How to generalise demonic composition

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Abstract

Demonic composition is defined on the set of binary relations over the non-empty set X, Rel_X, and is a variant of standard or "angelic" composition. It arises naturally in the setting of the theory of non-deterministic computer programs, and shares many of the nice features of ordinary composition (it is associative, and generalises composition of functions). When equipped with the operations of demonic composition and domain, Rel_X is a left restriction semigroup (like PT_X, the semigroup of partial functions on X), whereas usual composition and domain give a unary semigroup satisfying weaker laws.

By viewing Rel_X under a restricted version of its usual composition and domain as a constellation (a kind of "one-sided" category), we show how this demonic left restriction semigroup structure arises on Rel_X, placing it in a more general context. The construction applies to any unary semigroup with a "domain-like" operation satisfying certain minimal conditions which we identify.

In particular it is shown that using the construction, any Baer $*$-semigroup S can be given a left restriction semigroup structure which is even an inverse semigroup if S is $*$-regular. It follows that the semigroup of $n \times n$ matrices over the real or complex numbers is an inverse semigroup with respect to a modified notion of product that almost always agrees with the usual matrix product, and in which inverse is pseudoinverse (Moore-Penrose inverse).

1 Introduction

Ordinary or "angelic" composition of binary relations is well-known. For $s, t \in Rel_X$, the set of all binary relations on the non-empty set X,

$$s;t = \{(x, y) \in X \times X \mid \exists z \in X : (x, z) \in s, (z, y) \in t\}.$$

This operation of course generalises composition of (partial) functions and shares some of its nice features, for example it is associative. So $(Rel_X, ;)$ is a semigroup, having the semigroup of partial functions on X, $(PT_X, ;)$, as a subsemigroup, with the full transformation semigroup T_X a subsemigroup of PT_X.

However, there is a less familiar way to compose binary relations, so-called "demonic composition". For $s, t \in Rel_X$, define

$$s \circledast t = \{(x, y) \in s; t \mid (x, z) \in s \Rightarrow z \in dom(t)\}.$$

The terminology comes from computer science thought experiments when one thinks about "nondeterministic programs": angelic composition does whatever can be done, while demonic composition assumes a "demon" will ruin things if it is possible for this to happen. This can be made more rigorous: roughly, angelic composition is important when thinking about partial correctness of "non-deterministic" programs, and demonic for total correctness (termination); see for example [4] for more details.

As an example, consider $s, t \in Rel_X$ where $X = \{a, b, c\}$: let

$$s = \{(a, a), (a, b), (b, c)\}, t = \{((a, c), (c, a), (c, b)\}.$$

Then we have that $s; t = \{(a, c), (b, b), (b, a), (c, a)\}$ while $s \circledast t = \{(b, b), (b, a)\}$.

Like angelic composition, demonic composition reduces to standard composition when applied to partial functions. Similarly, when applied to left total binary relations (those having domain all of X), angelic and demonic composition evidently agree. Less obviously, demonic composition is an associative operation. For all these reasons, one can argue that demonic composition is just as worthy a generalisation of functional composition as is angelic composition.

How is the semigroup $(Rel_X, \circledast)$ different to $(Rel_X, ;)$? To help clarify this, consider the operation of domain D on Rel_X. For $s \in Rel_X$, we define

$$D(s) = \{(x, x) \mid x \in dom(s)\},$$

the restriction of the identity function to $dom(s)$. So D is also well-defined on PT_X (and even on T_X, but is rather uninteresting there).

In fact the unary semigroup of partial functions over X equipped with domain, $(PT_X, ;, D)$, is a *left restriction semigroup*. This class of unary semigroups has received considerable attention in recent years, and forms a finitely based variety. It has a "Cayley theorem": every left restriction semigroup embeds in one of the form PT_X.

Now $(Rel_X, ;, D)$ is not a left restriction semigroup, although it does satisfy some weaker laws. However, by contrast $(Rel_X, \circledast, D)$ *is* a left restriction semigroup; see [4] for example. It then follows that left restriction semigroups also have a Cayley theorem in terms of binary relations equipped with demonic composition and domain. (By contrast, no similar Cayley theorem is possible for any finitely axiomatised class of unary semigroups in terms of binary relations under angelic composition and domain; see [10].)

What is the relationship between the usual and demonic compositions on Rel_X? In fact it can be shown that for all $s, t \in Rel_X$,

$$s \circledast t = A(s; A(t)); s; t \text{ where } A(s) = \{(x, x) \in X \times X \mid x \notin dom(s)\}.$$

Here, $A(s)$ is the so-called *antidomain* of s, a unary operation which has been considered in both relational and functional settings: see [16] and [13].

Why does this work – is something more general going on? It is the purpose of this work to show that the answer is “yes” and to apply the general approach to a range of examples. One surprising consequence is that full matrix semigroups over the real or complex numbers have previously unnoticed inverse semigroup structure when the definition of matrix product is adjusted for some pairs of matrices, with inverse given by the Moore-Penrose matrix inverse (which is usual inverse if it exists).

A key concept in this work is that of a constellation (a kind of “one-sided category”). A left restriction semigroup becomes a constellation when one retains the domain operation D and also defines the partial product $s \circ t := st$ if and only if $sD(t) = s$, moreover it gives a so-called inductive constellation as in [7]. It was shown in [7] that the categories of left restriction semigroups and inductive constellations (with suitable morphisms) are isomorphic: left restriction semigroups and inductive constellations are essentially “the same things”.

We could just as well make the above partial product definition in a general unary semigroup and: (a) ask exactly when doing so yields a constellation; (b) characterise those constellations that arise in this way, generalising the correspondence between left restriction semigroups and inductive constellations set out in [7]; and (c) ask exactly when the unary semigroup gives an inductive constellation, meaning that its multiplication can be adjusted by extending the constellation product to be a total operation in a different way to the original so as to give a left restriction semigroup. We address all of (a), (b) and (c) in what follows, but (c) will dominate. In particular, the answer to (c) provides the systematic process by which demonic composition of binary relations comes about, and it also gives rise to unexpected left restriction and even inverse semigroup structure in other familiar examples of semigroups.

2 Preliminaries

2.1 Some properties of idempotents

Throughout this subsection, S is a semigroup with set of idempotents $E(S)$. The *natural right quasiorder* on $E(S)$ is given by $e \leq_r f$ if and only if $e = ef$, and the *natural left quasiorder* on $E(S)$ is given by $e \leq_l f$ if and only if $e = fe$; both are indeed quasiorders. The *natural order* $\leq$ on $E(S)$ is the intersection of $\leq_l$ and $\leq_r$, so that $e \leq f$ if and only if $e = ef = fe$, and is a partial order on $E(S)$, with respect to which 1 is the top element.

Let E be a nonempty subset of $E(S)$. All of the following concepts are defined in [19], although many of them were first defined earlier than that. E is *right pre-reduced* if $\leq_r$ is a partial order on E (that is, for all $e, f \in E$, $e = ef$ and $f = fe$ imply $e = f$), *left pre-reduced* if $\leq_l$ is a partial order, and *pre-reduced* if it is both left and right pre-reduced. E is said to be *right reduced* if $e = ef$ implies $e = fe$, *left reduced* if $e = fe$ implies $e = ef$, and *reduced* if it is both left and right reduced. So E is right reduced if and only if $\leq_r \subseteq \leq_l$ (which happens if and only if $\leq_r$ is the natural order on E), left reduced if and only if the opposite inclusion holds, and reduced if the two quasiorders are equal. Obviously, if E is (left/right) reduced, then it is (left/right) pre-reduced, although the converses fail in general.

2.2 Left restriction semigroups and beyond

The class of left restriction semigroups may be defined to be the class of unary semigroups with unary operation D satisfying the following laws:

- $D(x)x = x$;
- $D(x)D(y) = D(y)D(x)$;
- $D(D(x)y) = D(x)D(y)$;
- $xD(y) = D(xy)x$.

This is the axiomatisation appearing in [7], although other (equivalent) formulations have been used. Other laws follow easily such as

- $D(xy) = D(xD(y))$ (the left congruence condition).

Suppose S is a left restriction semigroup. It follows that $D(S) = \{D(s) \mid s \in S\}$ is a semilattice under multiplication, and that $D(x)$ is the smallest $e \in D(S)$ such that $es = s$. An important example of a left restriction semigroup is the semigroup under composition of partial functions $X \to X$ for some non-empty set X, denoted PT_X and equipped with domain operation given by $D(s) = \{(x, x) \mid x \in dom(s)\}$. Every left restriction semigroup embeds in PT_X for some X, as was first shown in [21], also shown in [11] and [15].

An *inverse semigroup* is a semigroup S in which for every $x \in S$ there is a unique $x' \in S$ satisfying $xx'x = x$ and $x'xx' = x'$. It is well-known that every inverse semigroup S is a left restriction semigroup if one sets $D(s) = ss'$ for all $s \in S$. Conversely, it was shown in Corollary 2.13 of [11] that a left restriction semigroup arises in this way from an inverse semigroup if and only if for all $s \in S$, there is "true inverse" $s' \in S$ such that $D(s) = ss'$ and $D(s') = s's$; then s' is the inverse of s in the inverse semigroup S. Every inverse semigroup embeds in the invberse semigroup I_X for some X, the subsemigroup of PT_X consisting of the injective partial functions on X and equipped with inversion.

