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Dual breaking of symmetries in algebraic models

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Abstract. The general features of the dual symmetry breaking in the algebraic structure models are discussed. Dual breaking indicates here simultaneous dynamical and spontaneous breaking. Several examples are considered, including the multiconfigurational dynamical symmetry (MUSY), which is the common intersection of the shell, collective and cluster models for the multi shell problem.

1 Introduction

The basic equation of the time-independent quantum mechanical description is the eigenvalue-equation of the energy. It has two important objects: the (H Hamiltonian) operator and its eigenvectors. When both of them are symmetric (see below for more details), one has an exact symmetry [1]. When the Hamiltonian contains a symmetry breaking interaction, then we speak about a dynamical breaking of the symmetry. In this case the operator is not symmetric, of course, but the eigenvectors may remain symmetric. This situation requires some special symmetry breaking interaction. When this is the case (non-symmetric operator with symmetric eigenvectors), one has a dynamical (or dynamically broken) symmetry. [2–4]. On the other hand, when the operator is symmetric, but the eigenvector (of the ground state) is not symmetric (in the strict sense), then the symmetry is spontaneously broken [5].

We consider here continuous symmetries which are characterized by Lie groups and their associated Lie algebras. (For simplicity we denote both the group and its algebra by the same letter.) The symmetry of the operators and eigenvectors can be expressed as follows. [1]. An operator H is symmetric, or in other words it is a scalar, if $[H, X_q] = 0$, where X_q is any element of the symmetry algebra. The state ψ is symmetric in the strict sense if $X_q\psi = 0$. [1] We call a set of states ψ_i symmetric in the loose sense if they transform according to a definite irreducible representation (irrep) of the symmetry group: $T\psi_i = \Sigma_j T_{ji}\psi_j$, where T_{ji} are elements of the representation matrix of the transformation T.

In this paper a symmetric state is meant in the loose sense, in general. When it is symmetric in the strict sense, we say it explicitly. For illustration: with respect to

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[1] This equation is equivalent to the more familiar condition: the state ψ transforms according to the identity representation $T\psi = \psi$. Here T is the operator of a symmetry transformation, representing an element of the Lie group, and for the identity representation $T = 1$.

the rotation a state is strictly symmetric if it has angular momentum $L = 0$, and it is symmetric in the loose sense if it has well-defined L value.

Obviously, these two different kinds of symmetry breaking, the dynamical and the spontaneous ones, do not take place at the same time when the symmetry is characterised by a single group G. (The H operator can not be symmetric and nonsymmetric at the same time.)

When, however, a group-chain

$$G \supset G' \tag{1}$$

is relevant for the description of the system, then the situation is different. This is the case when the Hamiltonian H splits up into a G-symmetric, and a G'-symmetric parts:

$$H = \alpha H_G + \beta H_{G'}. \tag{2}$$

In order to be $G(G')$-symmetric H_G ($H_{G'}$) can contain the generators of G (G') only via the invariant (or Casimir) operators of the group.

The first part of the Hamiltonian H_G is invariant with respect to G, including its G' subgroup, of course. (The invariant operator of G commutes with all the elements of its algebra, including the ones of its G' subalgebra.)

Due to the presence of the second part $H_{G'}$, however, the total Hamiltonian is not G-symmetric, H has only G' as an exact symmetry. $H_{G'}$ breaks the G-symmetry dynamically. The result is the dynamical symmetry mentioned above: all the representation labels of group-chain (1) are good quantum numbers, thus the degeneracy corresponding to G splits up, but no mixing is taking place between the basis states of different quantum numbers [2–4].

On the other hand G' may be spontaneously broken. The spontaneous breaking is taking place in the eigenvalue-equation of the G-symmetric part of the Hamiltonian H_G. As already mentioned, this operator has a G' symmetry as well (since it is a subgroup of the symmetry-group G). As a consequence the eigenvectors of H_G transform, in general, according to the irreducible representations of G as well as G'. When, however, a degeneracy is present a different situation may appear. A definite irrep of the larger group may contain several irreps of its subgroup. Thus it may very well happen that different states transform according to a definite irrep of G, but according to different irreps of G'. If a linear combination of such states appears, it transforms according to a single irrep of G, but does not necessarily transform according to any specific irrep of G'. This results in a spontaneous breaking of the G' symmetry in the eigenvalue-problem of the operator H_G.

