The theory of Wiener–Itô integrals in vector valued Gaussian stationary random fields. Part I

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Abstract: The subject of this work is the multivariate generalization of the theory of multiple Wiener–Itô integrals. In the scalar valued case this theory was described in paper [9]. The proofs of the present paper apply the technique of that work, but in the proof of some results new ideas were needed. The motivation for this study was a result in paper [1] of Arcones which contained the multivariate generalization of a non-central limit theorem for non-linear functionals of Gaussian stationary random fields presented in paper [5]. However, the formulation of Arcones' result was incorrect. To present it in a correct form the multivariate version of the theory explained in [9] has to be worked out, because the notions introduced in this theory are needed in its formulation. This is done in the present paper. In its continuation, in paper [11] it is explained how to work out a method with the help of the results in this work that enables us to prove non-Gaussian limit theorems for non-linear functionals of vector valued Gaussian stationary random fields. The right version of Arcones' result presented also in the introduction of this work will be formulated and proved with its help in paper [12].

1 Introduction. An overview of the results.

Let $X(p) = (X_1(p), \dots, X_d(p))$, $p \in \mathbb{Z}^\nu$, where $\mathbb{Z}^\nu$ denotes the lattice points with integer coordinates in the ν-dimensional Euclidean space $\mathbb{R}^\nu$, be a d-dimensional real valued Gaussian stationary random field with expectation $EX(p) = 0$, $p \in \mathbb{Z}^\nu$. We define the notion of Gaussian property of a random field in the usual way, i.e. we demand that all finite sets $(X(p_1), \dots, X(p_k))$, $p_j \in \mathbb{Z}^\nu$, $1 \le j \le k$, be a Gaussian random vector, and we call a random field $X(p)$, $p \in \mathbb{Z}^\nu$, stationary if for all $m \in \mathbb{Z}^\nu$ the random field $X^{(m)}(p) = X(p+m)$, $p \in \mathbb{Z}^\nu$, has the same finite dimensional distributions as the original random field $X(p)$, $p \in \mathbb{Z}^\nu$. In most works only the case $\nu = 1$ is considered, but since

we can prove our results without any difficulty for stationary random fields with arbitrary parameter $\nu \geq 1$ we consider such more general models.

Our goal is to work out a good calculus which provides such a representation of the non-linear functionals of our vector valued Gaussian stationary random field which helps us in the study of limit theorems for such functionals. To understand what kind of limit theorems we have in mind take the following example.

Let us have a function $H(x_1, \dots, x_d)$ of d variables, and define with the help of a d-dimensional vector valued Gaussian stationary random field

$$X(p) = (X_1(p), \dots, X_d(p)), \quad p \in \mathbb{Z}^\nu,$$

and this function the random variables $Y(p) = H(X_1(p), \dots, X_d(p))$ for all $p \in \mathbb{Z}^\nu$. Let us introduce for all $N = 1, 2, \dots$ the normalized sum

$$S_N = A_N^{-1} \sum_{p \in B_N} Y(p) \tag{1.1}$$

with an appropriate norming constant $A_N > 0$, where

$$B_N = \{p = (p_1, \dots, p_\nu) \colon \ 0 \leq p_k < N \text{ for all } 1 \leq k \leq \nu\}. \tag{1.2}$$

We are interested in a limit theorem for these normalized sums S_N with an appropriate norming constant A_N as $N \to \infty$. In particular, we want to know when we get a classical central limit theorem with the natural normalization $A_N = N^{\nu/2}$ and when appear new kind of limit theorems. These questions were studied in the special scalar valued case $d = 1$ in papers [2] and [5]. Arcones investigated the multivariate generalization of the results in these papers.

He proved the multivariate version of the result in paper [2] which states that if the covariance function of the underlying Gaussian field tends to zero sufficiently fast at infinity, and the function $H(x_1, \dots, x_d)$ has some nice properties, then the central limit theorem holds with the classical normalization. (He considered only the case $\nu = 1$, but this restriction has no great importance.) In Theorem 6 of his paper he also formulated a result about a non-central limit theorem under appropriate conditions. But there are some serious problems with that result. Arcones wanted to prove a multivariate generalization of the result in paper [5], but to do this he should have solved some problems whose discussion he omitted.

The Gaussian limit theorem can be proved in the multivariate case by means of a natural generalization of the method in paper [2], or one can apply some more powerful new method, (see e.g. [13]), but in the proof of the multivariate generalization of the non-central limit theorem 6 in paper [1] some new problems appear whose solution demands hard work.

The first problem is related to the formulation of the result. In paper [5] the limit distribution is presented by means of a multiple Wiener-Itô integral with respect to the random spectral measure of a one-dimensional stationary (generalized) Gaussian random field. This random integral was introduced in

the paper of Dobrushin [4], and it is explained in more detail in my Lecture Note [9]. But this notion was worked out in Dobrushin's paper only for scalar valued random fields, and the limit distribution in Theorem 6 of Arcones' paper is presented with the help of Wiener–Itô integrals with respect to random spectral measures corresponding to vector valued stationary Gaussian random fields. Such integrals were not defined before, and their definition is far from trivial. The goal of the present paper is to fill this gap. Here the multivariate random spectral measures will be introduced together with the multiple Wiener–itô integrals with respect to them, and their most important properties will be proved. This is needed for the right formulation and proof of Arcones' result. I shall formulate the right version of this result in the introduction of this paper, but its proof will be given only in paper [12] with the help of the results in this work and its continuation [11].

To understand what kind of problems we meet in this paper let us first consider briefly how the theory of Wiener–Itô integrals was worked out for scalar valued random fields by Itô in [8] and Dobrushin in [4].

Itô considered a Gaussian random field in [8] whose elements could be expressed as random integrals with respect to a Gaussian orthogonal random measure. He also defined multiple random integrals (called later Wiener–Itô integrals in the literature) with respect to this orthogonal random measure, and expressed all square integrable random variables measurable with respect to the σ-algebra generated by the elements of the Gaussian orthogonal random measure as a sum of such multiple integrals. The introduction of this integral turned out to be useful, because it helped in the study of non-linear functionals of the Gaussian random field defined by means of this integral. In particular, Itô found a very useful relation, called Itô's formula in the literature, between the multiple random integrals he defined and Hermite polynomials.

