City Research Online

City, University of London Institutional Repository

This is the accepted version of the paper.

This version of the publication may differ from the final published version.

Permanent repository link: https://openaccess.city.ac.uk/id/eprint/24780/

Link to published version:

City Research Online: http://openaccess.city.ac.uk/   publications@city.ac.uk

Equally Diversified or Equally Weighted?

Gianluca Fusai[1]

Domenico Mignacca[2]

Andrea Nardon[3]

Ben Human[4]

August 6, 2020

Abstract

We show how to decompose the portfolio volatility into undiversified volatility and diversification components. Our decomposition has a clear statistical interpretation because it relates the diversification component to the partial covariances, i.e. the covariances between the residuals of the regressions of the weighted asset returns with respect to the portfolio return. On this basis, we advocate the construction of an equally diversified portfolio. An empirical analysis illustrates the superior out-of-sample performance of the equally diversified portfolios with respect to the equally weighted portfolio.

[1]**Corresponding Author**. DISEI, Università del Piemonte Orientale, Italy, and Cass Business School, City, University of London. Email: gianluca.fusai@uniupo.it
[2]Executive Director Investment Risk, Qatar Investment Authority
[3]Partner, Head of Quants, Sarasin & Partners LLP
[4]Quant Analyst, Sarasin & Partners LLP

1 Introduction

In order to build a portfolio that "is not heavily exposed to individual shocks", one popular strategy is to select mean-variance efficient portfolios. However, they are characterized by very extreme weights and disappointing out-of-sample performances respect for example to an equally weighted (EW) portfolio, see DeMiguel et al. [1]. On the other side, to hold an EW portfolio is not necessarily a guarantee of diversification. For an extreme example, suppose an investor is optimizing a portfolio with only two assets: a broad market index, such as the SP 500, and one of its constituents (e.g. the IBM stock). The (true) optimal portfolio would likely be very different from the equally weighted one. In a more realistic example, a portfolio equally weighted in 10 high-tech firms would be less diversified than one invested in ten stocks from very different industries. In this paper, we aim to quantify the level of diversification of a given portfolio. We introduce a new diversification index by decomposing the portfolio volatility into undiversified volatility and a diversification component. The diversification component offsets the undiversified part leaving as a final result the portfolio volatility itself. Our decomposition has a clear statistical interpretation because it relates the diversification component to the so called partial covariances, i.e. the covariances between the residuals of the regressions of the weighted asset returns with respect to the portfolio return. The undiversified component is related to the sum between the individual asset variance and the (partial) variance of the residual of the just mentioned regressions. Moreover, the risk that can be diversified is measured with reference to the partial variances, i.e. the risk orthogonal to the portfolio return, and not with respect to the variance of the assets. The idiosyncratic risk is completely offset by the component related to the partial covariances.

2 The Diversification Index

We consider a portfolio with n risky securities and its volatility σ is the risk measure

$$\sigma = \sqrt{\mathbf{w}'\mathbf{\Sigma}\mathbf{w}} \tag{1}$$

where $\boldsymbol{\Sigma}$ is the $n \times n$ covariance matrix with elements $\sigma_{ij}, i, j = 1, \cdots, n$ and $\mathbf{w}$ is the vector of weights with elements w_i.

We consider the n linear regressions of each weighted asset return with respect to the portfolio return

$$w_j r_j = \alpha_j + \beta_j r_p + \varepsilon_j, j = 1, \cdots, n. \tag{2}$$

where α_j is the average return not related to the portfolio, β_j is the beta of the weighted return with respect to the portfolio and ε_j is the diversifiable risk. At portfolio level, we have that $\sum_j \alpha_j = \sum_j \varepsilon_j = 0, \sum_j \beta_j = 1$. The coefficients of the least-square fit are

$$\hat{\beta}_j = \frac{w_j \sigma_j \rho_{jp}}{\sigma} \tag{3}$$

where ρ_{jp} is the correlation between the returns of asset j and the portfolio. It is convenient to introduce the quantity $\gamma_j = w_j \sigma_j \rho_{jp}$ and, given that $\sum_j \gamma_j = \sigma$, we can interpret it as measure of the risk contribution of asset j to the overall portfolio risk (and therefore $\hat{\beta}_j$ as percentage contribution). This decomposition of the portfolio volatility is very popular in the asset management industry: the risk contribution is the basis of risk parity portfolios for which weights are chosen so that the ex-ante risk contributions of each asset are equal, i.e. $\gamma_j = \sigma/n, \quad \forall j$, see Maillard et al. [6]. The decomposition of the volatility as sum of the individual risk contributions holds because the adopted risk measure is homogeneous of degree one and we can apply the Euler's theorem. Therefore, the risk contribution of asset j has the equivalent mathematical representation as weighted partial derivative of the portfolio volatility and, according to the regression, it also measures the exposure of each weighted return to the portfolio return.

