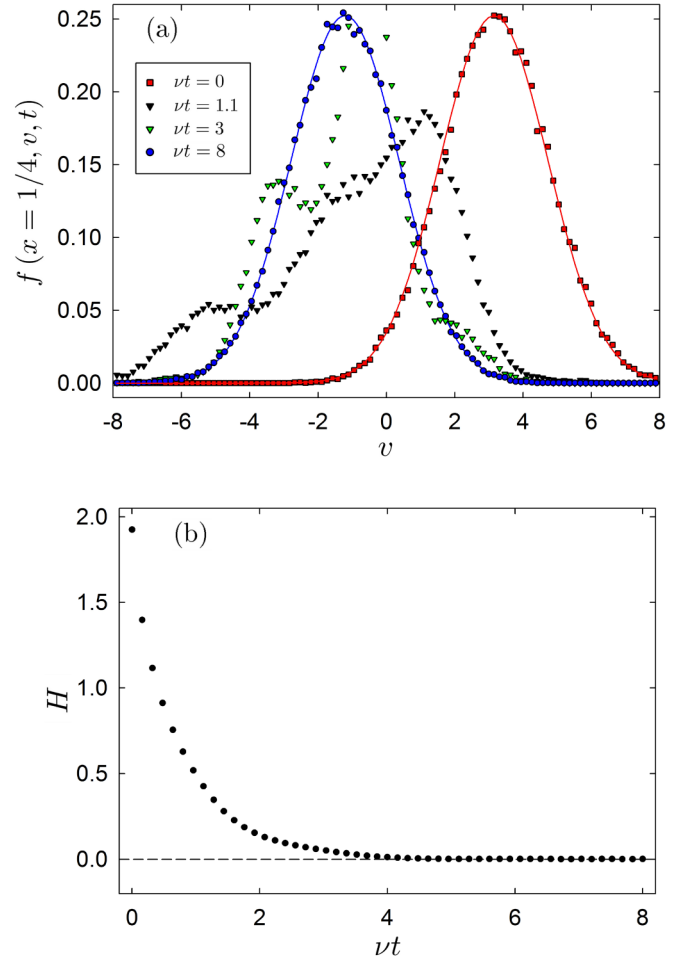
the Gram-Charlier series does not converge when the tails of the distribution decay to 0 more slowly than the square root of the Gaussian. Nevertheless, when all the coefficients γ_n defined in Eq. (46) exist and are finite, we still expect the H theorem to hold. We illustrate this situation with an initial exponential distribution; specifically, we consider

$$f(x,v,0) = \frac{1}{\sqrt{2T(t=0)}} \exp\left[\frac{\sqrt{2}|v - u(x,t=0)|}{\sqrt{T(t=0)}}\right], \quad (50)$$

with $u(x,t=0) = u_{st}(x) + 4.4\,\sin(2\pi x)$ and $T(t=0) = 0.1\,T_{st}$. Consistent with our expectation, we can see in Fig. 4 that indeed the H functional also monotonically decreases.