Later Dobrushin worked out a version of this theory in [4], where he studied non-linear functionals of a stationary Gaussian random field. In such a random field a spectral and a random spectral measure can be defined in such a way that the elements of the stationary Gaussian random field can be expressed in a special form of (one-fold) random integrals with respect to the random spectral measure. These random integrals can be considered as the Fourier transforms of the random spectral measure. Dobrushin defined also multiple random integrals with respect to this random spectral measure, and studied their properties. He proved that these random integrals defined with respect to the random spectral measure have similar properties as the multiple integrals introduced by Itô. In particular, he proved Itô's formula for this new type of random integrals. This enabled him to express all square integrable random variables measurable with respect to the σ-algebra generated by the elements of the original stationary Gaussian random field as a sum of multiple random integrals with respect to the random spectral measure. He also found a simple and useful formula for the calculation of the shift transforms of a random variable which is presented as a sum of multiple random integrals. With the help of these results the normalized random sums S_N defined in (1.1) can be expressed in a simple and useful form if the underlying stationary Gaussian random field is scalar valued (i.e. $d = 1$).

This representation of the normalized random sums S_N made possible to prove the limit theorems in [5].

We want to prove the generalization of the results in [5] for non-linear functionals of vector valued stationary Gaussian random fields. The first step of this program is to work out the multivariate version of Dobrushin's theory, and this is the subject of the present paper.

First we have to define the spectral and random spectral measure of vector valued stationary Gaussian random fields, and this is the subject of Sections 2 and 3. To do this the multivariate version of some classical results has to be proved. In the scalar valued case a spectral measure can be defined whose Fourier transform is the correlation function of the stationary random field we are working with. In the case of a vector valued stationary random field of dimension d the correlation function is a $d \times d$ dimensional matrix valued function. It can be shown that there exists a $d \times d$ dimensional matrix valued measure on the d dimensional torus $[-\pi, \pi)^d$ for which each coordinate of the matrix valued correlation function is the Fourier transforms of the corresponding coordinate of this matrix valued measure. This measure is called the spectral measure of the random field. In the scalar valued case, i.e. if $d = 1$ the spectral measure is a positive measure, while in the vector valued case it is a positive semidefinite matrix valued measure. A more detailed description of these results together with their proofs is given in Section 2.

In Section 3 the so-called random spectral measure corresponding to a vector valued stationary Gaussian random field is defined. It is a vector valued random measure with the same dimension d as the underlying vector valued stationary Gaussian random field. Its distribution is determined by the spectral measure of the underlying random field. A random integral can be defined with respect to the coordinates of the random spectral measure, and each coordinate of the elements of the underlying vector valued Gaussian random field can be expressed by means of an appropriate random integral with respect to the corresponding coordinate of the random spectral measure. Because of the form of this integral this result can be interpreted so that the underlying stationary Gaussian random field is the Fourier transform of the random spectral measure corresponding to it. The construction of the random spectral measure and the description of its most important properties is given in Section 3.

Moreover, we need later the notion of spectral measures and random spectral measures corresponding to stationary generalized random fields, and they are introduced in Section 4. In the main text of this paper a more detailed, precise definition of these notions will be given. We have to define these objects, because we can formulate the limit in the limit theorems we are interested in in this paper by means of multiple random integrals with respect to the random spectral measures corresponding to stationary generalized random fields.

Then I define the multiple Wiener–Itô integrals with respect to the coordinates of a vector valued random spectral measure in Section 5, and I also prove there their most important properties. In Section 6 I prove an important result, called the diagram formula which enables us to express the product of two multiple Wiener–Itô integrals as the sum of appropriately defined multiple

Wiener–Itô integrals. The present paper contains these results.

In the continuation of this paper, in work [11] I work out the basic tools needed in the proof of such non-central limit theorems as the multivariate generalization of the limit theorem in [5]. First I prove, with the help of the above mentioned diagram formula, an important result about the relation between multiple Wiener–Itô integrals and Wick polynomials of Gaussian vectors. Wick polynomials are the several dimensional generalizations of Hermite polynomials, and the result mentioned before is the natural multivariate generalization of Itô's formula. Besides, [11] contains a formula that enables us to express the shift transforms of a random variable given in the form of a sum of multiple random variables in a useful form. These results enable us to rewrite the normalized random sums S_N defined in (1.1) in a form which helps in the study of limit theorems. They enabled me to formulate and prove in [12] the right version of Theorem 6 in Arcones' paper [1].

Next I briefly describe the right version of Arcones' non-central limit theorem. In its formulation we consider d-dimensional stationary Gaussian random fields

$$X(p) = (X_1(p), \dots, X_d(p)), \quad EX_j(p) = 0 \text{ for all } 1 \le j \le \nu \text{ and } p \in \mathbb{Z}^\nu,$$

whose covariance function $r_{j,j'}(p) = EX_j(0)X_{j'}(p)$, $1 \le j, j' \le d$, $p \in \mathbb{Z}^\nu$, is such a matrix valued function whose coordinates decrease asymptotically polynomially at infinity with some power $0 < \alpha < \nu$. More generally, this behaviour may be slightly modified by multiplication with a slowly varying function. More explicitly, we demand that

$$\lim_{T \to \infty} \sup_{p\colon\ p \in \mathbb{Z}^\nu,\, |p| \ge T} \frac{\left| r_{j,j'}(p) - a_{j,j'}(\frac{p}{|p|})|p|^{-\alpha} L(|p|) \right|}{|p|^{-\alpha} L(|p|)} = 0 \tag{1.3}$$

for all $1 \le j, j' \le d$, where $0 < \alpha < \nu$, $L(t)$, $t \ge 1$, is a real valued function, slowly varying at infinity, bounded in all finite intervals, and $a_{j,j'}(t)$ is a real valued continuous function on the unit sphere $\mathcal{S}_{\nu-1} = \{x\colon\ x \in \mathbb{R}^\nu,\ |x| = 1\}$, and the identity $a_{j',j}(x) = a_{j,j'}(-x)$ holds for all $x \in \mathcal{S}_{\nu-1}$ and $1 \le j, j' \le d$.

For the sake of simpler discussion we also demand that

$$EX_j^2(0) = 1 \text{ for all } 1 \le j \le d, \text{ and } EX_j(0)X_{j'}(0) = 0 \text{ if } j \ne j',\ \ 1 \le j, j' \le d. \tag{1.4}$$

This is not an essential restriction, as it is explained in [12].

We want to describe the limit behaviour of some non-linear functionals of such a random field. To do this first we describe the asymptotic behaviour of its spectral measure. To formulate such a result let us introduce the following notation.