An important and frequent example of the splitting of the Hamiltonian as described by Eq. (2) is provided by the separation of the intrinsic (fast) and collective (slow) degrees of freedom [2,6,7].

In case of a dynamically symmetric Hamiltonian this separation is exact, i.e. there is no mixing term in the energy. Then G is the group (or group-chain) of the intrinsic motion and G' is that of the collective part. Detailed examples will be presented below.

The dynamical breaking of the G symmetry can provide us with a (non-trivial, i.e. non-degenerate) spectrum, while the spontaneously broken G' symmetry results in the non-symmetric ground state.

The dual (dynamical and spontaneous) symmetry-breaking is in fact very frequent in the algebraic structure models, though not so much discussed until now. In this paper we show how it takes place first in the Elliott-model [8–11], which is the prototype of the algebraic structure models, then we mention a few other cases. Finally a detailed example is presented in relation with the multiconfigurational dynamical symmetry (MUSY) [12], which is the connecting symmetry of the shell, collective and cluster models for the multi-major-shell problem [13,14].

2 Elliott model

The Elliott model [8–11] is an $L-S$ coupled shell model, in which the spin-isospin degrees of freedom are characterized by Wigner's $U^{ST}(4)$ group. The space part has an $U(3)$ symmetry due to the fact that the shell potential is taken to be that of the harmonic oscillator, and the nucleon-nucleon force is quadrupole type. The Hamiltonian is:

$$H = H_{HO} + \kappa Q^a \cdot Q^a. \tag{3}$$

Here HO stands for the harmonic oscillator, and Q^a is the mass quadrupole operator acting within a single shell, which is obtained by summing the quadrupole operators of the nucleons. In Elliott's representation the building blocks of the physical operators are the creation and annihilation operators of oscillator quanta but it can be formulated also in the language of the nucleon creation and annihilation operators [15].

The physically relevant group-chain of the model is:

$$\{U^{ST}(4) \supset U^S(2) \otimes U^T(2)\} \otimes \{U(3) \supset SU(3) \supset SO(3)\} \tag{4}$$

The $U^S(2)$, $U^T(2)$, and $SO(3)$ symmetries are exact symmetries of the model. The symmetries of the larger groups are dynamically broken.

In its original form the model deals with a single major shell problem. When it is the sd shell, i.e. $2\hbar\omega$ oscillator shell for example, then it has 6 single particle orbitals. Together with the 4 spin-isospin degrees of freedom it results in 24 single-particle states. Therefore, in this case the chain (4) turns out to be a subgroup chain of the $U(24)$ dynamical group (including a $U(6) \supset U(3)$ part in the space sector).

In fact, for many physical problems we can restrict ourselves to a single irrep of the spin-isospin groups, and apply the Hamiltonian of Eq. (3). Then the relevant group-chain simplifies to the second part of (4): $U(3) \supset SU(3) \supset SO(3)$.

With the invariant operators of these groups

$$C^{(1)}_{U3} = n, \quad C^{(2)}_{SU3} = \frac{3}{4} L \cdot L + \frac{1}{4} Q^a \cdot Q^a, \quad C^{(2)}_{SO3} = L \cdot L, \tag{5}$$

one can rewrite the Hamiltonian. Here $C^{(i)}$ stands for the Casimir operator of degree i of the algebra indicated as a subscript, and $n = n_1 + n_2 + n_3$ is the number of oscillator quanta.

The algebraic form of the Hamiltonian is

$$H = C^{(1)}_{U3} + \alpha C^{(2)}_{SU3} + \delta C^{(2)}_{SO3}. \tag{6}$$

It is seen that it separates into an intrinsic and a collective parts:

$$H = H_{intr} + H_{coll}, \tag{7}$$

where the collective part is the rotational term

$$H_{coll} = \delta C^{(2)}_{SO3} = \delta L \cdot L, \tag{8}$$

while the intrinsic part is

$$H_{intr} = C^{(1)}_{U3} + \alpha C^{(2)}_{SU3}. \tag{9}$$

This latter part determines the bandheads, while the rotational term splits (and shifts) the bands.

The intrinsic Hamiltonian $H_{intr} = C^{(1)}_{U3} + \alpha C^{(2)}_{SU3}$ is invariant under the rotation. Its eigenstates, including the ground state, however, may have a deformed shape. Only the completely filled shell has spherical symmetry, all other nuclei have deformed ground state. It means that the spherical symmetry of the Schrödinger equation of the intrinsic system is spontaneously broken. In such a case the ground state of the intrinsic Hamiltonian is degenerate, and has no definite angular momentum, i.e. it is not rotationally symmetric (even in the loose sense).