B. Numerical results in the uniformly heated system

To conclude, we put forward the results of simulations for the uniformly heated system. Specifically, our simulations have been done for $\nu = 20$, $a = 0$ (no shear), and $\xi = 50$. In
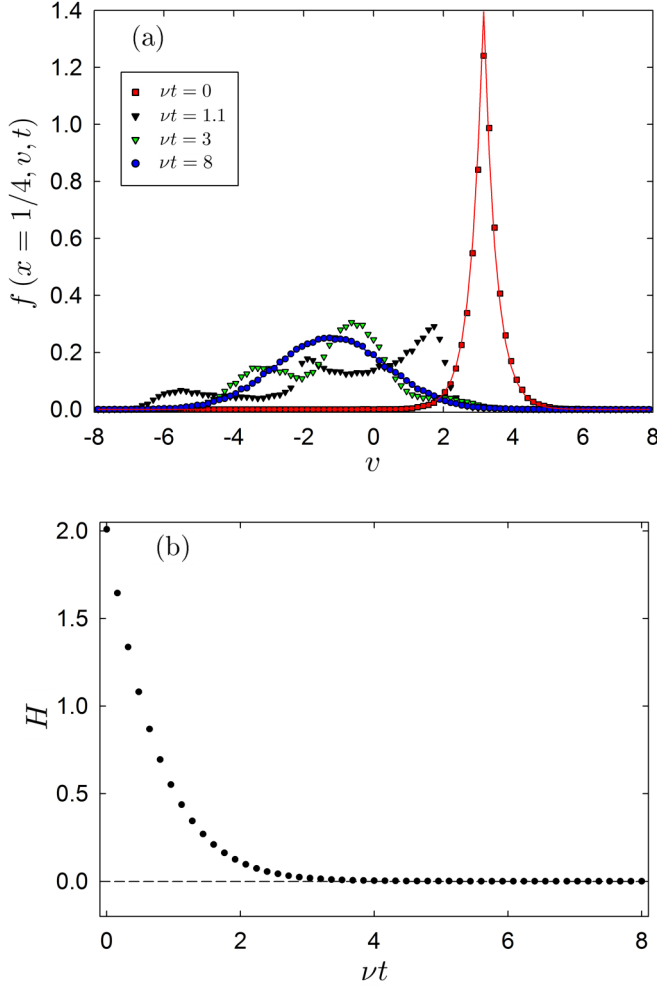
FIG. 4. The same plots as in Fig. 2, but starting from an initial PDF with a divergent Gram-Charlier series. Concretely, the plots correspond to an exponential initial distribution centered in $u(x, t = 0) = u_{\rm st}(x) + 4.4 \sin(2\pi x)$ and with $T(t = 0) = 0.1\, T_{\rm st}$.

order not to overload the reader with too many examples, we only present the more complex case in Fig. 5: the relaxation towards the steady state from an initial exponential distribution, as given by Eq. (50). In particular, we consider that $u(x, t = 0) = 4.4 \sin(2\pi x)$ and $T(t = 0) = 0.1\, T_{\rm st}$. Note that the perturbation from the steady values is the same as in Fig. 4 for the sheared case. Again, we observe the monotonic relaxation of H towards the stationary value, consistent with our theoretical result, even for a initial distribution for which the Gram-Charlier series does not converge.

VI. CONCLUSIONS

Within a simplified model for a granularlike velocity field, we have analytically shown that the nonequilibrium steady state that the system eventually reaches in the long-time limit is globally stable. This has been done for quite a general situation, in which energy may be injected into the system both through the boundaries and by a heating mechanism that acts in the bulk. The proof is valid for both spatially homogeneous situations (such as the uniformly heated system)
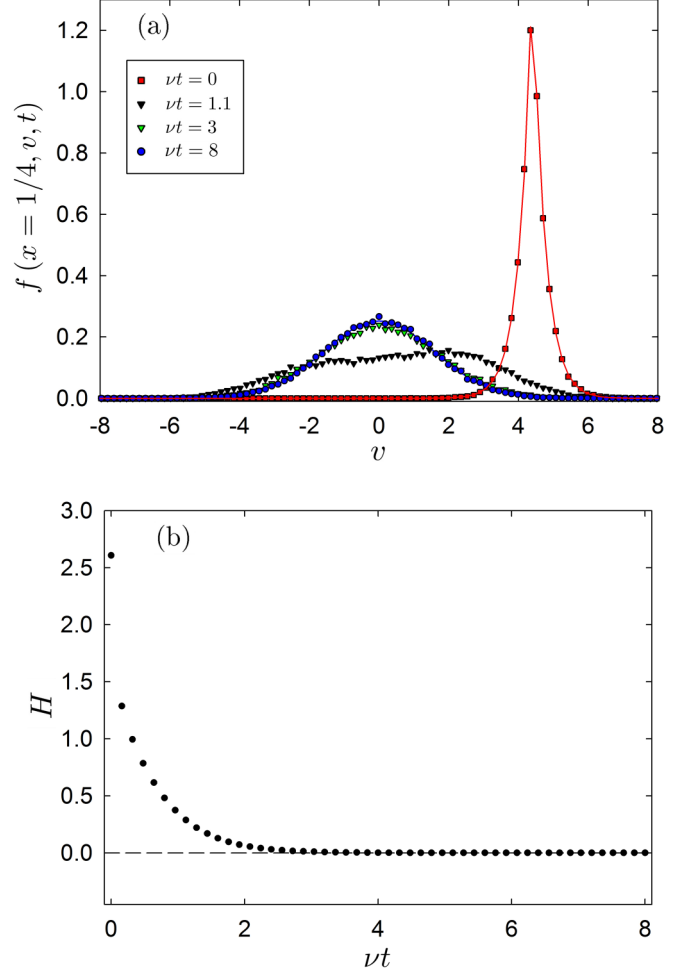