Given a vector valued stationary random field $X(p) = (X_1(p), \dots, X_d(p))$, $p \in \mathbb{Z}^\nu$, with expectation zero and covariance function $r_{j,j'}(p) = EX_j(0)X_{j'}(p)$, $1 \le j, j' \le d$, $p \in \mathbb{Z}^\nu$, that satisfies relation (1.3) let us consider its matrix

valued spectral measure $G = (G_{j,j'})$, $1 \le j, j' \le d$, on the torus $[-\pi, \pi)^\nu$. Take its rescaled version $G^{(N)} = (G^{(N)}_{j,j'}$, $1 \le j, j' \le d$,

$$G^{(N)}_{j,j'}(A) = \frac{N^\alpha}{L(N)} G_{j,j'}\left(\frac{A}{N}\right), \quad A \in \mathcal{B}^\nu, \quad N = 1, 2, \dots, \quad 1 \le j, j' \le d, \quad (1.5)$$

concentrated on $[-N\pi, N\pi)^\nu$ for all $N = 1, 2, \dots$, where $\mathcal{B}^\nu$ denotes the σ-algebra of the Borel measurable sets on $\mathbb{R}^\nu$. In the next result we give the limit of the matrix valued measures $G^{(N)}$, as $N \to \infty$. Since the coordinates of the matrices $G^{(N)}$ are non-probability measures, and their limits are non-finite measures we have to introduce the right form of convergence which will be applied in the limit theorem we shall describe. In paper [12] the so-called vague convergence of complex measures are defined, (more precisely its definition is recalled). In this definition also the notion of complex measures with locally finite measures appear whose definition is explained in Section 4 of this paper. This notion was introduced, because they are needed in the study of spectral measures of stationary generalized fields, and we want to work with such objects. In the presentation of the limit theorem I want to discuss we need the result of Proposition 1.1 of [12] whose formulation applies the above notions. This Proposition 1.1 agrees with the following result.

Proposition 1.1. *Let $G = (G_{j,j'})$ be the matrix valued spectral measure of a d-dimensional vector valued stationary random field whose covariance function $r_{j,j'}(p)$ satisfies relation (1.3) with some parameter $0 < \alpha < \nu$. Then for all pairs $1 \le j, j' \le d$ the sequence of complex measures $G^{(N)}_{j,j'}$ defined in (1.5) with the help of the complex measure $G_{j,j'}$ tends vaguely to a complex measure $G^{(0)}_{j,j'}$ on $\mathbb{R}^\nu$ with locally finite total variation. These complex measures $G^{(0)}_{j,j'}$, $1 \le j, j' \le d$, have the homogeneity property*

$$G^{(0)}_{j,j'}(A) = t^{-\alpha} G^{(0)}_{j,j'}(tA) \quad \text{for all bounded } A \in \mathcal{B}^\nu,\ 1 \le j, j' \le d, \text{ and } t > 0. \quad (1.6)$$

The complex measure $G^{(0)}_{j,j'}$ with locally finite variation is determined by the number $0 < \alpha < \nu$ and the function $a_{j,j'}(\cdot)$ on the unit sphere $S_{\nu-1}$ introduced in formula (1.3).

There exists a vector valued Gaussian stationary generalized random field on $\mathbb{R}^\nu$ with that matrix valued spectral measure $(G^{(0)}_{j,j'})$, $1 \le j, j' \le d$, whose coordinates are the above defined complex measures $G^{(0)}_{j,j'}$, $1 \le j, j' \le d$.

In the non-central limit theorem I shall describe the limit of such random variables S_N is investigated which are defined by formulas (1.1) and (1.2) with the help of a vector valued stationary Gaussian random field whose correlation function satisfies relations (1.3) and (1.4) and an appropriate norming constant A_N. To give a complete definition of these random variables we must tell what kind of functions $H(x_1, \dots, x_d)$ we apply in their definition. I shall choose functions of the following form in this definition. $H(x_1, \dots, x_d)$ depends on a

previously fixed constant k, and it has the form

$$H(x_1,\dots,x_d)=\sum_{\substack{(k_1,\dots,k_d),\ k_j\ge 0,\ 1\le j\le d,\\ k_1+\dots+k_d=k}} c_{k_1,\dots,k_d}H_{k_1}(x_1)\cdots H_{k_d}(x_d) \tag{1.7}$$

with some coefficients $c_{k_1,\dots,k_d}$, where $H_k(\cdot)$ denotes the k-th Hermite polynomial with leading coefficient 1.

The limit distribution of the above introduced random variable S_N is desribed in Theorem 1.2A of [12]. This theorem is written down in the following Theorem 1.2. The limit in this result is presented by means of a multiple Wiener–Itô integral with respect to the random spectral measure corresponding to the matrix valued spectral measure $(G^{(0)}_{j,j'})$, $1\le j,j'\le d$, which appeared in Proposition 1.1. Let me remark that because of the homogeneity property (1.6) of this measure $G^{(0)}_{j,j}(\mathbb{R}^\nu)=\infty$ for any $1\le j\le d$. Hence this matrix valued spectral measure can be defined only as the spectral measure of a generalized and not as the spectral measure of an ordinary vector valued stationary random field.

Theorem 1.2. *Fix some integer $k\ge 1$, and let $X(p)=(X_1(p),\dots,X_d(p))$, $p\in\mathbb{Z}^\nu$, be a vector valued Gaussian stationary random field whose covariance function $r_{j,j'}(p)=EX_j(0)X_{j'}(p)$, $1\le j,j'\le d$, $p\in\mathbb{Z}^\nu$, satisfies relation (1.3) with some $0<\alpha<\frac{\nu}{k}$ and relation (1.4). Let $H(x_1,\dots,x_d)$ be a function of the form given in (1.7) with the parameter k we have fixed in the formulation of this result. Define the random variables $Y(p)=H(X_1(p),\dots,X_d(p))$ for all $p\in\mathbb{Z}^\nu$ together with their normalized partial sums*

$$S_N=\frac{1}{N^{\nu-k\alpha/2}L(N)^{k/2}}\sum_{p\in B_N}Y(p),$$

where the set B_N was defined in (1.2). These random variables S_N, $N=1,2,\dots$, satisfy the following limit theorem.