In many cases the action of the angular momentum operators on such a state results in different states of the same energy. These excitation quanta of zero energy represent the Goldstone bosons of the Elliott model [7]. Such bosons appear, whenever a global continuous symmetry is spontaneously broken [16]. This is the case with the rotational symmetry of the intrinsic Hamiltonian.

The Elliott model was extended both in its model space and concerning its interactions. Spectra were calculated with more realistic interactions, including SU(3) symmetry-breaking terms, like e.g. pairing force. Since, however, we are interested here in the symmetry properties of the model, we do not discuss these cases. As for the larger model space is concerned, some examples will be discussed in detail below. Here we mention two examples.

One of the recent no-core shell models is formulated with Elliott SU(3) basis [17, 18]. This model is applied both with model interaction, as well as with realistic interactions, which are obtained either from scattering data or from effective field theory. In describing the experimental data, these latter calculations represent the bigger news. From the viewpoint of the symmetries of the model the situation is the following. In the large model space obviously new symmetries can be there. This problem has not been studied yet systematically from the mathematical viewpoint. From the physical viewpoint, however, it is very remarkable that the SU(3) and Sp(6,R) symmetries were found to emerge from ab initio calculations, which apply large model space and realistic interactions [19]. The space symmetries of the original model are still there, and they play the same role. SU(3) is responsible for the intrinsic shape, and SO(3) is representing the collective part. The SU(3) is dynamically broken, while SO(3) is spontaneously broken in the eigenvalue problem of the intrinsic Hamiltonian. [2]

A simplified version of the no-core SU(3) shell model is the symmetry adapted quartet model [20]. It is applicable for the description of the nuclei with N=Z=even neutron and proton numbers. The building blocks of the model are the quartets formed by two protons and two neutrons with permutational symmetry of {4} or spin-isospin symmetry {1,1,1,1}. The space of this model is obtained by truncating the space of the no-core shell model to the Wigner-scalar sector. It is applied mainly in schematic studies, like symmetry-based unified description of shell and cluster spectra. In this approach again the $SU(3) \supset SO(3)$ space symmetries are playing the same role, as discussed before.

3 Other models

Here we consider some other models related to the shell structure, collective behavior, and clusterization, or connecting these phenomena to each other. Each model is similar to the Elliott model in its algebraic method: both the basis states and the operators carry some group symmetries. Some of them are direct extensions of the

[2] In fact, in the no-core model the proton-neutron formalism is used instead of the isospin scheme of Eq. (4), but the two sets are closely related to each other, and their choice does not disturb the space symmetries.

Elliott model. We start with the ones, which account for the quadrupole collectivity in close connection with the shell structure, and consider other phenomena afterwords.

Symplectic shell model

The description of the electromagnetic transitions without an effective charge requires the incorporation of the major shell excitations; i.e. a vertical extension of the SU(3) shell model. For this purpose the symplectic group proved to be very useful (see e.g. [21]), and in the formalism developed in [22] the symplectic shell model has been widely applied.

The Sp(6,R) group is generated by the position vectors and their canonically conjugate momenta of the nucleons. [3] An alternative set of its generators is expressed in terms of harmonic oscillator operators. Let $c^\dagger_{\alpha i}$ be the creation operator of an oscillator quantum in the direction i belonging to the nucleon α, and $c_{\alpha i}$ the corresponding annihilation operator. Then the 9 number conserving $C_{ij} = \frac{1}{2}\Sigma_\alpha^{A-1}(c^\dagger_{\alpha i}c_{\alpha j} - c_{\alpha j}c^\dagger_{\alpha i})$ operators generate the U(3) group. The 6 creation $B^\dagger_{ij} = \frac{1}{2}\Sigma_\alpha^{A-1}c^\dagger_{\alpha i}c\dagger_{\alpha j}$ and 6 annihilation $B_{ij} = \frac{1}{2}\Sigma_\alpha^{A-1}c_{\alpha i}c_{\alpha j}$ operators ladder by 2 or -2 quanta. Their appropriate linear combinations are U(3) and spherical tensors. In particular, the creation operators are $[2, 0, 0]$ U(3) tensors, therefore, their products also carry U(3) labels: $[n^e_1, n^e_2, n^e_3]$. Since these operators commute with each other, only the symmetrically coupled products are non-vanishing. (Their coupling always produces a set of unique irreps thus there is no need to introduce an additional multiplicity label.) Note that all the symplectic generators are fully symmetric one-body operators, so they conserve the nuclear permutational symmetry. Therefore, if the band-head U(3) irrep is Pauli-allowed, then so are all others in the symplectic band.