Whilst the coefficients γ_j are measuring the risk contribution of each asset, the variance of the residuals of the regressions is a measure of the risk unrelated to the portfolio risk and can be taken as a measure of how well the portfolio is diversified. The reason for this choice is now detailed.

Let us consider the covariance $\sigma_{i,j.r_p}$ between residuals of regressions i and j

$$\sigma_{i,j.r_p} = cov(w_i r_i - \hat{\beta}_i r_p, w_j r_j - \hat{\beta}_j r_p) \tag{4}$$

This quantity is called partial covariance and it equals

$$\sigma_{i,j.r_p} = w_i w_j \sigma_{i,j} - \gamma_i \gamma_j \tag{5}$$

Therefore, we can decompose the covariance between two weighted stock returns as follows

$$cov(w_i r_i, w_j r_j) = \gamma_i \gamma_j + \sigma_{i,j.r_p} \tag{6}$$

and, setting $i = j$, we also have the variance decomposition

$$var(w_i r_i) = \gamma_i^2 + \sigma_{i.r_p}^2 \tag{7}$$

where $\sigma_{i.r_p}^2$ is called partial variance.

Formulas (6) and (7) provide a decomposition of the covariances (and variances) of the weighted returns: the first component is the product $\gamma_i \gamma_j$, that, once summed across all i and j, builds to up to the portfolio variance; the second term is the idiosyncratic covariance, $\sigma_{i,j.r_p}$, that allows us to diversify away, at stock and portfolio level, the partial variances. Indeed, summing over all indexes i and j, we have

$$\sigma^2 = \sum_{i,j} cov(w_i r_i, w_j r_j) = \sum_{i,j} \left(\gamma_i \gamma_j + \sigma_{i,j.r_p} \right) = \sum_i \gamma_i \sum_j \gamma_j + \sum_{i,j} \sigma_{i,j.r_p} = \sigma^2 + \sum_{i,j} \sigma_{i,j.r_p} \tag{8}$$

and it is clear that the sum of partial covariances must be equal to zero. In particular, we can split it into two components, and, after normalizing with respect to the portfolio volatility, we have

$$\sigma = \sigma + \frac{1}{\sigma} \left(\sum_i \sigma_{i.r_p}^2 + \sum_{i,j,i \neq j} \sigma_{i,j.r_p} \right) \tag{9}$$

that confirms that, at portfolio level, the partial variances are offset by the partial covariances. Due to the homogeneity of the risk contribution, a similar decomposition holds also at asset level

$$\gamma_j = \gamma_j + \frac{1}{\sigma} \left(\sigma_{j.r_p}^2 + \sum_{k,j \neq k} \sigma_{j,k.r_p} \right) \tag{10}$$

i.e. the partial variance of asset j is cancelled by the partial covariances with all the remaining

assets. Therefore, according to (10) we can call $\gamma_j + \frac{\sigma^2_{j.r_p}}{\sigma}$ the undiversified risk contribution of asset i and γ_i the (diversified) risk contribution. According to (9), $\sigma + \frac{\sum_i \sigma^2_{i.r_p}}{\sigma}$ is the undiversified portfolio risk. Combining an asset with the remaining ones, the portfolio risk becomes σ.

On this basis, we introduce the Quantitative Diversification Index measuring the diversification contribution of asset j

$$QDX_j = \frac{\sigma^2_{j.r_p}}{\sigma} \tag{11}$$

and rewrite (10) as

$$\gamma_j = \gamma_j + QDX_j + DIV_j \tag{12}$$

and (9) as

$$\sigma = \sigma + QDX + DIV \tag{13}$$

where $QDX = \sum_{i=1} QDX_i$, $DIV_j = \frac{1}{\sigma} \sum_{i,i \neq j} \sigma_{i,j.r_p}$ and $DIV = \sum_{i=1} DIV_i$. A few comments are useful.