Let $Z_{G^{(0)}}=(Z_{G^{(0)},1},\dots,Z_{G^{(0)},d})$ be a vector valued random spectral measure which corresponds to the matrix valued spectral measure $(G^{(0)}_{j,j'})$, $1\le j,j'\le d$, defined in Proposition 1.1 with the help of the matrix valued spectral measure $G=(G_{j,j'})$, corresponding the covariance function $r_{j,j'}(p)$ we are working with. Then the sum of multiple Wiener–Itô integrals

$$\begin{aligned} S_0 \;=\; & \sum_{\substack{(k_1,\dots,k_d),\ k_j\ge 0,\ 1\le j\le d,\\ k_1+\dots+k_d=k}} c_{k_1,\dots,k_d}\int\prod_{l=1}^{\nu}\frac{e^{i(x_1^{(l)}+\dots+x_k^{(l)})}-1}{i(x_1^{(l)}+\dots+x_k^{(l)})} \\ & \qquad Z_{G^{(0)},j(1|k_1,\dots,k_d)}(\,dx_1)\dots Z_{G^{(0)},j(k|k_1,\dots,k_d)}(\,dx_k) \end{aligned} \tag{1.8}$$

exists. (These Wiener–Itô integrals are defined in Section 5 of this paper.) Here we use the notation $x_p=(x_p^{(1)},\dots,x_p^{(\nu)})$, $p=1,\dots,k$, and define the indices $j(s|k_1,\dots,k_d)$, $1\le s\le k$, as $j(s|k_1,\dots,k_d)=r$ if $\sum_{u=1}^{s-1}k_u<r\le\sum_{u=1}^{s}k_u$,

$1 \le s \le k$. *(For* $s = 1$ *we apply the notation* $\sum_{u=1}^{0} k_u = 0$ *in the definition of* $j(1|k_1, \dots, k_d)$*.) The normalized sums* S_N *converge in distribution to the random variable* S_0 *defined in (1.8) as* $N \to \infty$*.*

The indexation of the terms $Z_{G^{(0)},j(s|k_1,\dots,k_d)}(\,dx_s)$ in formula (1.8) can be explained in a simpler way. In the first k_1 arguments $x_1, \dots, x_{k_1}$ we wrote $Z_{G^{(0)},1}(\,dx_s)$, $1 \le s \le k_1$, in the next k_2 terms we wrote $Z_{G^{(0)},2}(\,dx_s)$, $k_1 + 1 \le s \le k_1 + k_2$, and so on. In the last k_d terms we wrote $Z_{G^{(0)},d}(\,dx_s)$, $k_1 + \dots + k_{d-1} + 1 \le s \le k$.

Actually a more general limit theorem is also proved in [12], but its proof is based on the result of Theorem 1.2. It is worth comparing Theorem 1.2 with its scalar valued version (i.e. with the result in the case $d = 1$ proved in [5]).

In paper [5] a result similar to Theorem 1.2 is proved in the scalar valued case. In that result $CH_k(x)$, $C \neq 0$, i.e. the k-th Hermite polynomial multiplied with a non-zero coefficient C plays the same role as the function $H(\cdot)$ defined in (1.7) in Theorem 1.2, and the condition $k\alpha < \nu$ has to be imposed. The limit is given by formula (1.8) in the case $d = 1$ with $H(x) = CH_k(x)$. Let me remark that the Wick polynomials, i.e. the multivariate generalizations of Hermite polynomials appeared in Theorem 1.2 in a hidden way. (See e.g.Section 2 of [9] for the definition of Wick polynomials.) Indeed, the random variables $Y(p) = H(X_1(p), \dots, X_d(p))$, $p \in \mathbb{Z}^\nu$, defined with the help of the function $H(\cdot)$ introduced in formula (1.7) are Wick polynomials of order k because of the relation (1.4). (See Corollary 2.3 in [9].) This indicates that the role of Hermite polynomials in results about scalar valued stationary Gaussian random fields is taken by Wick polynomials in the their vector valued counterparts. The next results also show such a correspondence.

The limit theorem in [5] remains valid if we replace the function $CH_k(x)$ in it with such a function $H(x)$ whose expansion with respect to the Hermite polynomials contains only terms $H_{k'}(x)$ of order $k' \ge k$, and the term $H_k(x)$ has a non-zero coefficient. The limit is the same as in the case when we take only the first term const.$H_k(x)$ in the expansion of the function $H(x)$. Similarly, Theorem 1.2 formulated above in the multivariate case remains valid if such a random random variable $H(X_1(0), \dots, X_d(0))$ is taken whose expansion with respect to Wick polynomials starts with a non-zero Wick polynomial of order k, and $k\alpha < \nu$. The limit does not change if we take only the term of order k of $H(X_1(0), \dots, X_d(0))$ in this expansion.

Let me finally remark that the Theorem holds only under the condition $k\alpha < \nu$. In the case $k\alpha > \nu$ the central limit theorem holds for S_N with the usual norming constant $A_N = N^{\nu/2}$. This follows from a slight generalization of the (correct) results in Arcones' paper [1]. In the boundary case $k\alpha = \nu$ the central limit theorem holds again for S_N, but in this case the norming constant may have the form $A_N = N^\nu L'(N)$ with a slowly varying function $L'(N)$ tending to infinity as $N \to \infty$. Let me also remark that the definition of the limit distribution in Theorem 1.2 given in formula (1.8), is meaningful only for $k\alpha < \nu$. This formula contains a multiple Wiener–Itô integral, and we have to check whether this Wiener–Itô integral is meaningful. It is explained at the beginning

of Section 5 that the multiple Wiener–Itô integrals are defined only with such kernel functions which satisfy an integrability condition. (This condition is formulated in property (b) in the definition of a class of functions $\mathcal{K}_{n,j_1,\dots,j_n}$.) It can be seen that the Wiener–Itô integral appearing in formula (1.8) is meaningful if $k\alpha < \nu$, because this integrability condition is satisfied in this case. On the other hand, this integral cannot be defined if $k\alpha \ge \nu$, because in this case this integrability condition is violated.

1.1 A more detailed description of the results.

Next I give a more detailed overview about the results of this paper.

First I characterize the distribution of the vector valued Gaussian stationary random fields $X(p) = (X_1(p), \dots, X_d(p))$, $p \in \mathbb{Z}^\nu$, with expectation zero. This is the subject of the second section of this work. Because of the Gaussian and stationary property of such a random field its distribution is determined by the correlation function $r_{j,j'}(p) = EX_j(0)X_{j'}(p)$ for all $1 \le j, j' \le d$ and $p \in \mathbb{Z}^\nu$. We are interested in the description of those functions $r_{j,j'}(p)$ which can appear as the correlation function of a vector valued stationary random field.

In the scalar valued case a well-known result solves this problem. The correlation function $r(p) = EX(0)X(p)$, $p \in \mathbb{Z}^\nu$, of a stationary field $X(p)$, $p \in \mathbb{Z}^\nu$, can be represented in a unique way as the Fourier transform of a spectral measure, and the spectral measures can be characterized. Namely, we call the finite (non negative), even measures on the torus $[-\pi, \pi)^\nu$ spectral measures. For any correlation function $r(p)$ of a stationary field there is a unique spectral measure μ such that $r(p) = \int e^{i(p,x)}\mu(\,dx)$ for all $p \in \mathbb{Z}^\nu$, and for all spectral measures μ there is a (Gaussian) stationary random field whose correlation function equals the Fourier transform of this spectral measure μ.