A number of weakenings of the left restriction semigroup axioms are often considered. The class of *LC-semigroups* was defined in [11] to consist of those unary semigroups S with unary operation D such that $D(S) = \{D(s) \mid s \in S\}$ is a subsemigroup which is a semilattice, with $D(x)$ the smallest $e \in D(S)$ under the (meet-)semilattice order such that $es = s$. These are given an equational axiomatisation in [11], generalising the above axioms for left restriction semigroups. The same concept was also considered by Batbedat in [2] where they were called type SL γ-semigroups. There is an obvious dual concept of RC-semigroup defined in terms of a range operation R. If S is both an LC-semigroup and an RC-semigroup with $D(S) = R(S)$, we call it a two-sided C-semigroup.

More general still are D-semigroups, introduced in [17]. S is one such if it is a unary semigroup with unary operation D such that $D(S)$ as above consists of idempotents of S with the property that for all $s \in S$, $D(s)$ is the smallest $e \in D(S)$ under the natural order $e \leq f$ if and only if $e = ef = fe$. Letting $D(s)$ be this smallest e, these have a finite equational axiomatisation as a class of unary semigroups, and it follows that $D(S)$ is left reduced. These are defined and studied in [17].

Even more general still are the *generalised D-semigroups* considered in [19], which are unary semigroups with unary operation D which are such that $D(S) = \{D(s) \mid s \in S\}$ consists of idempotents of S with the property that for all $s \in S$, $D(s)$ is the smallest $e \in D(S)$ under the left quasiorder $e \leq_l f$ if and only if $e = fe$ such that $es = s$. These also have a finite equational axiomatisation as a variety of unary semigroups, and $D(S)$ is left pre-reduced, so $\leq_l$ is a partial order. In [19], generalised D-semigroups were defined in terms of a generalised Green's relation that does not concern us here.

A generalised D-semigroup can satisfy the *left congruence condition*: $D(xy) = D(xD(y))$. An example of an LC-semigroup satisfying this law which is not a left restriction semigroup is $(Rel_X, ;, D)$, the semigroup of binary relations under composition and equipped with domain defined as for PT_X.

Two-sided C-semigroups satisfying the left and right congruence conditions were considered in [14], where they were called Ehresmann semigroups. Again, a significant example is $(Rel_X, ;, D, R)$. This leads us to the following.

Definition 2.1 *An LC-semigroup satisfying the left congruence condition is a* left Ehresmann semigroup*.*

2.3 Constellations and left restriction semigroups

Given a set P equipped with a partial binary operation $\circ$, we say $e \in S$ is a *right identity* if for all $a \in P$, if $a \circ e$ exists then it equals a.

Constellations are partial algebras equipped with a partial binary operation and a unary "domain" operation D. Axioms for constellations can be given as follows. For all $x, y, z \in P$,

(C1) if $x \circ (y \circ z)$ exists then so does $(x \circ y) \circ z$, and then the two are equal;

(C2) if $x \circ y$ and $y \circ z$ exist then $x \circ (y \circ z)$ exists;

(C3) for each $x \in P$, $D(x)$ is the unique right identity in P such that $D(x) \circ x = x$.

These are the laws obtained in [8]. The second law can be strengthened to an "if and only if", resulting in one of the original defining axioms given in [7]. It also follows that $D(P) = \{D(s) \mid s \in P\}$ is the set of all right identity elements in P, and that $e \circ e$ exists (hence equals e) for all $e \in D(P)$.

The following is noted in [7].

Proposition 2.2 *If P is a constellation then for $s, t \in P$, $s \circ t$ exists if and only if $s \circ D(t)$ does.*

There is a natural quasiorder on $D(P)$ given by $e \leq f$ if and only if $e \circ f$ exists. This extends to the natural quasiorder on P given by $s \leq t$ if and only if $D(s) \circ t$ exists and equals s, or equivalently, $s = e \circ t$ for some $e \in D(P)$. As noted in [8], if one of these quasiorders is a partial order, so is the other, and then P is said to be *normal*, a condition evidently equivalent to the assertion that for all $e, f \in D(P)$, if $e \circ f$ and $f \circ e$ exist then $e = f$. In this case, we call $\leq$ the *natural order* on P.

The relevant notion of morphism for constellations is that of a *radiant*, defined in [7]. If P, Q are constellations, then the function $\rho : P \to Q$ is said to be a radiant if

1. for all $s, t \in P$, if $s \circ t$ exists, then so does $\rho(s) \circ \rho(t)$, and indeed $\rho(s \circ t) = \rho(s) \circ \rho(t)$, and
2. for all $s \in P$, $\rho(D(s)) = D(\rho(s))$.

In [7], it was noted that a left restriction semigroup (such as PT_X) can be made into a constellation by defining $s \circ t := st$ but only when $sD(t) = s$ (which says that the range of s is contained in the domain of t), and by retaining D. (The natural order in the left restriction semigroup then coincides with the natural order on the derived constellation.) A constellation arises from a left restriction semigroup in this way if and only if it is inductive in the sense of [7], as is shown there.

The inductive property was shown in [8] to be equivalent to the following conditions on a constellation P.

> (O4) If $e \in D(P)$ and $a \in P$, then there is a maximum $x \in P$ with respect to the natural quasiorder $\leq$ on P, such that $x \leq a$ and $x \circ e$ exists, the *co-restriction* of a to e, denoted $a|e$; and
>
> (O5) for $x, y \in P$ and $e \in D(P)$, if $x \circ y$ exists then $D((x \circ y)|e) = D(x|(D(y|e))$.

Every inductive constellation is normal. Note that the co-restriction $a|e$ in (O4) is fully determined (if it exists) by the natural order $\leq$ on P.

We next present a new characterisation of the inductive property that is especially useful in what follows.

Lemma 2.3 *The constellation $(P, \circ, D)$ is inductive if and only if it is normal and for all $s \in P$ and $e \in D(P)$, there exists $s \cdot e \in D(P)$ such that for all $t \in P$, $(t \circ s) \circ e$ exists if and only if $t \circ (s \cdot e)$ exists. In that case, the co-restriction $s|e$ is $(s \cdot e) \circ s$ for all $s \in P$ and $e \in D(P)$.*

Proof. Suppose P is inductive. Then $D(P)$ is normal. Let $s \cdot e = D(s|e)$, so that $s|e = (s \cdot e) \circ s$ (since $s|e \leq s$) and $((s \cdot e) \circ s) \circ e$ exists. If $t \in P$ is such that $(t \circ s) \circ e$ exists, then $D(t) = D(t \circ s) = D((t \circ s) \circ e)) = D(t|D(s|e)) = D(t|(s \cdot e)) = D((t \cdot (s \cdot e)) \circ t) = t \cdot (s \cdot e)$, so $t = D(t) \circ t = (t \cdot (s \cdot e)) \circ t = t|(s \cdot e)$, so $t \circ (s \cdot e)$ must exist. Conversely, if $t \circ (s \cdot e)$ exists, then because $s|e = (s \cdot e) \circ s$

exists, $t \circ (s|e) = t \circ ((s \cdot e) \circ s) = (t \circ (s \cdot e)) \circ s = t \circ s$ exists, but also $(s|e) \circ e$ exists, and so $t \circ (s|e) \circ e) = (t \circ (s|e)) \circ e = (t \circ s) \circ e$ exists. So $(t \circ s) \circ e$ exists if and only if $t \circ (s \cdot e)$ exists.

Conversely, suppose that for all $s \in P$ and $e \in D(P)$, there exists $s \cdot e \in D(P)$ such that for all $t \in P$, $(t \circ s) \circ e$ exists if and only if $t \circ (s \cdot e)$ exists. Note that $(s \cdot e) \circ D(s)$ exists, as is clear on letting $t = s \cdot e$: $(s \cdot e) \circ (s \cdot e)$ exists so $((s \cdot e) \circ s) \circ e$ and in particular $(s \cdot e) \circ s$ exists. For $a \in P$ and $e \in D(P)$, we show that $a|e = (a \cdot e) \circ a$ is the co-restriction of a to e. By assumption, $a|e \leq a$ and $(a|e) \circ e = ((a \cdot e) \circ a) \circ e$ exists because $(a \cdot e) \circ (a \cdot e)$ obviously exists. If $x \leq a$ and $x \circ e$ exists, then $x = D(x) \circ a$ and $(D(x) \circ a) \circ e$ exists, so $D(x) \circ (a \cdot e)$ exists, so $x = D(x) \circ a = (D(x) \circ (a \cdot e)) \circ a = D(x) \circ ((a \cdot e) \circ a) = D(x) \circ (a|e)$, so $x \leq a|e$. This establishes (O4).

For (O5), suppose $x, y \in P$ are such that $x \circ y$ exists. We must show that $D((x \circ y)|e) = D(x|(D(y|e))$. Recalling that $a|e = (a \cdot e) \circ a$ for all $a \in P$ and $e \in D(P)$, we have that

$$D((x \circ y)|e) = D(((x \circ y) \cdot e) \circ (x \circ y)) = D((x \circ y) \cdot e) = (x \circ y) \cdot e,$$

while

$$D(x|(D(y|e)) = D((x \cdot D((y \cdot e) \circ y)) \circ x) = D(x \cdot D(y \cdot e)) = x \cdot (y \cdot e).$$

So we must show that $f = (x \circ y) \cdot e = x \cdot (y \cdot e) = g$. To do this we show $f \leq g$ and $g \leq f$, or that both $f \circ g$ and $g \circ f$ exist.

But by assumption, $((x \circ y) \cdot e) \circ (x \cdot (y \cdot e))$ exists if and only if $((((x \circ y) \cdot e) \circ x) \circ (y \cdot e)$ exists, which exists if and only if $(((x \circ y) \cdot e) \circ x) \circ y) \circ e$ exists, which, because $x \circ y$ exists, itself exists if and only if $(((x \circ y) \cdot e) \circ (x \circ y)) \circ e$ exists, which exists if and only if $((x \circ y) \cdot e) \circ ((x \circ y) \cdot e)$ exists, that is, $f \circ f$ exists, which it does. So $f \circ g$ exists.