The model has a rich group structure, one of its physically important subgroup chain, called shell model chain, is associated with Elliott's SU(3):

$$\begin{array}{ccccc} Sp(6,R) & \supset & U(3) & \supset SU(3) \supset SO(3) & \\ |[n^s_1, n^s_2, n^s_3], [n^e_1, n^e_2, n^e_3], \rho, & & [n_1, n_2, n_3], & (\lambda, \mu), K \;, \quad L & \rangle . \end{array} \tag{10}$$

Here $[n^s_1, n^s_2, n^s_3]$ denotes the symplectic bandhead, which is a U(3) irrep, being a lowest-weight Sp(6,R) state, while ρ distinguishes multiple occurance of $[n_1, n_2, n_3]$ in the product $[n^e_1, n^e_2, n^e_3] \otimes [n^s_1, n^s_2, n^s_3]$. Note that this basis is not orthonormal; such a basis can be constructed inductively, by diagonalizing the norm matrix (provided by the inner products of the basis states) in each major shell.

In collective terms the symplectic model includes monopole and quadrupole vibrations as well as vorticity degrees of freedom for the description of the rotational dynamics in a continuous range from the irrotational flow to rigid rotor.

In [24] it was found that a realistic description of the low-lying spectrum can be obtained with a Hamiltonian, containing three parts, each with well-defined physical content: a harmonic oscillator term, responsible for the shell structure, a two-nucleon quadrupole interaction $Q^c \cdot Q^c$, accounting for the major shell excitations, and a residual interaction that reproduces the dynamics of a quantum rotor in an irrep of SU(3). This latter part has an $\Sigma_i L_i L_i$ term as well as a third $(\Sigma_{ij} L_i Q^c_{ij} L_j)$ and fourth order $(\Sigma_{ijk} L_i Q^c_{ij} Q^c_{jk} L_k)$ contribution and matrix elements which are much smaller than those of the first two parts. Q^c is the collective quadrupole operator, including the Q^a algebraic quadrupole operator of the Elliott model (of Eq. (3)), and creation and annihilation operators of the $2\hbar\omega$ excitations.

[3] We note here that the language of the literature is not univocal, some authors denote this group by $Sp(3, R)$.

The Hamiltonian of the SU(3) dynamical symmetry is obtained as the limit of this general Hamiltonian, by i) substituting the collective quadrupole operator by the algebraic one, and ii) neglecting the higher order terms of the residual interaction. The dynamically symmetric Hamiltonian has the same algebraic structure (characterized by the group-chain $SU(3) \supset SO(3)$) like the Elliott model, therefore, the role of the symmetry-breaking is the same, as before.

Recently the Sp(6,R) symplectic shell model has been extended to a two-component, i.e. proton-neutron model with Sp(12,R) group structure [25]. The Sp(12,R) model is suitable for the description of the quadrupole collectivity in heavy nuclei, by incorporating major shell excitations (describing E2 transitions without effective charge). The shell model classification of the states are provided by the group-chain:

$$Sp(12, R) \supset U(6) \supset SU_p(3) \otimes SU_n(3) \supset SU(3) \supset SO(3). \qquad (11)$$

The embedding of the crucial $SU(3) \supset SO(3)$ group-chain into a larger group does not influence the role of the SU(3) and SO(3) symmetries (of the intrinsic and collective degrees of freedom), thus the role of their breaking remains untouched.