FIG. 5. Numerical results for the uniformly heated system. The plots are analogous to those in Fig. 2. (a) Time evolution of the PDF and (b) time evolution of the H functional. The system is initially prepared with an exponential PDF centered in $u(x, t = 0) = 4.4 \sin(2\pi x)$ and with $T(t = 0) = 0.1\, T_{\rm st}$. Data are averaged over 3000 trajectories in a system with $N = 330$ sites.

and inhomogeneous situations (such as the USF and Couette states).

The proof of global stability is based on showing that the H functional $H = \int dx\, dv\, f \ln(f/f_{\rm st})$ (the Kullback-Leibler divergence between the time-evolving one-particle PDF $f(x, v, t)$ and its value in the stationary state $f_{\rm st}(x, v)$ [50]) is nonincreasing in the infinite-time limit. Thus, we do not need H to be a "good" Lyapunov functional for all times in order to prove global stability. In conclusion, global stability and the validity of an H theorem do not seem to be unavoidably connected.

Moreover, we have analytically shown that the Boltzmann functional $H_B = \int dx\, dv\, f \ln f$ cannot be, in general, a Lyapunov functional for systems with nonconservative interactions. Close to the steady state, we have proven that dH_B/dt contains nonvanishing terms that are linear in the deviations $\Delta f = f - f_{\rm st}$. Therefore, a reversal of the sign of Δf entails a reversal of the sign of dH_B/dt. This general analytical proof of the inadequacy of H_B as a Lyapunov functional is in agreement with previous results in some specific cases [7,11].

We have also succeeded in demonstrating that the *H* functional is nonincreasing and thus a "good" Lyapunov functional for some specific driving mechanisms. Our proof is not restricted to spatially homogeneous situations and is applicable to two relevant physical cases: the approach to (i) the USF state and (ii) the NESS corresponding to the uniformly heated case. The proof involves a suitable expansion of the one-particle PDF in Hermite polynomials, which is a generalization of the well-known Sonine-Laguerre expansion in kinetic theory. Although the proof is only rigorous for PDFs having a convergent series expansion, we expect it to remain valid for more general PDFs. In fact, we have numerically validated this expectation in some specific situations.

The analytical results presented here are thus in agreement with the numerical evidence in Refs. [7,8] and advance the understanding of this field in a twofold way. First, an analytical proof, which was lacking, is provided for a simplified model. Second, spatially inhomogeneous situations are considered, both in the time evolution and in its steady state.

Some limitations of our results have to be underlined, though. First, the simplifications introduced in the model make it impossible to address the problem of the stability of the homogeneous cooling state in the undriven system at the level of the kinetic equation, as discussed in Ref. [14]. Second, for the driving mechanisms for which we can analytically prove that *H* is a "good" Lyapunov functional, the steady distribution is exactly Gaussian. Nevertheless, we think that this is not a fundamental point and expect that the kind of expansion-based proof presented here may be extended to other situations. A particularly appealing case is the approach to the Couette NESS, for which the stationary PDF is nonGaussian in the model [14].

On a different note, our kinetic equation, (14), shows some resemblance to evolution equations for the one-particle PDF found in other physical contexts, such as the Vlasov equation in plasma physics or astrophysics [51–53] or the nonlinear (in the distribution function) Fokker-Planck equation for systems of infinitely many coupled nonlinear oscillators exhibiting phase transitions [54]. It is a driftlike term depending on a certain average of the PDF that all these different problems share. For both the Vlasov and the nonlinear Fokker-Planck equations, the existence of a Lyapunov functional has been proved by considering a variant of the functional $H[f]$ defined in Eq. (4) [55–57]. Thus, an interesting prospect is to investigate whether this kind of approach may be extended to our class of models with nonconservative interactions.