In Section 2 we prove a similar result for vector valued stationary random fields. In the case of a vector valued Gaussian stationary random field $X(p) = (X_1(p), \dots, X_d(p))$, $p \in \mathbb{Z}^\nu$, we have for all pairs of indices (j, j'), $1 \le j, j' \le d$, a unique complex measure $G_{j,j'}$ on the torus $[-\pi, \pi)^\nu$ with finite total variation such that $r_{j,j'}(p) = EX_j(0)X_{j'}(p) = \int e^{i(p,x)}G_{j,j'}(\,dx)$ for all $p \in \mathbb{Z}^\nu$. This can be interpreted so that the correlation function $r_{j,j'}(p)$, $1 \le j, j' \le d$, $p \in \mathbb{Z}^\nu$, is the Fourier transform of a matrix valued measure $(G_{j,j'})$, $1 \le j, j' \le d$, on the torus $[-\pi, \pi)^\nu$. We want to give, similarly to the scalar valued case, a complete description of those matrix valued measures on the torus $[-\pi, \pi)^\nu$ for which the correlation function of a vector valued Gaussian stationary random field can be represented as its Fourier transform. Such matrix valued measures will be called matrix valued spectral measures.

As I have mentioned, the coordinates of a matrix valued spectral measure are complex measures with finite total variation. The scalar valued counterpart of this condition is the condition that the spectral measure of a scalar valued stationary random field must be finite. Another important property of a matrix valued spectral measure is that it must be positive semidefinite. The meaning of this property is explained before the formulation of Theorem 2.2, and Lemma 2.3 gives a different, equivalent characterization of this property. Let me remark

that in the scalar valued case the spectral measure must be a measure (and not only a complex measure), and this fact corresponds to the above property of matrix valued spectral measures. Finally, a matrix valued spectral measure must be even. This means that its coordinates are even, i.e. for all $1 \le j, j' \le d$ and measurable sets A on the torus $G_{j,j'}(-A) = \overline{G_{j,j'}(A)}$, where the overline indicates complex conjugate.

Theorem 2.2 states that the above properties characterize the matrix valued spectral measures. Let me remark that there are papers, (see e.g. [3], [7] or [14]) which contain the above results, although in a slightly different formulation, at least in the case $\nu = 1$. Nevertheless, I worked out their proof, since I applied a different method which is used also in the later part of the paper.

In Section 3 I introduce the vector valued random spectral measures corresponding to a matrix valued spectral measure $(G_{j,j'})$, $1 \le j, j' \le d$. To do this first I consider a vector valued stationary Gaussian random field $X(p) = (X_1(p), \dots, X_d(p))$, $p \in \mathbb{Z}^\nu$, with spectral measure $(G_{j,j'})$, $1 \le j, j' \le d$, and show that a vector valued random measure $Z_G = (Z_{G_1}, \dots, Z_{G_d})$ can be defined on the measurable subsets $A \subset [-\pi, \pi)^\nu$ of the torus which have some nice properties. A random integral can be defined with respect to the coordinates of this random measure, and the coordinates $X_j(p)$, $1 \le j \le d$, $p \in \mathbb{Z}^\nu$, of the random field $X(p)$ can be expressed as the Fourier transforms of the appropriate coordinate Z_{G_j} of this random measure. More explicitly, $X_j(p) = \int e^{i(p,x)} Z_{G,j}(\, dx)$ for all $p \in \mathbb{Z}^\nu$ and $1 \le j \le d$. I remark that the random variables $Z_{G,j}(A)$, $1 \le j \le d$, $A \subset [-\pi, \pi)^\nu$, are complex valued.

I have listed some properties of this random measure $(Z_{G,1}, \dots, Z_{G,d})$. These properties determine its distribution, and they depend only on the spectral measure $(G_{j.j'})$, $1 \le j, j' \le d$, of the underlying random field $X(p)$, $p \in \mathbb{Z}^\nu$. We shall call the vector valued random measures with these properties a vector valued random spectral measure corresponding to the matrix valued spectral measure $(G_{j,j'})$, $1 \le j, j' \le d$. We can prove that the Fourier transform of all vector valued random spectral measures corresponding to a matrix valued spectral measure can be defined, and it is a vector valued Gaussian stationary random field with this matrix valued spectral measure.

Besides the above results I also proved some important properties of the random integrals with respect to a vector valued spectral measure in Section 3. I characterized those functions which can be integrated with respect to these random spectral measure, and also described those functions whose integrals are real valued random variables. In particular, I proved that if a vector valued Gaussian stationary random field $X(p) = (X_1(p), \dots, X_d(p))$, $p \in \mathbb{Z}^\nu$, is given, we fix some parameter $1 \le j \le d$, and take the real Hilbert space consisting of the closure of finite linear combinations $\sum_k c_k X_j(p_k)$ with real number valued coefficient c_k in the Hilbert space of square integrable random variables, then each element of this Hilbert space can be expressed as the integral of a function on the torus $[-\pi, \pi)^\nu$ with respect to the random spectral measure $Z_{G,j}$. The functions taking part in the representation of this Hilbert space also constitute a real Hilbert space. A more detailed formulation of this result is given in Lemma 3.2.

It may be worth discussing the relation of the results in Section 3 to their scalar valued correspondents. The results about the existence of random spectral measures for scalar valued Gaussian stationary random fields give a great help in proving the results in Section 3. In particular, these results provide the definition of the random spectral measures $Z_{G,j}$, and determine their distribution for all $1 \le j \le d$. The definition of $Z_{G,j}$, and the properties determining its distribution depend only on the measure $G_{j,j}$. On the other hand, we had to carry out some additional work to prove those properties of a vector valued spectral random measure which determine the joint distribution of their coordinates. The non-diagonal elements $G_{j,j'}$ with $j \neq j'$ of the matrix valued spectral measure $(G_{j,j'}, 1 \le j, j' \le d$, appear at this point of the investigation.

The fourth section deals with a special subject, and our motivation to study it demands some explanation. Here we consider vector valued Gaussian stationary generalized random fields.