Similarly, because $(x \cdot (y \cdot e)) \circ x$ exists (since $(x \cdot h) \circ x = x|h$ exists for all $h \in D(P)$), and $x \circ y$ exists, we have that $(x \cdot (y \cdot e)) \circ (x \circ y)$ exists, and so $g \circ f = (x \cdot (y \cdot e)) \circ ((x \circ y) \cdot e)$ exists if and only if $((x \cdot (y \cdot e)) \circ (x \circ y)) \circ e$ exists, if and only if $(((x \cdot (y \cdot e)) \circ x) \circ y) \circ e$ exists, if and only if $((x \cdot (y \cdot e)) \circ x) \circ (y \cdot e)$ exists, if and only if $g \circ g = (x \cdot (y \cdot e)) \circ (x \cdot (y \cdot e))$ exists, which it does. □

When a left restriction semigroup S is viewed as a constellation $(S, \circ, D)$ as above, then the latter is inductive, and for $s, t \in S$, $s \circ t = st$ whenever the former is defined and $s|e = se$ for all $e \in D(S)$. Conversely, it was shown in [7] that an inductive constellation P can be made into a left restriction semigroup by retaining D and extending the partial multiplication to an everywhere-defined one by defining, for all $s, t \in P$, $s \otimes t := (s|D(t)) \circ t$. Indeed it was shown there that the categories of inductive constellations and left restriction semigroups (with the natural notions of morphisms for both) are isomorphic.

3 The most general unary semigroups that give constellations

In this section, our aim is to find the most general unary semigroup that can give rise to a constellation, in which the constellation product is a restricted version of the semigroup product, defined by analogy with the left restriction semigroup case as follows.

Definition 3.1 *Let S be a unary semigroup with unary operation D. Define the partial product $\circ$ as follows: for all $s, t \in S$,*

$$s \circ t := st \text{ whenever } sD(t) = s,$$

the restricted product *on S.*

This is exactly how one defines the constellation product in a left restriction semigroup, but also for the more general types of unary semigroups as in [18].

Some observations. Given a unary semigroup $(S, \cdot, D)$, if $(S, \circ, D)$ is to be a constellation, it must be such that the existence of $a \circ b$ coincides with the existence of $a \circ D(b)$ (using Proposition 2.2), and when the latter exists, it should equal a. This forces $a \circ D(b)$, hence also $a \circ b$, to exist if and only if $aD(b) = a$ in the unary semigroup. Since $D(a) \circ a$ exists and equals a, necessarily $D(a)D(a) = D(a)$, and $D(a)a = a$. But since also $D(a)$ is the unique element of the form $D(x)$ for which $D(x) \circ a$ exists and equals a, it follows that if $D(x)D(a) = D(x)$ and $D(x)a = a$, then $D(x) = D(a)$. Then it is also necessary to satisfy the first two constellation laws, namely

- if $a \circ b$ and $b \circ c$ exist, then so does $a \circ (b \circ c)$;
- if $a\dot{(}b \circ c)$ exists then so does $(a\dot{b}) \circ c$ and the two are equal.

We therefore make a first-pass definition of our most general unary semigroups.

Definition 3.2 *We say S is a* demigroup *if it is a unary semigroup with unary operation D for which*

1. *$D(x)^2 = D(x)$;*
2. *$D(x)x = x$;*
3. *$D(x)D(y) = D(x)$ and $D(x)y = y$ imply $D(x) = D(y)$;*
4. *if $xD(y) = x$ and $yD(z) = y$ then $xD(yz) = x$;*
5. *if $xD(yz) = x$ and $yD(z) = y$ then $xD(y) = x$.*

We have just argued that these various laws are necessary, but in fact they are sufficient.

Proposition 3.3 *If S is a demigroup and we define the restricted product $\circ$ as indicated above, then $(S, \circ, D)$ is a constellation.*

Proof. The final two laws spell out the first two constellation laws. It remains to show that for each $x \in S$, $D(x)$ is the unique right identity e in the constellation for which $e \circ x$ exists and equals x.

Suppose $x \in S$. Then $D(x) \circ x$ exists beause $D(x)^2 = D(x)$, and equals $D(x)x = x$. Suppose e is another right identity for which $e \circ x$ exists and equals x. Since e is a right identity, it is such that if $a \circ e$ exists for some $a \in S$, then it equals a, so if $aD(e) = a$ then $ae = a$. Now $D(e)D(e) = D(e)$, so neessarily $D(e)e = D(e)$, But $D(e)e = e$, so $D(e) = e$. Now $e \circ x$ exists so $eD(x) = e$, so $D(e)D(x) = D(e)$, and also $e \circ x = x$ so $D(e)x = ex = x$, so by the third law, $e = D(e) = D(x)$, as required. □

The above laws may be streamlined considerably.

Proposition 3.4 *The unary semigroup $(S, \cdot, D)$ is a demigroup if and only if it satisfies the following equational laws.*

1. *$D(x)^2 = D(x)$;*
2. *$D(x)x = x$;*
3. *$D(xy) = D(xD(y))$ (left congruence condition).*

In this case, $D(D(x)) = D(x)$ for all $x \in S$.

Proof. Assume S is a demigroup. The first two laws above are immediate. For the third, first note that for all $a, b \in S$, $D(aD(b))D(aD(b)) = D(aD(b))$ and $aD(b)D(b) = aD(b)$, so letting $x = D(aD(b))$ and $y = aD(b)$, we have that $xD(y) = x$ and $yD(z) = y$, so we may infer from the fourth demigroup law that $xD(yz) = x$, that is, $D(aD(b)) = D(aD(b))D(aD(b)b) = D(aD(b))D(ab)$. So $D(xD(y))D(xy) = D(xD(y))$ is a law. But then for all x, y, z, $D(xD(y))xy = D(xD(y))xD(y)y = xD(y)y = xy$. So from the third law for demigroups, we have $D(xD(y)) = D(xy)$.

Conversely, suppose the above laws hold on S. Then for $x \in S$, $D(x) = D(D(x)x) = D(D(x)D(x)) = D(D(x))$, as claimed. The first two laws for demigroups are immediate. For the third, suppose $D(x)D(y) = D(x)$ and $D(x)y = y$. Then $D(x) = D(D(x)) = D(D(x)D(y)) = D(D(x)y) = D(y)$.

Next suppose $xD(y) = x$ and $yD(z) = y$. Then $xD(yz) = xD(yD(z)) = xD(y) = x$. So the fourth law holds.

For the fifth law, if $xD(yz) = x$ and $yD(z) = y$ then $xD(y) = xD(yD(z)) = xD(yz) = x$. □

So a demigroup is simply a semigroup equipped with a projection D into its set of idempotents such that $D(x)x = x$ and $D(xy) = D(xD(y))$ for all x, y. Consistent with previous usage, the last of these laws is often referred to as the left congruence condition, since for unary semigroups with unary operation D, the condition is equivalent to the fact that the relation ρ given by $x\rho y$ if and only if $D(x) = D(y)$ is a left congruence.

Definition 3.5 *If $(S, \cdot, D)$ is a demigroup, then $(S, \circ, D)$, where $\circ$ is the restricted product, is the* derived constellation*.*

Example 3.6 *Not all demigroups are generalised D-semigroups.*

Not every generalised D-semigroup is a demigroup since the former need not satisfy the left congruence condition $D(xy) = D(xD(y))$, and the converse fails also. Let S be a right zero semigroup and define $D(x) = x$ for all $x \in S$. This evidently satisfies the laws (as does any band in which one defines $D(s) = s$ for all s), but $D(S) = S$ is not left pre-reduced. However, it is still the case that for $x \in S$, $D(x)$ is *one of* the smallest $g \in D(S)$ (with respect to $\leq_l$) for which $gx = x$, since $e \leq_l f$ for all $e, f \in S = D(S)$. For an example where this fails, consider the semigroup $S = \{e, f, a\}$ with Cayley table below.

$\cdot$	e	f	a
e	e	a	a
f	e	f	a
a	e	a	a

.

Define $D(a) = f$, $D(e) = e, D(f) = f$. The result is easily seen to be a demigroup. Then $D(S) = \{e, f\}$ has $e \leq_l f$, and $ea = fa = a$ yet $D(a) = f$, so $(S, \cdot, D)$ cannot be a generalised D-semigroup.

An important possible property of a constellation is normality, defined earlier. Every inductive consatellation is normal, and more generally, by a result in [8] normal constellations are precisely those which can be faithfully embedded in the derived constellation of PT_X. The following is easily seen.

Proposition 3.7 *If S is a demigroup then its derived constellation is normal if and only if $D(S)$ is right pre-reduced.*

4 Digression: when is a constellation the derived constellation of a demigroup?

Before moving on to determining exactly when the derived constellation of a demigroup is inductive, we pause to characterise those constellations that are derived constellations of demigroups. This would then provide the most general possible result along the lines of the results given in [7] (for left restriction semigroups) and then [18] (for D-semigroups). Since this is not our main focus, proofs (which follow the earlier cases fairly closely) are either briefly presented or omitted entirely, with a reference given to the earlier cases.

In the previously considered less general cases, such a characterisation has been achieved via the notion of co-restriction on a constellation as in [18], and the same applies here.

Definition 4.1 *A* generalised co-restriction constellation *is a constellation P equipped with an operation of co-restriction $|: P \times D(P) \to P$, satisfying the following laws: for all $s, t \in S$ and $e, f \in D(P)$:*

(GR1) $s \circ e$ exists if and only if $s|e = s$;

(GR2) if $e|f = f$ then $(s|e)|f = s|f$;

(GR3) $(s|D(t|e)) \circ (t|e) = ((s|D(t)) \circ t)|e$.