Contracted symplectic model

In the limit of large number of oscillator quanta ($n_s = n_1^s + n_2^s + n_3^s$) the symplectic shell model reduces to a collective model with a simpler structure. The dynamical group of the model simplifies to $U_b(6) \otimes U_s(3)$, i.e. to a compact group, as opposed to the noncompact Sp(6,R) of the shell model. Technically it is achived by replacing the raising and lowering operators by boson creation and annihilation operators: $b_m^{\dagger(l)} = (1/\epsilon)B_m^{\dagger(l)}$, $b_m^{(l)} = (1/\epsilon)B_m^{(l)}$, where ϵ denotes $[\frac{4}{3}n_s]^{\frac{1}{2}}$. This model is called U(3) boson model [23], or contracted symplectic model [24]. Mathematical justification for the symplifying assumptions is provided through the application of the group deformation mechanism. This model is more easily applicable, e.g. it has an orthonormal set of basis states:

$$\begin{array}{ccccccc} U_s(3) & \otimes & U_b(6) & \supset U_s(3) \otimes U_b(3) & \supset & U(3) \supset SU(3) \supset SO(3) & (12) \\ |[n_1^s, n_2^s, n_3^s], & & [n_b, 0, 0, 0, 0, 0], & [n_1^b, n_2^b, n_3^b], \rho, & & [n_1, n_2, n_3]\ , (\lambda, \mu), K\ ,\ L\ \rangle. & \end{array}$$

The united U(3) group is generated by the sum of the operators corresponding to the subgroups $U_s(3) \otimes U_b(3)$:

$$n = n_s + 2n_b,\ Q = Q_s + Q_b,\ L = L_s + L_b. \qquad (13)$$

The $U_s(3)$ is Elliott's shell model symmetry of the $0\hbar\omega$ shell, and $U_b(6)$ is the group of the six dimensional oscillator, generated by the bilinear products of the ($l = 0$ and 2) boson creation and annihilation operators. It is realized in a similar way as the U(6) group of the interacting boson model (IBM) [2], nevertheless physically it is different, because in the case of the contracted symplectic model the bosons are associated to intershell excitations, not to intrashell ones.

As for the symmetries and their breaking is concerned it can be summarized in a way similar to that of the Elliott model again. Analytical solution of the energy eigenvalue problem can be obtained when the Hamiltonian is expressed in the invariant operators of the group-chain (12). When the full Hamiltonian is considered, then SO(3) is an exact symmetry, those of the larger groups are dynamically broken. When dividing the full Hamiltonian into an intrinsic and collective parts then $L \cdot L$ is the collective one, and the rest belong to the intrinsic. The intrinsic Hamiltonian has a spontaneously broken rotational symmetry.

Interacting boson model
The interacting boson model [2] is a successful algebraic model of the quadrupole collectivity, which was applied very widely. The basic building blocks of the model are the monopole and quadrupole bosons. Microscopically they correspond to nucleon pairs in the valence shell with $L = 0$ and $L = 2$ angular momentum. The dynamical algebra of the model is U(6), and it has three dynamical symmetries: $U(6) \supset G \supset SO(3)$, where $G : U(5) \supset O(5)$, corresponding to the anharmonic vibrator, $G : SU(3)$, describing the rigid rotor, and $G : O(6) \supset O(5)$, for the gamma unstable limit. The intrinsic and collective features of this model have been worked out in great detail (see e.g. [2,6]), here we recall only a very brief summary.

As mentioned above the dynamical symmetry with the SU(3) algebra describes the rigid rotation. In this case the scenario of the dual breaking is the same, as in the case of the Elliott-model and its extensions. The SU(3) invariant operator of the Hamiltonian is the intrinsic part, and the SO(3) part (L^2) is the collective one. The presence of the L^2 operator breaks the SU(3) symmetry dynamically. In return, in the eigenvalue problem of the intrinsic Hamiltonian the SO(3) symmetry is spontaneously broken: the ground state is usually not spherically symmetric [2,6].

In case of the two other dynamical symmetries, which include quadrupole vibrations as well, the collective Hamiltonian contains the Casimir-operator of the O(5) algebra, too. The intrinsic Hamiltonian is provided by the invariant operator of the first subgroup of U(6), i.e. U(5) or O(6), respectively. Therefore, the U(5) and O(6) symmetries are broken dynamically by both the O(5) and SO(3) invariants. And in the eigenvalue problems of the intrinsic operators not only the SO(3), but also the O(5) symmetries are spontaneously broken [6].

In case of the IBM the separation of the Hamiltonian into intrinsic and collective parts is worked out in detail for the general case as well, when the Hamiltonian contains contributions from each of the three dynamical symmetries. In this case the collective part contains a further term belonging to the $\overline{O(6)}$ group, which is mathematically equivalent to the $O(6)$ of the dynamical symmetry, but it is generated by a different set of operators.