We could have considered the continuous time version of vector valued stationary random fields where the parameter set is $t \in \mathbb{R}^\nu$ and not $p \in \mathbb{Z}^\nu$. Here we did not discuss such models, we have considered instead vector valued Gaussian stationary generalized random fields. This means a set of random vectors $(X_1(\varphi), \dots, X_d(\varphi))$ with some nice properties which are indexed by an appropriately chosen class of functions. The precise definition of this notion is given in Section 4. We have constructed a large class of Gaussian stationary generalized random fields, presented their matrix valued spectral measures, and constructed the vector valued random spectral measures corresponding to them. In [9] the notion of Gaussian stationary generalized random fields was introduced and investigated in the scalar valued case. Some useful results were proved there. It was shown (with the help of some important results of Laurent Schwartz about generalized functions) that in the scalar valued case the class of Gaussian, stationary generalized random fields constructed in such a way as it was done in the present paper contains all Gaussian stationary generalized random fields. (Here I consider two random fields the same if their finite dimensional distributions agree.) Similarly, it is very likely that also in the multivariate case all generalized stationary generalized Gaussian random fields can be constructed by the method described in this paper. But I did not study this question, because I was interested in a different problem.

Although the theory of generalized random fields is an interesting subject in itself, I investigated it for a different reason. I was interested in the matrix valued spectral measures of vector valued Gaussian stationary generalized random fields and the vector valued random spectral measures corresponding to them and not in the Gaussian, stationary generalized random fields which were needed for their construction. They behave similarly to the analogous objects corresponding to (non-generalized) Gaussian stationary random fields. We can work with them in the same way. Nevertheless, there is a difference between these new spectral and random spectral measures and their previously defined counterparts which is very important for us. Namely, the coordinates of a matrix valued spectral measure corresponding to a non-generalized random field are complex measures with finite total variation, while in the case of generalized

random fields the matrix valued spectral measures need not satisfy this condition. It is enough to demand that the corresponding matrix valued measures have locally finite total variation, and the matrix valued spectral measures are semidefinite matrix valued measures with moderately increasing distribution at infinity. (The definition of these notions is contained in Section 4.)

The above facts mean that we can work with a much larger class of random spectral measures after the introduction of Gaussian stationary generalized random fields and random spectral measures corresponding to them. This is important for us, because in the limit theorems we are interested in the limit can be expressed by means of multiple Wiener–Itô integrals with respect to random spectral measures constructed with the help of vector valued Gaussian stationary generalized random fields. Theorem 1.2 discussed in this introduction is an example for such a limit theorem.

Sections 2—4 contain the main results about the linear functionals of vector valued Gaussian stationary random fields. They are also needed in the study of their non-linear functionals , and this is the subject of Sections 5 and 6. The results of these sections help us to work out some tools which are useful in the study of limit theorems with a new type of non-Gaussian limit.

In Section 5 multiple Wiener–Itô integrals are defined with respect to the coordinates of a vector valued random spectral measure $(Z_{G,1}, \dots, Z_{G,d})$. We define for all numbers $n = 1, 2, \dots$, and parameters $j_1, \dots, j_n$ such that $1 \le j_k \le d$ for all $1 \le k \le n$ and all functions $f \in \mathcal{K}_{n,j_1,\dots,j_n}$, where $\mathcal{K}_{n,j_1,\dots,j_n}$ is a real Hilbert space defined in Section 5, an n-fold Wiener–Itô integral

$$I_n(f|j_1, \dots, j_n) = \int f(x_1, \dots, x_n) Z_{G,j_1}(\,dx_1) \dots Z_{G,j_n}(\,dx_n),$$

and prove some of its basic properties. The definition and proofs are very similar to the definition and proofs in scalar valued case, only we have to apply the properties of vector valued random spectral measures.

There is one point where we have a weaker estimate than in the scalar valued case. We can give an upper bound on the second moment of a multiple Wiener–Itô integral with the help of the L_2 norm of the kernel function of this integral in the way as it is formulated in formula (5.6), but we can state here only an inequality and not an equality. The behaviour of Wiener–Itô integrals with respect to a scalar valued random spectral measure is different. If we integrate in this case a symmetric function, and we may restrict our attention to such integrals, then we have equality in the corresponding relation. This weaker form of the estimate (5.6) has the consequence that in certain problems we can get only weaker results for Wiener–Itô integrals with respect to the coordinates of a vector valued random spectral measure than for Wiener–Itô integrals with respect to scalar valued random spectral measures. But this will cause no serious problem in our study about multiple Wiener–Itô integrals with respect to vector valued random spectral measures.

Multiple Wiener–Itô integrals were introduced in order to express a large class of random variables with their help. More precisely, we are interested in the following problem. Let us have a vector valued Gaussian stationary random

field $X(p) = (X_1(p), \dots, X_d(p))$, $p \in \mathbb{Z}^\nu$. Their elements can be expressed as the Fourier transforms of a vector valued random spectral measure $Z_G = (Z_{G,1}, \dots, Z_{G,d})$. Let us consider the real Hilbert space $\mathcal{H}$ defined in the second paragraph of Section 5 with the help of this vector valued stationary Gaussian random field. We would like to express the elements of this Hilbert space in the form of a sum of multiple Wiener–Itô integrals with respect to the coordinates of the vector valued spectral measure Z_G. This problem together with the study of a theory useful in the investigation of limit theorems for non-linear functionals of vector valued stationary Gaussian random fields will be the subject of the second part of this work [11]. But to carry out this program we still need the proof of an important result about multiple Wiener–Itô integrals discussed in Section 6 of this work.

In Section 6 I formulate and prove the multivariate version of a classical result. I describe the product of two multiple Wiener–Itô integrals as the sum of multiple Wiener–Itô integrals with respect to the coordinates of a vector valued random spectral measure. The formulation and proof of this result is similar to that of the corresponding result in the scalar valued case. In this result we define the kernel functions of the Wiener–Itô integrals appearing in the sum expressing the product of two Wiener–Itô integrals with the help of some diagrams. Hence this result got the name diagram formula. I wrote down the formulation of the diagram formula in the case of vector valued random spectral measures in detail. On the other hand, I gave only a sketch of its proof, because it is actually an adaptation of the original proof with a rather unpleasant notation. I concentrated on the points which explain why the diagram formula has such a form as we claim. Besides, I tried to explain those steps of the proof where we have to apply some new ideas. I hope that the interested reader can reconstruct the proof on the basis of these explanations by looking at the original proof.

Section 6 also contains a corollary of the diagram formula, where I formulate this result in a special case. I formulated this corollary, because in this work we need only this corollary of the diagram formula.