First. The novel idea here is that in measuring the diversification effect of an asset we need to control for the portfolio return, so that the diversification contribution of an asset is related to the partial covariance of that asset with the remaining assets rather than being measured by the covariances among weighted returns. This is also confirmed when we notice that in our decomposition, the covariances among residuals counter-balance the specific variances: indeed, at an asset level, $QDX_i + DIV_i = 0$ and, at a portfolio level, $QDX + DIV = 0$, so $\sigma + QDX$ can be interpreted as the un-diversified portfolio risk.

Second. We can rewrite QDX_i in terms of the correlation of the asset with the portfolio

$$QDX_i = \frac{w_i^2 \sigma_i^2 \left(1 - \rho_{ip}^2\right)}{\sigma}, i = 1, \cdots, n \tag{14}$$

According to intuition, diversification should be related to the elimination of the risk unrelated to the portfolio. This is confirmed by the above expression: an asset having a low correlation with the portfolio provides diversification benefits. Viceversa, if the asset has a large correlation with the portfolio, it cannot help in diversifying away the risk of the remaining assets.

Third. Diversification does not mean low volatility. Indeed, according to formula (13) we can build portfolios with the same level of risk σ but very different values of QDX, or viceversa portfolios having different levels of risk but similar diversification levels. In order to reduce the portfolio risk we exploit the covariances between asset returns. In order to control the level of diversification of the portfolio we need to control the partial covariances: diversification is not related to the low correlation between assets, but to the variance of the residuals in a factor model, where the factor is the portfolio return. The decomposition does not hold anymore using different portfolios.

Fourth. The use QDX can be motivated using a second order Euler decomposition formula, see Mignacca and Fusai [8]. In other words, our decomposition is not arbitrary and it is valid whenever the risk-measure is homogeneous of order one. In addition, this implies that our measure can promptly be extended to other risk measures, such as Value at Risk and Expected Shortfall.

Finally, our analysis complements the one in Hallerbach [3]. In particular, Hallerbach observes that the orthogonal projection of the weighted asset returns into the space spanned by the portfolio return is linear whenever the returns belong to the class of multivariate elliptical distributions, for which the Gaussian distribution is a member of. This is a rich family of distributions that exhibit heavy tails and can be very useful for capturing extreme events. To be in an elliptical world is very important. Indeed, if we adopt an homogeneous risk-measure such as VaR and Expected Shortfall, the diversification measure is related to the second partial derivatives of the risk function, and from the statistical point of view to conditional covariances that, in the ellipitcal world, can be computed analytically. On the other side, the decomposition of the portfolio volatility in (12) and (13) holds whatever the joint distribution of asset returns: it only requires the homogeneity of the risk measure, and its calculation is based on the partial covariances rather than on conditional covariances[5].

---

[5]Except for the multivariate normal distribution, in the elliptical class conditional covariances are random, depending on the conditioning variable, whilst partial covariances are deterministic.

3   Diversification Parity

Given that the contribution of each asset to the overall portfolio diversification is measured by QDX_j, in the vein of *risk parity portfolios* in Maillard et al. [6] and the *principal portfolios* Meucci [7], we build a portfolio where the diversification contributions are similar across assets: we minimize the cross-section dispersion of the variances of the idiosyncratic risks. A low diversified portfolio is one with idiosyncratic variances concentrated in few assets. We aim to solve

$$\hat{\mathbf{w}}_{QDX} = argmin_{\mathbf{w}} \quad \sum_{j=1}^{n} \left(R_j - \frac{1}{n} \right)^2 \tag{15}$$

where R_j is the ratio between the quantity of diversification relative to the single asset and their sum, i.e.

$$R_j = \frac{QDX_j}{QDX}.$$

The balance and the no short-sell constraints can be also added. In practice, we look for a portfolio allocation such that $R_j = \frac{1}{n}, \forall j$. It is also useful to use the concept of effective number of bets, see Meucci [7], via

$$\mathcal{N}^{(\mathbf{w},\Sigma)} = \exp \left(-\sum_{i=1}^{n} R_j \ln(R_j) \right). \tag{16}$$

$\mathcal{N}^{(\mathbf{w},\Sigma)}$ achieves the maximum value of n when the diversification parity condition holds, and takes the minimum value of 1 for a portfolio fully invested in one asset. Moreover, if the weight of a subset made of m assets is zero, the entropy measure has a maximum value of $n - m$. Therefore, this entropy measure takes values in $[1, n]$ and its value can be interpreted as the effective number of uncorrelated risks we are investing respect to the nominal number of constituents in a portfolio.