Compared to (R1)–(R5) in the definition of a co-restriction constellation in [17], where the derived constellations of D-semigroups were characterised, (GR1), and (GR2) are exactly (R1) and (R3) respectively, (GR3) fulfils a role similar to that of (R4) which is no longer assumed and also implies (R5), and no version of (R2) holds or is needed.

The proof of the following is routine.

Proposition 4.2 *If S is a demigroup then $\mathbf{P}(S)$ is a generalised co-restriction constellation, with co-restriction defined by $s|e = se$ for all $s \in S$ and $e \in D(S)$.*

The proof of the following is formally exactly the same as the proof of Lemma 4.4 in [18].

Lemma 4.3 *If P is a generalised co-restriction constellation, then for all $e \in D(P)$ and $s \in P$, $e|e = e$ and $(s|e) \circ e$ exists.*

Likewise, the pattern of the following proof follows that of Proposition 4.5 in [18], but we include it here because there are some small differences.

Proposition 4.4 *Every generalised co-restriction constellation P determines a demigroup $\mathbf{T}(P)$ with multiplication given by $x \otimes y := (x|D(y)) \circ y$ and with domain operation D the same as in P, and the derived constellation of $\mathbf{T}(P)$ is P.*

Proof. Note first that $\otimes$ is everywhere-defined. We show it is associative. For all $a, b, c \in S$,

$$\begin{aligned} a \otimes (b \otimes c) &= a \otimes ((b|D(c)) \circ c) \\ &= [a|(D(b|D(c)) \circ c)] \circ [(b|D(c)) \circ c] \\ &= [(a|D(b|D(c)) \circ (b|D(c))] \circ c \text{ by (C1)} \end{aligned}$$

whereas $(a \otimes b) \otimes c = ((a|D(b)) \circ b) \otimes c = (((a|D(b) \circ b)|D(c)) \circ c$. So it suffices to show that

$$(a|D(b|D(c))) \circ (b|D(c)) = (a|D(b) \circ b)|D(c).$$

But this is immediate from (GR3).

Clearly $D(a) \otimes a = (D(a)|D(a)) \circ a = D(a) \circ a = a$, $D^2(a) = D(a)$ for all $a \in S$, and

$$D(a \otimes b) = D((a|D(b)) \circ b) = D(a|D(b)) = D((a|D(b)) \circ D(b)) = D(a \otimes D(b))$$

for all $a, b \in S$.

Finally, for $a, b \in \mathbf{T}(P)$, their restricted product exists if and only if $a = a \otimes D(b) = (a|D(D(b))) \circ D(b) = a|D(b)$, but then this implies $a \circ D(b)$ exists by (GR1), and so $a \circ b$ exists, and in this case, both agree with $a \otimes b$. So the derived constellation of $\mathbf{T}(P)$ is nothing but P. □

Remaining facts now follow as in [18], and before that, [7]. In particular, the following makes use of (GR1)–(GR3) and the above proposition.

Proposition 4.5 *If S is a demigroup and P a generalised co-restriction constellation, then $\mathbf{T}(\mathbf{P}(S)) = S$ and $\mathbf{P}(\mathbf{T}(P)) = P$.*

To obtain a category isomorphism, we define the relevant notion of morphism for generalised co-restriction constellations. These are nothing but radiants $\rho : P \to Q$ additionally satisfying $\rho(s|e) = \rho(s)|\rho(e)$ for all $s \in P$ and $e \in D(P)$. That this yields a category is because the composite of co-radiants is a co-radiant; the elementary reasoning for this is as in [18] for the case considered there. The proof of the following is very similar to that of Theorem 4.8 in [18], and prior to that, Theorem 4.7 in [7].

Theorem 4.6 *The category of demigroups is isomorphic to the category of generalised co-restriction constellations and co-radiants.*

5 Demonization

We now turn to the main purpose of the current work.

In the previous section we found a type of constellation that corresponded $1:1$ with demigroups, where the constellation product is just the restricted product on the demigroup. However, we could also ask: given a demigroup S, when is the derived constellation inductive? For then, there is a derived left restriction semigroup structure definable on S as well.

If S is a demigroup, recalling that the restricted product $s \circ t$ exists in the constellation $(S, \circ, D)$ if and only if $sD(t) = s$, we immediately obtain the following consequence of Lemma 2.3 and Proposition 3.7.

Theorem 5.1 *Suppose S is a demigroup. Then the derived constellation $(S, \circ, D)$ is inductive if and only if $D(S)$ is right pre-reduced and for all $t \in S$ and $e \in D(S)$, there exists $t \cdot e \in D(S)$ such that for all $s \in S$, $ste = st$ and $sD(t) = s$ if and only if $s(t \cdot e) = s$, and then the co-restriction $s|e = (s \cdot e) \circ s = (s \cdot e)s$ for all $s \in S$ and $e \in D(S)$.*

This leads us to the following.

Definition 5.2 *A demigroup S is* inductive *if and only if the derived constellation is.*

So from Theorem 5.1, the demigroup S is inductive if and only if $D(S)$ is right pre-reduced and there is an "action" $\cdot : S \times D(S) \to D(S)$ of S on $D(S)$, such that for all $s, t \in S$ and $e \in D(S)$, $ste = st$ and $sD(t) = s$ if and only if $s(t \cdot e) = s$.

Definition 5.3 *Given an inductive demigroup S, extending the restricted product to a left restriction semigroup product gives an operation denoted by $\circledast$, and we denote by S_D the left restriction semigroup $(S, \circledast, D)$ itself, the* demonization *of S.*

Evidently, if S is an inductive demigroup, then S and S_D have the same derived constellation.

Applying Theorem 5.1 immediately gives the following.

Corollary 5.4 *If $(S, \times, D)$ is an inductive demigroup, then in the demonization S_D,*

$$s \circledast t = (s \cdot D(t))st \text{ for all } s, t \in S.$$

Before turning to examples, we list some elementary properties of multiplication in the demonization of an inductive demigroup.

Proposition 5.5 *Let S be an inductive demigroup.*

1. *For all $s, t \in S$, $D(s \circledast t) = s \cdot D(t)$.*
2. *For all $s, t \in S$, $s \circledast t = st$ if and only if $s \cdot D(t) = D(st)$. In that case, $sD(t) = D(st)s$.*
3. *For all $s, t \in S$, if $D(s) \leq_r D(t)$ then $D(s) \circledast t = D(s)t$.*
4. *If $D(S)$ is right reduced and for all $s, t \in S$, $D(st)D(s) = D(st)$ (for example, if S is left Ehresmann), then for all $s, t \in S$, $sD(t) = D(st)s$ if and only if $s \circledast t = st$.*

Proof. For 1, we have

$$\begin{aligned}
D(s \circledast t) &= D((s \cdot D(t))st) \\
&= D((s \cdot D(t))sD(t)) \\
&= D((s \cdot D(t))s) \\
&= D((s \cdot D(t))D(s)) \\
&= D(s \cdot D(t)) \\
&= s \cdot D(t),
\end{aligned}$$

as claimed.

For 2, if $st = s \circledast t$, then $st = (s \cdot D(t))st$, so $D(st) = D(s \circledast t) = s \cdot D(t)$ by the third part. Conversely, if $(s \cdot D(t)) = D(st)$, then $s \circledast t = (s \cdot D(t))st = D(st)st = st$. Under these equivalent conditions,

$$\begin{aligned} sD(t) &= D(sD(t))sD(t)) \\ &= D(st)sD(t) \\ &= (s \cdot D(t))sD(t) \\ &= (s \cdot D(t))s \\ &= D(st)s. \end{aligned}$$

For 3, if $D(s) \leq_r D(t)$ then $D(s) \cdot D(t) = D(s)$ and so $D(s) \circledast t = (D(s) \cdot D(t))D(s)t = D(s)D(s)t = D(s)t$.

Finally for 4, note that if the given conditions are assumed, and $sD(t) = D(st)s$, then because $D(st)sD(t) = D(sD(t))sD(t) = sD(t) = D(st)s$, so because $D(st)D(s) = D(st)$, we have that $D(st) \leq_r s \cdot D(t)$ and so $D(st)(s \cdot D(t)) = D(st)$, so $D(st) = (s \cdot D(t))D(st)$, so

$$\begin{aligned} s \circledast t &= (s \cdot D(t))st \\ &= (s \cdot D(t))D(st)st \\ &= D(st)st \\ &= st. \end{aligned}$$

□

We now compile some further facts about inductive demigroups.

Proposition 5.6 *Let S be an inductive demigroup.*

1. *For all $t \in S$ and $e \in D(S)$, $t \cdot e$ is the largest $f \in D(S)$ under $\leq_r$ such that $fte = ft$ and $f \leq_r D(t)$.*
2. *For all $e, f \in D(S)$, $e \cdot f$ is their meet $e \wedge f$ under $\leq_r$.*
3. *If $s \in S$ with $e, f \in D(S)$, and if $se = sf = s$, then $s(e \cdot f) = s$.*

Proof. For 1, let $s = f$ in Theorem 5.1.

For 2, note that $(e \cdot f) \leq_r D(e) = e$, so $e \cdot f \leq_r e$, and moreover $(e \cdot f)f = (e \cdot f)ef = (e \cdot D(f))eD(f) = (e \cdot D(f))e = e \cdot f$. So $e \cdot f$ is a lower bound of e, f. If $g \leq_r e, f$, then $ge = gef$ so by the first part, $g \leq_r e \cdot f$, so $e \cdot f$ is the greatest such lower bound.