2 Spectral representation of vector valued stationary random fields

Let $X(p) = (X_1(p), \dots, X_d(p))$, $p \in \mathbb{Z}^\nu$, where $\mathbb{Z}^\nu$ denotes the lattice of points with integer coordinates in the ν-dimensional Euclidean space $\mathbb{R}^\nu$, be a d-dimensional real valued Gaussian stationary random field with expected value $EX(p) = 0$, $p \in \mathbb{Z}^\nu$. Let us first characterize the covariance matrices $R(p) = (r_{j,j'}(p))$, $1 \le j, j' \le d$, $p \in \mathbb{Z}^\nu$, of this d-dimensional stationary random field, where $r_{j,j'}(p) = EX_j(0)X_{j'}(p) = EX_j(m)X_{j'}(p+m)$, $1 \le j, j' \le d$, $p, m \in \mathbb{Z}^\nu$.

In the case $d = 1$ we can characterize the function $R(p) = EX(0)X(p)$, (in this case $j = j' = 1$, so we can omit these indices) as the Fourier transform of an even, finite (and positive) measure G on the torus $[-\pi, \pi)^\nu$, called the spectral

measure. We are looking for the vector valued version of this result. Before discussing this problem I recall the definition of the torus $[-\pi,\pi)^\nu$.

The points of the torus $[-\pi,\pi)^\nu$ are those points $x=(x_1,\dots,x_\nu)\in\mathbb{R}^\nu$ for which $-\pi\le x_j\le\pi$ for all $1\le j\le\nu$. But if a coordinate of x in this set equals π, then we consider this point the same if we replace this coordinate by $-\pi$. In such a way we can identify all points of this set by a point of the set $[-\pi,\pi)^\nu\subset\mathbb{R}^\nu$. We define the topology on the torus on $[-\pi,\pi)^\nu$ as the topology induced by the metric $\rho(x,y)=\sum_{j=1}^{\nu}(|x_j-y_j| \bmod 2\pi)$ if $x=(x_1,\dots,x_\nu)\in[-\pi,\pi)^\nu$ and $y=(y_1,\dots,y_\nu)\in[-\pi,\pi)^\nu$. These properties of the torus $[-\pi,\pi)^\nu$ must be taken into account when we speak of the set $-A=\{-x\colon\ x\in A\}$ for a set $A\subset[-\pi,\pi)^\nu$ or of a continuous function on the torus $[-\pi,\pi)^\nu$.

Later we shall speak also about the torus $[-A,A)^\nu$ for arbitrary $A>0$. This is defined in the same way, only the number π is replaced by A in the definition.

It is natural to expect that there is a natural definition of even positive semidefinite matrix valued measures also in the d-dimensional case, $d\ge2$, and this takes the role of the spectral measure in the vector valued case. To define this notion first I prove a lemma. Before formulating it I recall the definition of a complex measure with finite total variation, since this notion appears in the formulation of the lemma. We say that a complex measure on a measurable space has finite total variation if both its real and imaginary part can be represented as the difference of two finite measures. I also recall Bochner's theorem, more precisely that version of this result that we shall apply in the proof.

Bochner's theorem. *Let $f(p)$, $p\in\mathbb{Z}^\nu$, be a positive definite function on $\mathbb{Z}^\nu$, i.e. such a function for which the inequality $\sum_{j=1}^{N}\sum_{j'=1}^{N}z_j\bar z_{j'}f(p_j-p_{j'})\ge0$ holds for any set of points $p_j\in\mathbb{Z}^\nu$, and complex numbers z_j, $1\le j\le N$, with some number $N\ge1$. Then there exists a unique finite measure G on the torus $[-\pi,\pi)^\nu$ such that*

$$f(p)=\int_{[-\pi,\pi)^\nu}e^{i(p,x)}G(\,dx)\quad\text{for all } p\in\mathbb{Z}^\nu.$$

If the function f is real valued, then the measure G is even, i.e. $G(-A)=G(A)$ for all $A\subset[-\pi,\pi)^\nu$.

Next I formulate the following lemma.

Lemma 2.1. *Let $X(p)=(X_1(p),\dots,X_d(p))$, $p\in\mathbb{Z}^\nu$, be a d-dimensional stationary Gaussian random field with expectation zero. Then for all pairs $1\le j,j'\le d$ the correlation function $r_{j,j'}(p)=EX_j(0)X_{j'}(p)$, $p\in\mathbb{Z}^\nu$, can be written in the form*

$$r_{j,j'}(p)=EX_j(0)X_{j'}(p)=EX_j(m)X_{j'}(m+p)=\int_{[-\pi,\pi)^\nu}e^{i(p,x)}G_{j,j'}(\,dx)\tag{2.1}$$

with a complex measure $G_{j,j'}$ on the torus $[-\pi,\pi)^\nu$ with finite total variation. The function $r_{j,j'}(p)$, $p\in\mathbb{Z}^\nu$, uniquely determines this complex measure $G_{j,j'}$ with finite total variation. It is even, i.e. $G_{j,j'}(-A)=\overline{G_{j,j'}(A)}$ for all measurable sets $A\subset[-\pi,\pi)^\nu$. The relation $G_{j',j}(A)=\overline{G_{j,j'}(A)}$ also holds for all $1\le j,j'\le d$ and $A\subset[-\pi,\pi)^\nu$.

Remark. Let us remark that given a d-dimensional stationary random field with expectation zero, there exist also such d-dimensional stationary random fields with expectation zero which are Gaussian, and have the same correlation function. As a consequence, in Lemma 2.1 we could drop the condition that the stationary random field we are considering is Gaussian. The same can be told about the other results of Section 2. I imposed this condition, because later, as we work with random spectral measures and random integrals with respect to them the Gaussian property of the underlying random field is important.

Proof of Lemma 2.1. By Bochner's theorem we may write

$$r_{j,j}(p)=\int_{[-\pi,\pi)^\nu} e^{i(p,x)}G_{j,j}(\,dx),\quad p\in\mathbb{Z}^\nu,$$

for all $1\le j\le d$ with some finite measure $G_{j,j}$ on $[-\pi,\pi)^\nu$. We find a good representation for $r_{j,j'}(n)$ if $j\ne j'$ with the help of following argument.