4   An Empirical Analysis

The backtesting period goes from February 1, 1995 through December 31, 2019 and considers the S&P 500 index as the investable universe[6]. Only securities that at rebalancing times were part of the benchmark have been considered. To validate the QDX methodology we have also considered other benchmarks such as MSCI Europe, MSCI Japan and MSCI US.

Beside the QDX and EW portfolios, we also consider other popular risk-based portfolio strategies, such as: b) Risk parity (RP), where each position has the same contribution to portfolio risk; c) Inverse volatility (IV) where assets are weighted in inverse proportion to their risk; d) Most Diversified (DR) which consists in maximising the ratio between the weighted average volatilities and the portfolio volatility.

In order to reduce the effect of sampling errors, we estimate the covariance matrix by taking a weighted average of the sample covariance matrix with Sharpe's single index model estimator, as in Ledoit and Wolf [4].

The portfolio is constructed and rebalanced with a semi-annual frequency, at the end of May and at the end of November. The portfolio is always fully invested and cash allocations as well as short positions are not permitted at any time.

From a theoretical point of view, diversification should work by moderating some of the negative returns during adverse markets, though not necessarily eliminating them and this can be relevant to long-term investment success. For this reason, in Table 1, aside traditional performance measures, we also consider conditional performance measures: i.e. we measure the average return and volatility *conditional* to the benchmark experimenting adverse movements, such as going below the 5% empirical percentile.

We can clearly identify a cluster, which contains the QDX, the EW, IV and the RP strategies: their performances have a correlation above 0.99. We can explain this on the basis of formula (7). According to it, the QDX parity portfolio assigns the weights so that $\sigma_{i.r_p}^2$ is the same across assets; the IV portfolio requires $w_i^2 \sigma_i^2$ to be equal across assets and the RP portfolio sets the portfolio weights so that assets have the same γ_i. The EW portfolio simply requires

---

[6]The datasets are sourced and maintained by the Quant Solutions team of Sarasin & Partners in London.

$w_i = 1/N$. In practice, these strategies are focusing on different components of the same formula and it is not a surprise that they have very similar metrics. They also considerably improve with respect to the wide market SP500 index and to the equally weighted portfolio in terms of larger realized return, lower volatility and larger Sharpe ratio, so confirming that we can achieve some benefit deviating from the EW strategy. Moreover, the QDX strategy appears to generate returns that are less leptokurtic (i.e. lower skewness and kurtosis) respect to the other strategies. The lowest turnover is achieved by the inverse volatility and the QDX strategies. Both in-sample and out-of-sample, the QDX strategy turns out to be the most diversified according the effective number of bets, computed either according to Meucci and to formula 16. The DR strategy has a different investment style respect to the other strategies, see Pola [9]. It returns a larger Sharpe ratio, a larger tracking error volatility, a lower Information ratio, a very large turnover. The DR portfolio suffers from high levels of concentration, having the lowest number of effective bets (in and out-of sample). The probability of beating the benchmark over mid-long horizons is significantly lower respect to the other strategies.

The benefits of diversification need to be assessed in the long run, and for this reason the empirical probability of beating the benchmark and the EW portfolio at different horizons (here 1 and 5 years) can be an useful metric: all strategies outperform the EW portfolio. Table 1 also reports performance measures conditional to the benchmark realizing extreme negative returns, i.e. below the 5% empirical percentile. In such a case, the three strategies, QDX, IV and RP, have very similar (conditional) average return, volatility, and a larger probability of beating the benchmark with respect to the EW and DR portfolios.