For 3, suppose $se = sf = s$. Then $sef = se$ so $s(e \cdot f) = s$. □

The first part of this last result shows that a semigroup can be inductive in at most one way, because $t \cdot e$ is determined by the ordering $\leq_r$ on $D(S)$.

One can give an equational characterisation of inductive demigroups.

Theorem 5.7 *Let S be a demigroup. Then it is inductive if and only if for all $s \in S$ and $e \in E$, there is $s \cdot e \in S$ satisfying, for all $s, t \in S$ and $e, f \in D(S)$:*

1. $D(s \cdot e) = s \cdot e$,

2. $se \cdot e = D(se)$,

3. $e \cdot f = f \cdot e$,

4. $(t \cdot e)t = (t \cdot e)te$ and

5. $(sD(t)) \cdot (t \cdot e) = (st) \cdot e$.

Proof. Suppose S is an inductive demigroup. The only one of the above laws which is not immediate either from the definition or from relevant parts of Proposition 5.6 is the last. But for this, it follows from 1 in Proposition 5.6 that the following are equivalent for $s, t \in S$ and $e, f \in D(S)$:

$$\begin{array}{lll} f & \leq_r & (sD(t)) \cdot (t \cdot e); \\ f & \leq_r & D(sD(t)) \text{ and } fsD(t)(t \cdot e) = fsD(t); \\ f & \leq_r & D(st) \text{ and } fsD(t)te = fsD(t)t; \\ f & \leq_r & D(st) \text{ and } fste = fst; \\ f & \leq_r & (st) \cdot e. \end{array}$$

Hence $(sD(t)) \cdot (t \cdot e) = (st) \cdot e$. So we turn to the converse: suppose the demigroup S satisfies the above laws 1 to 5.

First, if $e, f \in D(S)$ are such that $e \cdot f = e$, then $ef = eef = (e \cdot f)ef = (e \cdot f)e = ee = e$, using Law 4, and conversely if $ef = e$ then $e = D(e) = D(ef) = (ef) \cdot f = e \cdot f$ using Law 2. Hence if $e, f \in D(S)$ are such that $ef = e, fe = f$ then $e = e \cdot f = f \cdot e = f$ using what was just shown and Law 3, so $D(S)$ is right pre-reduced.

Next, suppose $x \in S$ and $e \in D(S)$. Using Laws 3 and 5,

$$(x \cdot e) \cdot D(x) = D(x) \cdot (x \cdot e) = (D(x)D(x)) \cdot (x \cdot e) = (D(x)x) \cdot e = x \cdot e,$$

so $(x \cdot e)D(x) = x \cdot e$ from what was shown in the previous paragraph.

Now suppose $x \in S$ and $e \in D(S)$. If $t \in S$ is such that $txe = tx$ and $tD(x) = t$ then

$$t \cdot (x \cdot e) = (tD(x)) \cdot (x \cdot e) = (tx) \cdot e = (txe) \cdot e = D(txe) = D(tx) = D(tD(x)) = D(t)$$

using Laws 2, 4 and 5, so

$$t(x \cdot e) = D(t)t(x \cdot e) = (t \cdot (x \cdot e))t(x \cdot e) = (t \cdot (x \cdot e))t = D(t)t = t$$

upon using Law 4. Conversely, if $t(x \cdot e) = t$ then $tD(x) = t(x \cdot e)D(x) = t(x \cdot e) = t$, and moreover $txe = t(x \cdot e)xe = t(x \cdot e)x = tx$ using Law 4. So by Theorem 5.1, S is inductive. □

The laws given above can be turned into equational laws on the whole of S by simply defining $s \cdot t := s \cdot D(t)$ for all $s, t \in S$. In this way, the class of inductive demigroups can be viewed as a variety of algebras with two binary operations and one unary operation.

Note that Example 3.6 is not inductive since there is no $g \in D(S)$ such that $gae = ga$.

Example 5.8 *A small inductive demigroup.*

Consider the monoid $S = \{1, e, f, g, s\}$, given by the following multiplication table.

$\times$	e	1	f	g	s
e	e	e	e	e	e
1	e	1	f	g	s
f	e	f	f	e	e
g	s	g	s	g	s
s	s	s	s	s	s

.

It is routine to check that $T = \{e, f, g, s\}$ is a subalgebra which is a semigroup, and so S is T with adjoined identity element. It is also routine to check that S is an inductive demigroup in which $D(S) = \{1, e, f, g\}$, which forms a lattice with respect to $\leq_r$ in which $e \leq_r f, g \leq_r 1$, with $D(s) = g$, and the action of S on $D(S)$ is as follows:

$\cdot$	e	1	f	g
e	e	e	e	e
1	e	1	f	g
f	e	f	f	e
g	e	g	e	g
s	g	g	g	g

.

S_D can be formed, and has the following multiplication table.

$\circledast$	e	1	f	g	s
e	e	e	e	e	e
1	e	1	f	g	s
f	e	f	f	e	e
g	e	g	e	g	s
s	s	s	s	s	s

.

The only change from the multiplication table for S is that $g \circledast e = g \circledast f = e$.

6 Two families of inductive demigroups

6.1 Special case: domain semirings

Definition 6.1 *A* domain semiring $(S, +, \times, D)$ *is an algebra in which*

- $(S, \times, D)$ *is a left Ehresmann semigroup with zero* 0 *and identity* 1 *both in* $D(S)$*,*

- $(S, +, \times)$ *is an idempotent semiring with bottom element* 0 *(so* $s + s = s$ *and* $s + 0 = s$ *for all* $s \in S$*), and*
- $(D(S), +, \times)$ *is a Boolean algebra (and it follows* 0 *is the bottom element and* 1 *the top).*

This is equivalent to the definition of Boolean domain semirings given in [5], for example.

It is well-known that $(Rel_X, \cup, \times, D)$ is a domain semiring. Denote by e' or $A(e)$ the complement of $e \in D(S)$; if $S = Rel_X$, this is the identity map on the complement of the domain of the restriction of the identity map e. Extend to give antidomain A on all of S by defining $A(s) = A(D(s))$, which on Rel_X is the identity map on the complement of the domain of $s \in Rel_X$. Note that demonic composition can be defined to be $s \circledast t = A(sA(t))st$ for all $s, t \in Rel_X$, as noted in [4], for example (see Definitions 5 and 8 there). In general we have the following.

Proposition 6.2 *Every domain semiring* S *is an inductive demigroup with*

$$s \cdot e := A(sA(e))D(s) \text{ for all } s \in S, e \in D(S).$$

Hence $S_D = (S, \circledast, D)$ *is a left restriction semigroup in which* $s \circledast t = A(sA(t))st$ *for all* $s, t \in S$.

Proof. Certainly S is a demigroup. For $s, t \in S$ and $e \in D(S)$, if $ste = st$ and $sD(t) = s$ then $D(sD(te')) = D(ste') = D(stee') = D(0) = 0$, so $sD(te') = 0$, and so

$$s = s1 = s(D(te')+A(te')) = sD(te')+sA(te') = 0+sA(te') = sD(t)A(te') = s(t \cdot e).$$

Also,

$$(s \cdot e)s = A(se')D(s)s1 = A(se')s(e+e') = A(se')s+A(se')se' = A(se')D(s)se+0 = (s \cdot e)se.$$

So S is inductive.

It now follows that $S_D = (S, \circledast, D)$ a left restriction semigroup, in which for all $s, t \in S$, $s \circledast t = (s \cdot D(t))st = A(sA(t))D(s)st = A(sA(t))st$ as claimed. □

In the case of the domain semiring $(Rel_X, \cdot, D)$, this yields the familiar demonic structure on Rel_X, and we have an alternative proof that $(Rel_X, \circledast, D)$ is a left restriction semigroup. In particular, this furnishes an alternative proof of the associativity of demonic composition, a non-trivial fact; for example, see the much-cited technical report [1] due to Backhouse. This example motivates our general use of the term "demonization" for S_D.

6.2 Special case: Baer $*$-semigroups

Another way to generalise Rel_X equipped with angelic composition and domain is via Baer $*$-semigroups.

Recall that an involuted semigroup is a semigroup S with a unary operation of involution *, satisfying $x^{**} = x$ and $(xy)^* = y^*x^*$. The set of projections $E^*(S) \subseteq E(S)$ in S is the set $E^*(S) = \{e \in E(S) \mid e^* = e\}$.

A *Baer $*$-semigroup* is an involuted semigroup with zero $(S, \cdot, ^*)$ in which the left annihilator $(0 : s) = \{x \in S \mid xs = 0\}$ of every $s \in S$ has a (necessarily unique) generator as a left ideal that is a projection; call this $A(s)$. So $A(s)^2 = A(s)^* = A(s)$, and $ts = 0$ if and only if $t = tA(s)$; that is, $(0 : s) = SA(s)$. The involution ensures that there is an equivalent dual definition in terms of right annihilators and right ideals. It is also well-known that S is a monoid in which $0^* = 0$ and $1^* = 1$, and that the set of elements of $A(S) = \{A(s) \mid s \in S\} \subseteq E^*(S)$ is an orthomodular lattice with respect to the natural order given by $e \leq f$ if and only if $e = ef = fe$, with top element 1, bottom 0, and $e \wedge f = eA(eA(f))$ for all $e, f \in A(S)$; see [6] where the dual right-sided version is developed. Because $e, A(eA(f)) \in E^*(S)$, it follows that $e \wedge f = A(eA(f))e$.