Octupole deformation
The algebraic description of the octupole deformation is developed also within the interacting boson model, by incorporating octupole bosons as well [2]. Here we consider very briefly a more recent approach [26] within the extended Elliott model. As the octupole operator is of negative parity at least two harmonic oscillator shells are required (with major quantum numbers N-1 and N). It turns out that the energy eigenvalue problem of the spherical shell model has an analytical solution generated by a spin-isospin-scalar octupole interaction, which tends to its geometric equivalent in the case of large N. The approximation is reasonably good even for low values of N, as long as the value of the orbital angular momenta are not too large.

The analytical solution is obtained for a case of a dynamical symmetry. In particular, the Hamiltonian is expressed as a combination of Casimir invariants of the algebra chain

$$U(\Omega^2) \supset G \supset SO(3), \tag{14}$$

where $\Omega = N + 1$, and

$$G = U_a(\Omega) \otimes U_b(\Omega) \supset Sp_a(\Omega) \otimes Sp_b(\Omega) \supset Sp(\Omega),$$
$$G = U_a(\Omega) \otimes U_b(\Omega) \supset SO_a(\Omega) \otimes SO_b(\Omega) \supset SO(\Omega),$$

depending on whether Ω is even or odd, respectively. Sp denotes the unitary symplectic algebra. For small dimensions they are isomorphic with orthogonal algebras: $Sp(2) \approx SO(3)$, $Sp(4) \approx SO(5)$.

In this case, too, the Hamiltonian can be completely separated into an intrinsic and a collective part $H = H_{intr} + H_{coll}$, and H_{coll} is the rotational term $H_{coll} = \delta C^{(2)}_{SO3} = \delta L \cdot L$.

For this shell-model-based algebraic description of the octupole deformation the details of the intrinsic state are not completely known yet. Nevertheless, the role of the rotational symmetry and its breaking is seen to be similar to that of the quadrupole case: the intrinsic Hamiltonian is spherically symmetric, but this symmetry breaks down spontaneously, due to the separation of the degrees of freedom, resulting in a deformed nuclear shape.

Cluster model: molecule-like configuration

The semimicroscopic algebraic cluster model [27] is a fully algebraic approach with transparent symmetry-properties. The internal structure of the clusters is described here by the Elliott model [8–10], therefore, this part of the wavefunction has a $\mathrm{U}^{ST}_C(4) \otimes \mathrm{U}_C(3)$ symmetry. The relative motion of the clusters is accounted for by the modified ($\mathrm{U}_R(4)$) vibron model [28].

The coupling between the relative motion and internal cluster degrees of freedom for a binary cluster system results in a group structure:

$$U^{ST}_{C_1}(4) \otimes U_{C_1}(3) \otimes U^{ST}_{C_2}(4) \otimes U_{C_2}(3) \otimes U_R(4). \tag{15}$$

The spin and isospin degrees of freedom are essential from the viewpoint of the construction of the model space. However, if one is interested only in a single supermultiplet $[\mathrm{U}^{ST}_C(4)]$ symmetry, which is typical in cluster problems, then the relevant group structure simplifies to that of the space part. It is characterized by the group-chain:

$$\begin{aligned} U_{C_1}(3) \otimes U_{C_2}(3) \otimes \ U_R(3) \supset U_C(3) \ \otimes U_R(3) \supset \\ U(3) \supset SU(3) \supset SO(3) \supset SO(2), \end{aligned} \tag{16}$$

here $\mathrm{U}_C(3)$ stands for the coupled space symmetry of the two clusters.

This basis is especially useful for treating the exclusion principle, since the U(3) generators commute with those of the permutation group, therefore, all the basis states of an irrep are either Pauli-allowed, or forbidden [29].

The exclusion of the Pauli-forbidden states amounts up to a truncation of the coupled U(3) basis from the side of the small number of oscillator quanta. Some major shells are completely missing, and from some other ones parts of the single-nucleon states are excluded. This is the modification [27] with respect to the original vibron model, as it is applied e.g. in molecular physics.

A Hamiltonian having the dynamical symmetry (16) gives a reasonably good description of several cluster spectra [30,14]. Such a Hamiltonian splits up into an intrinsic and a collective part, just like that of the Elliott model: $H = H_{intr} + H_{coll}$. The collective part is again the rotational term: $H_{coll} = \delta L \cdot L$. The intrinsic part gives the shape of the cluster configuration in the body-fixed system. It describes two clusters with arbitrary quadrupole deformation and arbitrary relative orientation. The geometric mapping of the semimicroscopic algebraic cluster model is discussed in detail in [31,32]. Examples for the deformed intrinsic states of some cluster configurations are presented in the next section.