Our work also opens the door to applying the ideas developed in this paper to more complex models, closer to real nonconservative systems, like granular gases. The pioneering numerical work in Refs. [7,8] strongly suggests that the *H* functional is a good Lyapunov functional for granular fluids. It seems worthwhile to try to analytically prove that this is indeed the case for the inelastic Boltzmann equation, at least for some specific situations. If nothing else, one would like to be able to show that the long-time solutions are globally stable by showing that *H* is asymptotically nonincreasing, similarly to what has been done here.

APPENDIX A: DERIVATION OF THE EXPRESSION FOR dH/dt IN A GENERAL DRIVEN STATE

Let us consider the three contributions to dH/dt in Eq. (29). We start with the diffusive one,

$$\frac{dH}{dt}\bigg|_{\text{diff}} = \int dx\, dv\, \mathcal{L}_{\text{diff}} f\, \ln\left(\frac{f}{f_{\text{st}}}\right) - \int dx\, dv\, \frac{f}{f_{\text{st}}} \mathcal{L}_{\text{diff}} f_{\text{st}},$$
(A1)

where $\mathcal{L}_{\text{diff}} f = \partial_x^2 f$ and we have used that $\int dx\, dv\, \partial_x^2 f_{\text{st}}$ vanishes identically. Integrating by parts the first term on the r.h.s. of Eq. (A1), the result is

$$\int dv\, \partial_x f \ln\left(\frac{f}{f_{\text{st}}}\right)\bigg|_0^1$$
$$- \int dx\, dv\, f \partial_x \ln f (\partial_x \ln f - \partial_x \ln f_{\text{st}}).$$
(A2)

Also integrating by parts the second term, one obtains

$$-\int dv\, \frac{f}{f_{\text{st}}} \partial_x f_{\text{st}}\bigg|_0^1 + \int dx\, dv\, f \partial_x \ln f_{\text{st}} (\partial_x \ln f - \partial_x \ln f_{\text{st}}).$$
(A3)

We assume that the boundary terms are equal to 0, that is,

$$\int dv\, \left[\partial_x f \ln\left(\frac{f}{f_{\text{st}}}\right) - \frac{f}{f_{\text{st}}} \partial_x f_{\text{st}}\right]_0^1 = 0.$$
(A4)

This is obviously true for Lees-Edwards and periodic boundary conditions [58]. Summing the two contributions to the diffusive term above, we have

$$\frac{dH}{dt}\bigg|_{\text{diff}} = -\int dx\, dv\, f (\partial_x \ln f - \partial_x \ln f_{\text{st}})^2,$$
(A5)

which is Eq. (30).

The noise term is treated along the same lines as above, but integrating by parts in v instead of x, since $\mathcal{L}_{\text{noise}} f = \frac{\xi}{2} \partial_v^2 f$. There in, the boundary terms vanish if f and f_{st} tend to 0 fast enough for $v \to \pm\infty$, and

$$\frac{dH}{dt}\bigg|_{\text{noise}} = -\frac{\xi}{2} \int dx\, dv\, f (\partial_v \ln f - \partial_v \ln f_{\text{st}})^2,$$
(A6)

which is Eq. (32).

Now we focus on the inelastic contribution,

$$\frac{dH}{dt}\bigg|_{\text{inel}} = \int dx\, dv\, \mathcal{L}_{\text{inel}} f\, \ln\left(\frac{f}{f_{\text{st}}}\right) - \int dx\, dv\, \frac{f}{f_{\text{st}}} \mathcal{L}_{\text{inel}} f_{\text{st}},$$
(A7)

in which $\mathcal{L}_{\text{inel}} f = \frac{\nu}{2} \partial_v[(v-u)f]$. Then

$$\left.\frac{dH}{dt}\right|_{\text{inel}} = \frac{\nu}{2} \int dx\, dv\, \partial_v[(v-u)f]\, \ln\left(\frac{f}{f_{\text{st}}}\right)$$
$$- \frac{\nu}{2} \int dx\, dv\, \partial_v[(v-u_{\text{st}})f_{\text{st}}]\, \frac{f-f_{\text{st}}}{f_{\text{st}}}. \quad \text{(A8)}$$