$(Rel_X, ;, \cup, D)$ is a domain semiring. However, it is also a Baer $*$-semigroup in which the involution is the operation of relational converse, and $A(s)$ is precisely as in the previous subsection, as is easily seen and in any case well-known. We consider other examples shortly.

We begin with some basic facts which have probably appeared elsewhere but which we include here for completeness.

Lemma 6.3 *If S is a Baer $*$-semigroup, then for all $x \in S$, $A^2(x)x = x$ and $A^3(x) = A(x)$.*

Proof. First, note that because $A(x)x = 0$ for any $x \in S$, $x^*A(x) = 0$ as well (upon taking involutions of both sides), and so $x^* \in (0 : A(x))$, and so $x^*A(A(x)) = x^*$. Again taking involutions we obtain $x = A(A(x))x$. Continuing, $A^3(x) = A(A((A(x)))A(x) = A(x)$ and so $A(x) \leq A^3(x)$. But then $A^3(x)x = A^3(x)A(A(x))x = 0x = 0$, so $A^3(x) \in (0 : x)$, and so $A^3(x)A(x) = A^3(x)$, so $A^3(x) \leq A(x)$. Hence $A(x) = A^3(x)$. □

Theorem 6.4 *If S is a Baer $*$-semigroup, then S is an inductive demigroup which is a D-semigroup, in which $D(S) = A(S)$, with $D(s) = A(A(s))$ for all $s \in S$ and with $s \cdot e = A(sA(e)) \wedge D(s)$ for all $s \in S$ and $e \in A(S)$. Hence $S_D = (S, \circledast, D)$ is a left restriction semigroup in which, for all $s, t \in S$, $s \circledast t = (s \cdot D(t))st = (A(sA(t)) \wedge D(s))st$.*

Proof. For all $x, y \in S$, $A(x)xy = 0$, so $A(x)A(xy) = A(x)$, and so on taking involutions, $A(xy)A(x) = A(x)$. Now let $D(x) = A(A(x))$ for each $x \in S$; then $D(D(x)) = D(x)$ since $A(A(A(x))) = A(x)$ as in Lemma 6.3. Then for each $x \in S$, $D(x) = A(A(x))x = x$ for all $x \in S$. Moreover if $ex = x$ for any $e \in A(S)$, then $A(x)A(e) = A(ex)A(e) = A(e)$, so $D(x)A(e) = A(A(x))A(e) = A(A(x))A(x)A(e) = 0$, and so $D(x)e = D(x)A(A(e)) = D(x)$. So S is a D-semigroup as in [17], hence certainly a demigroup.

For the left congruence condition, let $x, y \in S$. Then $A(xy)xy = 0$, so $A(xy)x \in (0 : y)$, and so $A(xy)xA(y) = A(xy)x$, and so $A(xy)xD(y) =$

$A(xy)xA(y)A(A(y)) = A(xy)x(A(A(y))A(y))^* = A(xy)x0^* = 0$, so $(xD(y))^*A(xy) = 0$, and so $(xD(y))^* \in (0 : A(xy))$, so $(xD(y))^*A(A(xy)) = (xD(y))^*$, so $D(xy)xD(y) = xD(y)$ and so $D(xD(y)) \leq D(xy)$. But as in any generalised D-semigroup, $D(xD(y))xy = D(xD(y))(xD(y)y = xD(y)y = xy$, so $D(xy) \leq D(xD(y))$. So $D(xy) = D(xD(y))$.

For all $x, y \in S$, because $A(A(y))y = y$ from the previous lemma, we obtain $A(xA(A(y))xy = A(xA(A(y)))xA(y))y = 0$, so $A(xA(A(y))x \in (0 : y)$ and so $A(xA(A(y))xA(y) = A(xA(A(y))x$. So $(x \cdot e)x = (x \cdot e)xe$ for all $x \in S$ and $e \in A(S)$.

Next, suppose $xye = xy$ for some $x, y \in S$ and $e \in E$ and that $xD(y) = x$. First note that $(eA(e))^* = A(e)e = 0$, so $eA(e) = 0^* = 0$. Then $xyA(e) = xyeA(e) = 0$, so $x \in (0 : yA(e))$ and so $x(y \cdot e) = x(A(yA(e)) \wedge D(y)) = x$. Hence S is inductive.

Because $A(x) = A^3(x)$ for all $x \in S$, it follows that $(e \cdot 0) \cdot 0 = A(A(e)) \wedge e = e \wedge e = e$ for all $e \in A(S)$. □

It was noted in Theorem 16 of [9] that one can obtain orthomodular lattices from generalisations of Baer ∗-semigroups that have no involution but retain a suitable notion of generalised complement. Theorem 6.4 will likely generalise to such settings also.

7 Further operations

We have seen that inductive demigroups carry structure making them behave like semigroups of partial functions equipped with domain. It is remarkable that other aspects of the structure of PT_X and important subsemigroups such as I_X, the injective partial functions on X, are frequently also present in the demonization of a particular demigroup.

7.1 Inverse

The left restriction semigroup I_X has true inverses and so is in fact an inverse semigroup equipped with inversion. So it is natural to ask when the demonization of an inductive demigroup is an inverse semigroup.

The following terminology corresponds to that used for generalised D-semigroups as in [19].

Definition 7.1 *If S is a demigroup with $s \in S$, call $s' \in S$ such that $D(s) = ss'$ and $D(s's) = s'$ a* D-inverse *of s. If a demigroup S is such that every $s \in S$ has a D-inverse, we call it* D-regular*, and if every $s \in S$ has a unique D-inverse s', we call it* D-inverse.

Since for $e \in D(S)$, $ee = D(e)$, e is a D-inverse for itself. In a D-inverse demigroup, $e' = e$ for all $e \in S$, and evidently $D(S) = \{ss' \mid s \in S\}$.

However, not every D-regular demigroup is D-inverse. For example, consider the band $S = \{e, f, 1\}$, the left zero semigroup on $\{e, f\}$ with adjoined identity

1, and define $D(x) = x$ for all $x \in S$. Then S is a demigroup, and both e, f have both themselves and each other as D-inverses. Note that $D(S)$ is not right pre-reduced, which turns out to be the key property.

The following is a generalisation of (part of) Proposition 4.3 in [19].

Proposition 7.2 *Suppose S is a D-regular demigroup. Then S is D-inverse if and only if $D(S)$ is right pre-reduced.*

Proof. Suppose S is a D-inverse demigroup. If $e, f \in D(S)$ are such that $ef = e$ and $fe = f$ then $D(e) = e = ef$ and $D(f) = f = fe$, so $e = e' = f$. So $D(S)$ is right pre-reduced.

Conversely, suppose $D(S)$ is right pre-reduced. If $s \in S$ has D-inverses t and u then $st = D(s) = su$ and $ts = D(t)$, $us = D(u)$. So $D(u)D(t) = usts = uD(s)s = us = D(u)$ and similarly $D(t)D(u) = D(t)$. Hence $D(t) = D(u)$ and so $t = D(t)t = tst = tsu = D(t)u = D(u)u = u$. □

Theorem 7.3 *The inductive demigroup S has demonization which is an inverse semigroup if and only if S is D-regular (equivalently, D-inverse).*

Proof. Suppose that $ss' = D(s)$ and $s's = D(s')$. Then $s \cdot D(s')$ is the largest $e \in D(S)$ under $\leq_r$ for which $e \leq_r D(s)$ and $es = esD(s') = ess's = eD(s)s = es$, so $e = D(s)$. Hence $s \circledast s' = (s \cdot D(s'))ss' = D(s)ss' = ss' = D(s)$. By symmetry, $s' \circledast s = D(s')$.

Conversely, suppose $s \circledast s' = D(s)$ and $s' \circledast s = D(s')$. Then $D(s) = (s \cdot D(s'))ss'$, so $(s \cdot D(s'))D(s) = D(s)$, yet $s \cdot D(s') \leq_r D(s)$, so $(s \cdot D(s'))D(s) = s \cdot D(s')$, and so $D(s) = s \cdot D(s')$; hence $D(s) = D(s)ss' = ss'$. By symmetry, $D(s') = s's$.

Hence, S is D-regular if and only if S_D is. So by Theorem 5.1 and Proposition 7.2, each is D-inverse if and only if the other is. But by Corollary 2.13 in [11], a left restriction semigroup S is D-inverse if and only if it is an inverse semigroup in which the D-inverse s' of s is its semigroup inverse and $D(s) = ss'$. □

The E-inverse semigroups considered in [19] are semigroups S with $E \subseteq E(S)$ and for all $s \in S$ there is a unique $s' \in S$ for which $ss's = s, s'ss' = s'$ and $ss', s's \in E$. These are generalised D-semigroups in which $D(s) = ss'$ and indeed $D(S) = E$ must be pre-reduced. In particular, when viewed as unary semigroups under D, they satisfy the first two demigroup laws. Bot note that for all $x, y \in S$, an E-inverse semigroup, $D(xD(y))xy = D(xD(y))xD(y)y = xD(y)y = xy$, so $D(xy) \leq_l D(xD(y))$, but also $D(xy)xD(y) = D(xy)xyy' = xyy' = xD(y)$, so $D(xD(y)) \leq_l D(xy)$, and so $D(xy) = D(xD(y))$. Hence E-inverse semigroups are nothing but D-inverse demigroups in which $D(S)$ is left pre-reduced, hence pre-reduced.

Important examples of E-inverse semigroups are $*$-regular Baer $*$-semigroups. A $*$-regular semigroup is an involuted semigroup S such that for all $s \in S$, there is a (necessarily unique) $s' \in S$ such that $ss's = s, s' = s'ss'$ and both $ss', s's \in E^*(S) = \{e \in E(S) \mid e^* = e\}$, the set of projections of S. Here s' is called the *Moore-Penrose inverse* of s. In this case, note that $E^*(S)$ is reduced, hence pre-reduced, and so S is $E^*(S)$-inverse.