4 Multiconfigurational dynamical symmetry

When major shell excitations are incorporated, then both the (symplectic) shell model, and the (contracted symplectic) collective model, as well as the (microscopic or semimicroscopic algebraic) cluster model has a set of basis states characterised by the irreps of the

$$U_x(3) \otimes U_y(3) \supset U(3) \supset SU(3) \supset SO(3) \supset SO(2) \tag{17}$$

group chain, as seen above. For the shell and collective models x stands for the band-head (valence shell), for the cluster model it refers to the internal cluster structure. y indicates in each case the major shell excitations; in the shell and collective model cases it takes place in steps of $2\hbar\omega$, connecting oscillator shells of the same parity, while in the cluster case it is in steps of $1\hbar\omega$, incorporating all the major shells. For the cluster model it has only completely symmetric (single-row Young-tableux) irreps: $[n, 0, 0]$, while in the case of the shell and collective models it can be more general. As a consequence the model space of the three models have a considerable overlap, but they are not identical.

A particularly simple Hamiltonian with dynamical symmetry (17) is

$$H = C^{(1)}_{U3} + \alpha C^{(2)}_{SU3} + \beta C^{(3)}_{SU3} + \delta C^{(2)}_{SO3}, \tag{18}$$

in which the contribution of the relative motion ($\mathrm{U}_R(3)$) and the internal cluster structure ($\mathrm{U}_C(3)$) appears only in the coupled (U(3)) form. Here $C^{(3)}_{SU3}$ is the third order Casimir operator of the SU(3) group. (It is needed in the description of the experimental spectra in order to distinguish between the prolate and oblate states.) In spite of its simple structure this Hamiltonian is able to account for some experimental spectra to a reasonable approximation [14,33]. In fact, it is able to describe the spectra of different cluster configurations and the shell (or quartet) model in a unified way [33,12]. Furthermore, since U(3) symmetry defines the quadrupole deformation, the spectrum of the dynamical symmetry of group-chain (17) and energy-operator (18) represents the common intersection of the shell, cluster and (quadrupole) collective model [13] for a multi-shell problem. A detailed example on the performance of this Hamiltonian was discussed in [14]. In particular, a prediction on a complete (high-lying) cluster spectrum was obtained from the description of the low-lying quartet (shell) spectrum, that turned out to be very similar to the experimental results. Here we concentrate on the intrinsic shapes of the shell and cluster states, which originate from the spontaneous breaking.

In the Hamiltonian (18) the intinsic and collective parts are the U(3)- and O(3)-related terms, respectively (similarly to some previous cases):

$$H_{intr} = C^{(1)}_{U3} + \alpha C^{(2)}_{SU3} + \beta C^{(3)}_{SU3},\ H_{coll} = \delta C^{(2)}_{SO3}. \tag{19}$$

The intrinsic Hamiltonian is spherically symmetric, but the cluster configuration is usually not, therefore, the rotational symmetry is spontaneously broken.

As for the shape of the intrinsic state of the cluster model, it has an interesting aspect, known as the cluster-shell duality. It means that the cluster and shell wave-function can be identical, as a consequence of the antisymmetrization. It is seen most easily in the SU(3)-models, that we discuss here.

It was shown by Wildermuth and Kanellopoulos [34] that the Hamiltonians of the two models can be rewritten into each other exactly for harmonic oscillator interactions. In fact the connection is valid also for the more general interactions of U(3) dynamical symmetry [35]. Thus any cluster state can be expanded in terms of shell basis of the same number of oscillator quanta. Furthermore, basis states of different U(3) representations are orthogonal to each other. Therefore, if a shell model state of a specific U(3) symmetry has a single multiplicity, then the cluster state (any cluster state) is identical with it (having only a single term in the expansion).

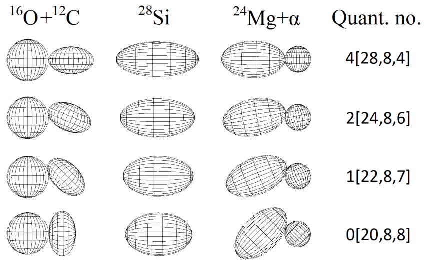

Fig. 1. Shape of some states in ^{28}Si. The quantum numbers in parenthesis are the U(3) labels, while the first integer shows the major shell excitation quanta. Note, that the multiplicity of these U(3) states in the shell basis is 1, therefore, the indicated shell, and both cluster configurations have wavefunctions with 100% overlap in each case, as a consequence of the antisymmetrization.