Table 2 reports performance measures conditional to the benchmarks (MSCI US, MSCI JAPAN and MSCI Europe) realizing very extreme negative return, i.e. below the 1%, 5% and 10% empirical percentiles. In all considered markets, at both an unconditional and conditional level, the QDX portfolio performs better respect to the EW strategy. These results confirm that equally diversifying can give marginal benefits with respect to equally weighting.

Leote de Carvalho et al. [5] and references therein show that different diversification strategies can be viewed as simple active strategies with respect to a few factors. Following the

Table 1: Performance measures of different diversification strategies, considering stocks within the SP500, EW: equally weighted portfolio. MV: global minimum variance portfolio; QDX: QDX parity portfolio; RP: risk parity portfolio; IV: inverse volatility portfolio; DR: maximum diversification portfolio. The backtesting period goes from February 1. 1995 through December 31. 2019.

	EW	QDX	RP	IV	DR	S&P 500
Average Return %	10.70%	10.96%	11.02%	10.92%	11.48%	9.62%
Volatility %	15.12%	14.36%	13.53%	13.96%	13.68%	14.68%
Sharpe Ratio	0.7078	0.7628	0.8145	0.7820	0.8395	0.6553
Skewness	-0.8882	-0.8994	-0.9742	-0.9505	-0.9255	-0.8933
Kurtosis	5.9254	5.8490	6.2121	6.0139	6.1573	4.7607
TEV %	5.51%	5.58%	6.34%	5.75%	12.29%	0.00%
Inf. Ratio	0.1969	0.2400	0.2216	0.2256	0.1518	
Max. DD%	31.12%	30.28%	29.06%	29.61%	28.38%	27.72%
Max. DD/Vol. %	2.0579	2.1078	2.1469	2.1212	2.0750	1.8886
Prob 1y over S&P	51.90%	56.06%	58.48%	59.52%	57.79%	0.00%
Prob 5y over S&P	83.40%	83.82%	83.82%	87.97%	85.06%	0.00%
Prob 1y over EW	0.00%	56.40%	51.90%	53.63%	50.17%	48.10%
Prob 5y over EW	0.00%	80.50%	77.59%	73.86%	80.50%	16.60%
Turnover	7.74	7.32	7.91	7.19	14.21	5.88
Effective Bets (Meucci)	394	395	403	396	234	
Effective Bets (formula 16)	282	387	270	338	15	
SP500 has performance below the 5% percentile						
Conditional Ann. Ret. %	-117.28%	-109.78%	-100.24%	-106.14%	-76.20%	-120.51%
Conditional Vol. %	16.51%	16.07%	17.22%	16.40%	23.54%	10.75%
Prob. Over SP500	46.67%	53.33%	73.33%	60.00%	73.33%	0.00%

Table 2: The Table returns the unconditional and conditional monthly average return. monthly volatility and (empirical) probability of beating the benchmark for the EW and QDX strategies and the benchmark itself. The conditioning is with respect to the benchmark having a return below its empirical percentiles (at level 1% and 5%). The same sample period as in Table has been considered.

Percentile		EW	QDX	S&P 500	EW	QDX	MSUS
Unconditional	Mon. Ret. %	0.89%	0.91%	0.80%	0.75%	0.77%	0.66%
	Mon. Vol. %	4.37%	4.15%	4.24%	4.47%	4.18%	4.11%
	Prob. over bncmk	49.33%	52.33%	0.00%	54.17%	55.09%	0.00%
5%	Mon. Ret. %	-9.77%	-9.15%	-10.04%	-10.58%	-9.93%	-9.91%
	Mon. Vol. %	4.77%	4.64%	3.10%	3.92%	3.86%	3.24%
	Prob. over bncmk	46.67%	53.33%	0.00%	36.36%	54.55%	0.00%
1%	Mon. Ret. %	-15.72%	-14.69%	-15.17%	-16.38%	-15.52%	-15.36%
	Mon. Vol. %	5.30%	5.13%	3.46%	6.98%	6.45%	4.79%
	Prob. over bncmk	66.67%	66.67%	0.00%	50.00%	50.00%	0.00%
		EW	QDX	MSJP	EW	QDX	MSEU
Unconditional	Mon. Ret. %	0.58%	0.59%	0.39%	0.58%	0.60%	0.41%
	Mon. Vol. %	5.01%	4.82%	5.12%	4.63%	4.40%	4.25%
	Prob. over bncmk	57.41%	56.48%	0.00%	60.65%	63.89%	0.00%
5%	Mon. Ret. %	-11.13%	-10.72%	-11.66%	-11.40%	-10.83%	-11.02%
	Mon. Vol. %	4.26%	3.91%	4.23%	2.97%	2.81%	1.96%
	Prob. over bncmk	72.73%	90.91%	0.00%	54.55%	72.73%	0.00%
1%	Mon. Ret. %	-17.74%	-16.71%	-18.85%	-16.02%	-15.03%	-14.26%
	Mon. Vol. %	7.54%	6.80%	6.79%	1.64%	1.63%	0.86%
	Prob. over bncmk	100.00%	100.00%	0.00%	50.00%	50.00%	0.00%