A *Rickart $*$-ring* S is an involuted associative ring with identity such that its multiplicative semigroup is a Baer $*$-semigroup. It is well-known that for such cases, $A(S) = E^*(S)$. A $*$-regular ring is an involuted associative ring with identity whose multiplicative semigroup is $*$-regular; these are automatically Rickart $*$-rings as well. (This is well-known but the proof is easy. For all $x \in R$, the following are equivalent for $y \in R$: $y \in (0 : x)$; $yx = 0$; $yxx' = 0$; $y(1 - xx') = y$; $y \in Re$ where $e = 1 - xx' \in E^*(S)$.) So by Theorem 6.4, S is an inductive demigroup with $D(s) = A(A(s))$ for all $s \in S$. But it is also D-regular and hence D-inverse since $ss' = D(s)$ and $s's = D(s')$; see 2.7 in [3] for example. It follows that S_D is an inverse semigroup in which inversion is Moore-Penrose inverse.

An example of a $*$-regular ring is $M_n(R)$ where R is the field of real or complex numbers. So, its multiplicative semigroup is a $*$-regular Baer $*$-semigroup. The Moore-Penrose inverse M' of a matrix M is such that if M is non-singular then $M' = M^{-1}$, and $MM' = M'M = 1$. The Moore-Penrose inverse is important in statistics and beyond.

Now multiplicatively, $M_n(R)$ is not an inverse semigroup. However, being a Baer $*$-semigroup, it is an inductive demigroup with $D(M) = A(A(M))$ as we have seen, with the action given by

$$M \cdot D(N) = A(MA(N)) \wedge D(M) = (1 - D(M(1 - D(N)))) \wedge D(M),$$

where $D(M) = MM'$ (M' the Moore-Penrose inverse of M) and $D(N) = NN'$, and then $M \circledast N = (M \cdot D(N))MN$. Hence $(M_n(R), \circledast, D)$ is a left restriction semigroup. Indeed because $D(M) = MM'$ and $D(M') = M'M$, D-regularity holds and so it is one arising from an inverse semigroup, in which inverse is Moore-Penrose inverse. So $(M_n(R), \circledast, {}')$ is an inverse semigroup!

Almost always, $M \circledast N = MN$ (since for this to hold, by Proposition 5.5 part 4, it is sufficient that N be nonsingular so that $D(N) = 1$, or more generally that $MD(N) = M$). However, it is not easy to write down an explicit formula for $M \circledast N$, even when $n = 2$. (The "formula" splits into cases, like the "formula" for Moore-Penrose inverse.) There is considerable interest in the structure of such inverse semigroups.

7.2 Intersection

Note that PT_X is closed under the operation of intersection $\cap$ given by

$$f \cap g = \{(x, y) \mid y = f(x) = g(x)\},$$

that is, intersection of the sets of ordered pairs defining the two functions. This is a semilattice operation on PT_X, and satisfies a small number of further equational laws that can be used to axiomatize subsemigroups of PT_X closed under intersection together with domain; see [21] and [12]. In the latter, the axioms were given as follows: $(S, \cdot, D, \cap)$ is an algebra in which

- $(S, \cdot, D)$ is a left restriction semigroup;
- $(S, \cap)$ is a semilattice in which $\cap$ is meet with respect to the natural order on $(S, \cdot, D)$; and
- $s(t \cap u) = (su) \cap (tu)$ for all $s, t, u \in S$.

Algebras of partial functions under composition and intersection are often called $\cap$-*semigroups* in the literature. Hence we make the following definition.

Definition 7.4 *An algebra $(S, \cdot, \cap, D)$ satisfying the above laws is a* left restriction $\cap$-semigroup*.*

A further natural natural question therefore is: when is the demonization of a demigroup a left restriction $\cap$-semigroup?

Definition 7.5 *A demigroup S has all lower equalizers (ALE) if, for all $s, t \in S$, there exists $s * t \in D(S)$ such that for all $u \in S$, ($us = ut$ and $uD(s) = uD(t) = u$) if and only if $u(s * t) = u$.*

Because of the right pre-reduced property of $D(S)$, the element $s * t$ in the above definition is unique if it exists. So a demigroup can have ALE in at most one way.

Theorem 7.6 *Let S be an inductive demigroup. Then S_D is a left restriction $\cap$-semigroup if and only if the demigroup S has ALE, and then $s * t = D(s \cap t)$ and $s \cap t = (s * t) \circledast s$.*

Proof. First suppose that S_D is a left restriction $\cap$-semigroup. For all $s, t \in S$, let $s * t = D(s \cap t) \in D(S)$. We note in passing that $(S, \circledast, *)$ is then nothing but a twisted agreeable semigroup in the sense of [12], and the following two laws hold: $s \circledast (t * u) = (s \circledast t) * (s \circledast u)$ and $D(s) = s * s$.

Choose $s, t \in S$. Then $s * t = D(s \cap t) \leq D(s) \circledast D(t) = D(s) \cdot D(t) = D(s) \wedge D(t)$ in the inductive demigroup S, so $s * t \leq_r D(s), D(t)$. So by 3 in Proposition 5.5,

$$(s * t)s = (s * t) \circledast s = D(s \cap t) \circledast s = s \cap t = D(s \cap t) \circledast t = (s * t) \circledast t = (s * t)t.$$

Next, suppose $u(s * t) = u$, for some $u \in S$. Then from what was just shown, we obtain that $us = u(s * t)s = u(s * t)t = ut$, and $uD(s) = u(s * t)D(s) = u(s * t) = u$, and similarly $uD(t) = u$.

Conversely, suppose $us = ut$ and $uD(s) = uD(t) = u$. Then obviously, $u \cdot D(s) = D(u)$, and so

$$u \circledast s = (u \cdot D(s))us = D(u)us = us = ut = \cdots = u \circledast t,$$

so using the above two laws for $*$,

$$u \circledast (s * t) = ((u \circledast s) * (u \circledast t)) \circledast u = ((u \circledast s) * (u \circledast s)) \circledast u = D(u \circledast s) \circledast u$$

$$= u \circledast D(s) = (u \cdot D(s))uD(s) = D(u)u = u,$$

and so $u = u \circledast (s * t) = (u \cdot (s * t))u(s * t)$, so $u(s * t) = u$. Hence the demigroup $(S, \cdot, D)$ has ALE with associated operation $*$.

Next suppose $(S, \cdot, D)$ has ALE with associated operation $*$. It follows that $s*t$ is the largest $e \in D(S)$ under $\leq_r$ such that $es = et$ and $e \leq_r D(s), D(t)$. We show that setting $s \cap t = (s*t) \circledast s$ makes the left restriction semigroup $(S, \circledast, D)$ into one with $\cap$. First, suppose $u \leq s, t$ under the standard order on $(S, \circledast, D)$. So $u = D(u) \circledast s = D(u) \circledast t$, so $D(u) \leq D(s), D(t)$ certainly, and so $D(u) \cdot D(s) = D(u) \cdot D(t) = D(u)$, and so $D(u) \circledast s = (D(u) \cdot D(s))D(u)s = D(u)s$, and similarly $D(u) \circledast t = D(u)t$, so $D(u)s = D(u)t$ and $D(s) \leq_r D(s), D(t)$ in $(S, \cdot)$, so $D(u) \leq_r (s * t)$. Hence

$$D(u) \circledast (s \cap t) = D(u) \circledast (s * t) \circledast s = D(u) \circledast s = u,$$

so $u \leq s \cap t$ under the standard order on $(S, \circledast, D)$. But also $s \cap t = (s*t) \circledast s \leq s$ and similarly $s \cap t \leq t$, so $s \cap t$ is the greatest lower bound of s, t in $(S, \leq)$.

To complete the proof that $(S, \circledast, D, \cap)$ is a left restriction $\cap$-semigroup, it remains to show that $s \circledast (t \cap u) = (s \circledast t) \cap (s \circledast u)$ for all $s, t, u \in S$. Now because $t * u \leq_r D(t)$, $(t * u) \circledast t = (t * u)t$ using 3 in Proposition 5.5, and noting also that

$$D((s \cdot D(t))st) = D((s \cdot D(t))sD(t)) = D((s \cdot D(t))s) = D((s \cdot D(t))D(s)) = s \cdot D(t),$$

we have that

$$\begin{aligned} LHS &= (s \cdot (D(t \cap u)))s(t \cap u) \\ &= (s \cdot (t * u))s((t * u) \circledast t) \\ &= (s \cdot (t * u))s(t * u)t \\ &= (s \cdot (t * u))st, \end{aligned}$$

while

$$\begin{aligned} RHS &= (s \circledast t) \cap (s \circledast u) \\ &= ((s \cdot D(t))st) \cap (s \cdot D(u))su) \\ &= ((s \cdot D(t))st) * (s \cdot D(u))su) \circledast (s \cdot D(t))st \\ &= ((s \cdot D(t))st) * (s \cdot D(u))su) \cdot (D(s \cdot D(t))st))(s \cdot D(t))st \\ &= ((s \cdot D(t))st) * (s \cdot D(u))su) \cdot (s \cdot D(t)))(s \cdot D(t))st \\ &= ((s \cdot D(t))st) * (s \cdot D(u))su) \cdot (s \cdot D(t)))st \\ &= ((s \cdot D(t))st) * (s \cdot D(u))su) \cdot D((s \cdot D(t))st))st \\ &= ((s \cdot D(t))st) * (s \cdot D(u))su)st, \end{aligned}$$

so it suffices to show that $s \cdot (t * u) = ((s \cdot D(t))st) * ((s \cdot D(t))su)$.