This duality was recognised in the early days of the cluster study for the ground state of some light nuclei [34]. But it turns out to be valid also for many excited bands, including the shape isomers, like superdeformed and hyperdeformed configurations, where the U(3) symmetry is especially good [36].

In case of the ^{28}Si nucleus e.g. [14] the following bands have a pure cluster (^{16}O+^{12}C, as well as ^{24}Mg+^{4}He) and shell configurations at the same time: 0(12,0)0^+, 1(14,1)1^-, 2(16,2)0^+, 4(20,4)0^+, where the quantum numbers are $n_x(\lambda,\mu)K^\pi$, with n_x denoting the excitation quantum with respect to the ground state. The 4(20,4)0^+ state has the superdeformed shape [37], with a small triaxiality, which was found recently [38]. The deformed cluster and shell configurations of the intrinsic states, that appear as a result of the spontaneous symmetry breaking are shown in figure 1. From the shell-model side the quadrupole shape is given by the U(3) quantum numbers of the state, as discussed before. The corresponding cluster configurations can be obtained from the Harvey prescription [39,40] and from the U(3) selection rule [41] which describes the structural aspect of the fusion (or fission) of a nucleus in terms of the harmonic oscillator basis. These shell and cluster configurations turn out to be identical with each other, due to the effect of the antisymmetrization.

5 Summary and conclusions

In this paper we have discussed the duality of symmetry breaking that shows up in the dynamical symmetry limits of the algebraic structure models. In such cases the Hamiltonian is expressed in terms of the invariant operators of a group-chain $G \supset G'$ (where G and G' stand for single groups or group chains). Then only the last group denotes an exact symmetry, the symmetries of the larger groups are dynamically broken, i.e. their representation labels are good quantum numbers of the states, but they are not degenerate. In addition to this dynamical symmetry breaking a spontaneous breaking is also taking place. It appears e.g. when the Hamiltonian can be split up into an intrinsic and collective parts: $H = \alpha H_G + \beta H_{G'}$. Then H_G has not only a G symmetry, but G' symmetry as well (since G' is a subgroup of G). Nevertheless, the eigenvectors of the intrinsic Hamiltonian are not necessarily G'-symmetric, when a degeneracy is present. Thus a spontaneous breaking is taking place in the eigenvalue equation of H_G: a symmetric Hamiltonian has a nonsymmetric ground state.

In the Elliott model the $SU(3) \supset SO(3)$ group-chain is relevant. The $SU(3)$ quantum numbers define the quadrupole shape of the intrinsic state, and the collective

Hamiltonian is the rotational term L^2. $SU(3)$ is dynamically broken, and $SO(3)$ is spontaneously broken in the eigenvalue equation of the intrinsic Hamiltonian. In several extensions of the model, like e.g. in the symplectic shell model, in the contracted symplectic collective model or in the semimicroscopic algebraic cluster model the same group chain appears, embedded into a larger group. As a consequence, the dynamical and spontaneous breaking in these models is taking place in the same way. In some other models, like interacting boson model, and models of octupole deformation a similar scenario is present, built on different algebraic structure.

When multi shell excitations are considered, then the common intersection of the shell, collective and cluster models turns out to be a $U_x(3) \otimes U_y(3) \supset U(3) \supset SU(3) \supset SO(3)$ dynamical symmetry. This symmetry is called multiconfigurational dynamical symmetry. In this case, too, the duality of the dynamical and spontaneous breaking is similar to that of the Elliott model. In relation with the deformation of the intrinsic states, which appears as a result of the spontaneous breaking, a further interesting duality can be seen in the MUSY. This is the cluster-shell duality. In particular, some intrinsic states (collective bands) of the shell and cluster models that we associate to rather different geometrical pictures become identical due to the antisymmetrization. Here we showed four intrinsic states of the ^{28}Si nucleus (with 0,1,2, and 4 excitation quanta, the latter one being the recently observed superdeformed state). In these cases the shell model and ^{24}Mg+^{4}He and ^{16}O+^{12}C cluster models have wavefunctions with 100% overlap.

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