above Authors, we have run a multi-factor regression of the excess returns of each strategy against different Fama-French (FF) factor portfolios (MKT-RF is the market-cap index return minus the U.S. one-month T-Bill rate, HML and SMB are the Value and Size factors, LBMHB and LRVMHRV are the Beta and residual Volatility Factors). The regression coefficients can be interpreted as weights of a multi-factor portfolio that closely tracks excess returns. Table 3 confirms the remarks in [5]: the DR is a defensive strategy with a negative exposure to the market; the remaining strategies (QDX, EW, RP and IV) look very similar in terms of factor exposures and this confirms that they represent a cluster. In conclusion, diversification is a relative concept depending on the dimension we are considering, i.e. asset or factor level, and the portfolio manager has to clarify if his/her aim is to build a balanced portfolio at asset or factor level. Forgetting one dimension, diversification benefits can be overstated.

Table 3: Factor Regression Coefficients for Risk-Based Strategies. Factors series have been downloaded from the French data library. Significance levels at 0.1%, 1% and 5% are marked by a, b and c.

	EW	QDX	RP	IV	DR	S&P 500
Intercept	0.0007	0.0009	0.0010	0.0009	0.0027	-0.0001
MKT-RF	0.0391	0.0288	0.0092	0.0181	-0.0777	0.0052
HML	0.2831^a	0.2681^a	0.2508^a	0.2511^a	0.1515^b	0.0339^b
SMB	0.1273^a	0.1091^a	0.1395^a	0.1202^a	0.1718^b	-0.1116^a
LBMHB	0.1152^a	0.1419^a	0.2196^a	0.1640^a	0.5396^a	0.0301^b
LRVMHRV	0.0499^c	0.0649^b	0.0420^c	0.0677^b	-0.1078^c	0.0691^a
R^2	68%	73%	73%	74%	57%	81%

al. [2] on SSRN.

References

[1] **DeMiguel, V., Garlappi, L., R. Uppal (2009)**, Optimal versus Naive Diversification: How Inefficient Is the 1/N Portfolio Strategy?, The Review of Financial Studies, Vol. 22, No. 5, May 2009, pp. 1915-1953.

[2] **Fusai G., Mignacca D., Nardon A., Human B. (2020)**, Equally weighted or equally diversified?, available at SSRN: https://ssrn.com/abstract=3628585.

[3] **Hallerbach W. G, (2017)**, If you have said A, you must also say B: Calculating diversified asset returns, The Journal of Wealth Management Fall 2017, 20 (2) pp. 76-81.

[4] **Ledoit, O. and M. Wolf (2004)**, A well-conditioned estimator for large-dimensional covariance matrices. Journal of Multivariate Analysis, 88(2), pp. 365–411.

[5] **R. Leote de Carvalho, X. Lu, P. Moulin (2012)**, Demystifying equity risk–based strategies: a simple alpha plus beta description, J. Portfolio Management, 38 (3), pp. 56-70

[6] **Maillard, S., T. Roncalli, and J. Teiletche (2010)**, The properties of equally weighted risk contribution portfolios. Journal of Portfolio Management 36(4), pp. 60–70.

[7] **Meucci, A. (2009)**, Managing Diversification, Risk, May, pp. 74-79.

[8] **Mignacca, D. and G. Fusai (2019)**, Diversify Diversification, working paper.

[9] **Pola, G. (2016)**, On entropy and portfolio diversification, Journal of Asset Management, vol. 17(4), pp. 218-228.