Suppose $g \in D(S)$ is such that $g \leq_r s \cdot (t*u)$. So $g \leq_r D(s)$ and $gs(t*u) = gs$, so $gsD(t) = gs$ since $t*u \leq_r D(t)$, and $gsD(u) = gs$, so $g \leq_r s \cdot D(t)$ and $s \cdot D(u)$. Hence

$$g(s \cdot D(t))st = gst = gsu = g(s \cdot D(u))su,$$

and

$$gD(s \cdot D(t))st) = gD((s \cdot D(t))sD(t)) = gD((s \cdot D(t))s)$$
$$= gD((s \cdot D(t))D(s)) = gD(s \cdot D(t)) = g(s \cdot D(t)) = g$$

and likewise $gD((s \cdot D(u))su) = g$, so $g \leq_r ((s \cdot D(t))st) * ((s \cdot D(u))su)$.

Conversely, suppose $g \leq_r ((s \cdot D(t))st * (s \cdot D(u))su)$, so $g(s \cdot D(t))st = g(s \cdot D(u))su$ and $g \leq_r D((s \cdot D(t))st) = s \cdot D(t)$ as before, and similarly $g \leq_r s \cdot D(u)$. So $gst = gsu$, and yet also $g \leq_r D(s)$, so to obtain $gs(t * u) = gs$, we now only need show that $gsD(t) = gsD(u) = gs$. But this follows since $g \leq_r s \cdot D(t)$ and $g \leq_r s \cdot D(u)$. □

Examples come from Rickart ∗-rings.

Proposition 7.7 *Let S be the multiplicative Baer ∗-semigroup of a Rickart ∗-ring R. Then S is an inductive demigroup with ALE, with $s * t = A(s - t) \wedge D(s) \wedge D(t)$, hence is such that S_D is a left restriction ∩-semigroup.*

Proof. S is an inductive demigroup by Theorem 6.4. Let $s * t$ be defined as above for all $s, t \in S$. Then $(s * t)(s - t) = 0$, so $(s * t)s = (s * t)t$.

Now suppose $u, s, t \in S$. If $u(s * t) = s$ then $us = u(s * t)s = u(s * t)t = ut$, and $uD(s) = u(s * t)D(s) = u(s * t) = u$ and similarly $uD(t) = u$. Conversely, if $us = ut$ and $uD(s) = uD(t) = u$ then $u(s - t) = 0$ so $uA(s - t) = u$, and so from part 3 of Proposition 5.6, $u(s * t) = u$. It follows from Theorem 7.6 that S has ALE. □

We can hope to combine the ideas of these last two subsections, and ask: when is the demonization of an inductive demigroup equipped with both inversion and intersection? In fact there are no further properties beyond what these two operations require seperately.

Definition 7.8 *A left restriction semigroup that is both inverse and a ∩-semigroup is an* inverse ∩-semigroup*.*

These axiomatise semigroups of injective partial functions equipped with inversion and intersection (and hence domain, which can be viewed as a derived operation), as shown in Section 7 of [12].

Corollary 7.9 *Let S be an inductive demigroup. Then S_D is an inverse ∩-semigroup if and only if the demigroup S is D-regulsr and has ALE.*

Examples come from ∗-regular rings.

Proposition 7.10 *Let S be the multiplicative Baer ∗-semigroup of a ∗-regular ring R. Then S is a D-regular inductive demigroup with ALE, and so S_D is an inverse ∩-semigroup.*

Proof. We have seen that S_D is a left restriction $\cap$-semigroup. But because S is D-inverse, it is an inverse $\cap$-semigroup. □

An example is the multiplicative Baer $*$-semigroup $M_n(R)$ where R is the ring of real or complex numbers, discussed earlier: its demonization is in fact an inverse $\cap$-semigroup.

8 Special Case: inductive demibands

A special case arises by assuming $D(S) = S$.

Definition 8.1 *A demigroup S is a* demiband *if $D(S) = S$. A demiband is* inductive *if it is inductive as a demigroup.*

Of course a demiband is a band, indeed because $D(s) = s$ for all $s \in S$, it is nothing more than this, for any band S is a demiband on which we define $D(s) = s$ for all $s \in S$, as is easily checked. In the inductive case, the right pre-reduced property of $D(S)$ becomes the right regularity law for bands: $xyx = yx$ for all x, y, which in turn is equivalent to the right reduced property ($ef = e \Rightarrow fe = e$).

In the demiband case, the equational laws given in Theorem 5.7 simplify somewhat as follows. (Note that D does not appear in these laws since $D(s) = s$ is a law.)

Proposition 8.2 *The algebra $(S, \times, \cdot)$ is an inductive demiband if and only if it satisfies the following laws.*

1. *$(S, \times)$ is a band and $(S, \cdot)$ is a commutative semigroup;*
2. $(s \cdot t)s = (s \cdot t)$
3. $(st) \cdot t = st$

Equivalently it is an algebra in which $(S, \times)$ is a right regular band and $\cdot$ is meet with respect to the partial order $\leq_r$ on $(S, \times)$.

Proof. First suppose $(S, \times, \cdot)$ is an inductive demiband. The first law is immediate, as is the second since $s \cdot t \leq_r s$. For the final one, $stt = st$ for all $s, t \in S$, so the largest $u \in S$ below st under $\leq_r$ such that $ustt = ust$ is st itself, giving the third law courtesy of Law 1 in Proposition 5.6.

Conversely, assume the above laws hold. Define $D(s) = s$ for all $s \in S$. The demigroup laws are immediate. For $x, y \in S$, if $xy = x$ then $x \cdot y = (xy) \cdot y = xy = x$, and conversely if $x \cdot y = x$ then $xy = (x \cdot y)y = (y \cdot x)y = (y \cdot x) = x \cdot y = x$. So $xy = x$ if and only if $x \cdot y = x$. So if $xy = x$ and $yx = y$ then $x = x \cdot y = y \cdot x = y$. So S is right pre-reduced.

Now for all $s, t \in S$, $(s \cdot t)st = (s \cdot t)t = (t \cdot s)t = t \cdot s = s \cdot t = (s \cdot t)s$ and $(s \cdot t)s = s \cdot t$. So if $u \in S$ is such that $u(s \cdot t) = u$ then $us = u$ and $ust = us$. On the other hand, if $us = u$ and $ust = us$ then $ut = u$, and so $u \cdot s = u$ and

$u \cdot t = u$. Hence $u \cdot (s \cdot t) = (u \cdot s) \cdot t = u \cdot t = u$, so $u(s \cdot t) = u$. Hence $(S, \times, \cdot)$ is an inductive demiband.

If $(S, \times, \cdot)$ is an inductive demiband, then as noted above, $(S, \times)$ is a right regular band, and by Law 2 in Propositon 5.6, $\cdot$ is meet with respect to $\leq_r$. Conversely, if $(S, \times)$ is a right regular band and $\cdot$ is meet with respect to the partial order $\leq_r$ on $(S, \times)$, then the first two laws given above are immediate, and because $(st)t = st$ so that $st \leq_r t$ for all $s, t \in S$, we obtain the third. This establishes the final claim. □

If S is an inductive demiband, then in S_D, $s \circledast t = (s \cdot t)st = (s \cdot t)$, as is the case for elements of $D(S_D)$ in general. The following is also clear.

Proposition 8.3 *Every inductive demiband has ALE, with $s * t = s \cdot t$; hence in S_D, $s \circledast t = s \cap t = s \cdot t$.*

Aside from the rather trivial cases of any semilattice in which one defines the second multiplication to equal the first, examples of inductive demibands are common. Consider PT_X, the set of partial functions on the non-empty set X. An operation of interest other than composition is *left restrictive multiplication*, defined as follows: for all $s, t \in PT_X$, $(s \odot t) = D(s)t$ (where composition is either angelic or demonic, it does not matter). This operation is motivated from computer science (see [20] for example), and is well-known to be associative, and $(PT_X, \odot)$ is a right regular band. Indeed it is a right normal band, satisfying the stronger law $(s \odot t) \odot u = (t \odot s) \odot u$. Conversely, it was shown in [22] that every right normal band embeds in one such.

Now $\leq_r$ in $(PT_X, \odot)$ is given by $s \leq_r t$ if and only if $s \odot t = s$, which is saying that $s \subseteq t$ when s, t are viewed as sets of ordered pairs. Meet under $\leq_r$ in $(PT_X, \odot)$ therefore exists and is nothing but *intersection* of (the sets of ordered pairs determined by) two partial functions, and indeed each of the inductive demiband laws in Proposition 8.2 is easily checked if we set $s \cdot t = s \cap t$.

Hence $(PT_X, \odot, \cap)$ is an inductive demiband. It is not a particularly typical one though, since aside from the right normal band law, it also satisfies the law $(s \odot t) \cap u = s \odot (t \cap u)$ which does not generally hold. For example, consider the band $S = \{0, e, f, 1\}$ consisting of the left zero semigroup $\{e, f\}$ with adjoined zero 0 and identity 1, a right normal band. Moreover S is easily seen to be an inductive demiband, in which $(S, \cdot)$ is the semilattice in which $0 \leq_r e, f \leq_r 1$, and then $(e1) \cdot f = e \cdot f = 0$ while $e(1 \cdot f) = ef = e$.

Finally, note that one can define $\odot$ on Rel_X and obtain a right regular band, and meet with respect to $\leq_r$ does exist, thus giving an inductive demiband structure. However, $\leq_r$ is not in general just relational inclusion, and correspondingly meet is not simply relational intersection.

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