Again, integrating by parts in v (here we do not write the boundary terms at $v \to \pm\infty$), the first term on the r.h.s. of Eq. (A8) is

$$- \frac{\nu}{2} \int dx\, dv\, (v-u)f\, (\partial_v \ln f - \partial_v \ln f_{\text{st}}), \quad \text{(A9)}$$

whereas the second term gives

$$\frac{\nu}{2} \int dx\, dv\, (v-u_{\text{st}})f\, (\partial_v \ln f - \partial_v \ln f_{\text{st}}). \quad \text{(A10)}$$

Summing up these two contributions, and taking into account that both u and u_{st} do not depend on v,

$$\left.\frac{dH}{dt}\right|_{\text{inel}} = \frac{\nu}{2} \int dx\, (u-u_{\text{st}}) \int dv\, f(\partial_v \ln f - \partial_v \ln f_{\text{st}}). \tag{A11}$$

Since $\int dv\, f\, \partial_v \ln f \equiv \int dv\, \partial_v f = 0$, this leads to Eq. (31).

APPENDIX B: SIMULATION STRATEGY

In the simulations, so as to generate a trajectory of the stochastic process, we proceed as follows. (i) A pair $(l, l+1)$ is chosen at random and undergoes the inelastic collision described by Eq. (5); (ii) all the particles are submitted to the stochastic thermostat according to (6) and (7); and (iii) time is incremented by $\delta\tau = -(N\omega)^{-1} \ln x$, with x being a homogeneously distributed random number in $(0, 1)$ [59–62]. This cycle (random choice of a pair and noise interaction followed by a time increment) is repeated until time exceeds some maximum time t_{max}.

Regarding the measurements of $f(x, v, t)$, we sample both position and velocity spaces by defining N_x bins of width Δx and N_v bins of width Δv. Of course, the product $N_x \Delta x = 1$, the whole lattice, whereas $N_v \Delta v$ gives the range of velocities

bounded by the cutoffs v_{min} and v_{max}. In our simulations, we control that the contribution to the PDFs coming from velocities outside the considered interval $[v_{\text{min}}, v_{\text{max}}]$ is negligible. With such a binning, we build up a histogram and therefrom the distribution function $f(x, v, t)$, which is represented by an $N_x \times N_v$ matrix for each time t. Both H and H_B are computed by numerically replacing the integral over x and v with sums over the prescribed bins.

APPENDIX C: DERIVATION OF EQ. (47)

In order to derive Eq. (47), we have to substitute the Gaussian stationary solution, (43), and the Gram-Charlier series, (44), into the three contributions to dH/dt, given by Eqs. (30)–(32). For the inelastic term, it is readily obtained that

$$\left.\frac{dH}{dt}\right|_{\text{inel}} = \frac{\nu}{2T_{\text{st}}} \int dx(u-u_{\text{st}})^2. \quad \text{(C1)}$$

For the diffusive and noise terms, the key ideas are a change in the integration over velocities from v to $c = (v-u)/\sqrt{T}$ and the use of the recursion relations and the orthogonality property of the Hermite polynomials [47]. Working along these guidelines, we arrive at

$$\left.\frac{dH}{dt}\right|_{\text{diff}} = - \int dx\, T\left(\frac{u'}{T} - \frac{u'_{\text{st}}}{T_{\text{st}}}\right)^2$$
$$- \frac{u'^2_{\text{st}}}{T^2_{\text{st}}} \int dx(u-u_{\text{st}})^2 + B_1(t), \quad \text{(C2)}$$

$$\left.\frac{dH}{dt}\right|_{\text{noise}} = -\frac{\xi}{2} \int dx\, T\left(\frac{1}{T} - \frac{1}{T_{\text{st}}}\right)^2$$
$$- \frac{\xi}{2T^2_{\text{st}}} \int dx(u-u_{\text{st}})^2 + B_2(t), \quad \text{(C3)}$$

where B_1 and B_2 are, respectively, the first and the second term in Eq. (49). The sum of the factors multiplying $\int dx(u-u_{\text{st}})^2$ vanishes upon taking into account the equation for the (spatially homogeneous) stationary temperature. Therefore, the sum of the remaining terms leads right to Eq. (47).

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