The function

$$\begin{aligned} q_{j,j'}(p) &= E[X_j(0)+iX_{j'}(0)][X_j(p)-iX_{j'}(p)]\\ &= E[X_j(0)+iX_{j'}(0)]\overline{[X_j(p)+iX_{j'}(p)]}, \end{aligned}$$

$p\in\mathbb{Z}^\nu$, is positive definite, hence it can be written in the form

$$E[X_j(0)+iX_{j'}(0)][X_j(p)-iX_{j'}(p)]=\int_{[-\pi,\pi)^\nu} e^{i(p,x)}H_{j,j'}(\,dx)$$

with some finite measure $H_{j,j'}$ on $[-\pi,\pi)^\nu$. Similarly,

$$E[X_j(0)+X_{j'}(0)][X_j(p)+X_{j'}(p)]=\int_{-[\pi,\pi)^\nu} e^{i(p,x)}K_{j,j'}(\,dx)$$

with some finite measure $K_{j,j'}$ on $[-\pi,\pi)^\nu$. Hence

$$\begin{aligned} EX_j(0)X_{j'}(p) &= \frac{i}{2}E[X_j(0)+iX_{j'}(0)][X_j(p)-iX_{j'}(p)]\\ &\quad+\frac{1}{2}E[X_j(0)+X_{j'}(0)][X_j(p)+X_{j'}(p)]\\ &\quad-\frac{(1+i)}{2}[EX_j(0)X_j(p)+EX_{j'}(0)X_{j'}(p)]\\ &= \int_{[-\pi,\pi)^\nu} e^{i(p,x)}G_{j,j'}(\,dx) \end{aligned}$$

with $G_{j,j'}(\,dx)=\frac{1}{2}[iH_{j,j'}(\,dx)+K_{j,j'}(\,dx)]-\frac{(1+i)}{2}[G_{j,j}(\,dx)+G_{j',j'}(\,dx)]$.

In such a way we have found complex measures $G_{j,j'}$ with finite total variation which satisfy relation (2.1). Since this relation holds for all $p \in \mathbb{Z}^\nu$, the function $r_{j,j'}(p)$, $p \in \mathbb{Z}^\nu$, determines the measure $G_{j,j'}$ uniquely.

Since $r_{j,j'}(p)$ is real valued, i.e. $r_{j,j'}(p) = \overline{r_{j,j'}(p)}$, it can be written both in the form

$$r_{j,j'}(p) = \int_{[-\pi,\pi)^\nu} e^{i(p,x)} G_{j,j'}(\,dx)$$

and

$$r_{j,j'}(p) = \int_{[-\pi,\pi)^\nu} e^{-i(p,x)} \overline{G_{j,j'}(\,dx)} = \int_{[-\pi,\pi)^\nu} e^{i(p,x)} \overline{G_{j,j'}(-\,dx)}.$$

Comparing these relations we get that $G_{j,j'}(A) = \overline{G_{j,j'}(-A)}$ for all measurable sets $A \subset [-\pi,\pi)^\nu$. Similarly, the relation $r_{j',j}(p) = r_{j,j'}(-p)$ implies that $G_{j',j}(A) = G_{j,j'}(-A) = \overline{G_{j,j'}(A)}$ for all measurable sets $A \subset [-\pi,\pi)^\nu$. Lemma 2.1 is proved.

Since all complex measures $G_{j,j'}$, $1 \le j, j' \le d$, have finite total variation by Lemma 2.1, there is a finite measure μ on the torus $[-\pi,\pi)^\nu$ such that all these complex measures $G_{j,j'}$ are absolutely continuous with respect to μ, and the absolute value of the Radon–Nikodym derivatives $g_{j,j'}(x) = \frac{dG_{j,j'}}{d\mu}(x)$ is integrable with respect to μ. The properties of the measures $G_{j,j'}$ proved in Lemma 2.1 imply that the $d \times d$ matrix $(g_{j,j'}(x))$, $1 \le j, j' \le d$, is Hermitian for almost all $x \in [-\pi,\pi)^\nu$ with respect to the measure μ. We shall call the matrix valued measure $(G_{j,j'}(A))$, $A \subset [-\pi,\pi)^\nu$, positive semidefinite if the matrix $(g_{j,j'}(x))$, $1 \le j, j' \le d$, is positive semidefinite for almost all $x \in [-\pi,\pi)^\nu$ with respect to μ. More precisely, we introduce the following definition.

Definition of positive semidefinite matrix valued, even measures on the torus. *Let us have some complex measures $G_{j,j'}$, $1 \le j, j' \le d$, with finite total variation on the σ-algebra of the Borel measurable sets of the torus $[-\pi,\pi)^\nu$. Let us consider the matrix valued measure $(G_{j,j'})$, $1 \le j, j' \le d$. We call this matrix valued measure positive semidefinite if there exists a (finite) positive measure μ on $[-\pi,\pi)^\nu$ such that all complex measures $G_{j,j'}$, $1 \le j, j' \le d$, are absolutely continuous with respect to it, and their Radon–Nikodym derivatives $g_{j,j'}(x) = \frac{dG_{j,j'}}{d\mu}(x)$, $1 \le j, j' \le d$, constitute a positive semidefinite matrix $(g_{j,j'}(x))$, $1 \le j, j' \le d$ for almost all $x \in \mathbb{Z}^\nu$ with respect to the measure μ. We call this positive semidefinite matrix valued measure $(G_{j,j'})$, $1 \le j, j' \le d$, on the torus even if $G_{j,j'}(-A) = \overline{G_{j,j'}(A)}$ for all measurable sets $A \subset [-\pi,\pi)^\nu$ and $1 \le j, j' \le d$.*

Later we shall speak also of positive semidefinite matrix valued, even measures on a torus $[-A, A)^\nu$ for arbitrary $A > 0$ which is defined in the same way, only the complex measures $G_{j,j'}$ and the dominating measure μ are defined on $[-A, A)^\nu$.

Remark. Here I am speaking about measures with finite total variation, although such (complex) measures are called generally bounded measures in the literature.

Actually, we know by Stone's theorem that any bounded signed measure can be represented as the difference of two bounded measures (with disjoint support). Nevertheless, I shall remain at this name, because actually we prove directly the finite total variation of the measures we shall work with in this paper. Besides, (in Section 4) I shall define complex measures on $\mathbb{R}^\nu$ with locally finite total variation, and I prefer such a name which refers to the similarity of these objects. (The complex measures with locally finite total variation are not measures in the original meaning of this word, only their restrictions to compact sets are complex measures.)

The next theorem about the characterization of the correlation function of a d-dimensional stationary Gaussian random field with zero expectation states that the correlation functions $r_{j,j'}(p)$, $1 \le j, j' \le d$, $p \in \mathbb{Z}^\nu$, can be given in the form (2.1) with the help of a positive semidefinite matrix valued, even measure $(G_{j,j'})$, $1 \le j, j' \le d$, on the torus $[-\pi, \pi)^\nu$. Moreover, it will be shown that we have somewhat more freedom when we choose a dominating measure μ in the definition of positive semidefinite matrix valued measures on the torus. If the coordinates of a matrix valued measure $(G_{j,j'})$, $1 \le j, k \le d$, are complex measures with finite total variation, and this matrix valued measure satisfies the definition of the positive semidefinite property with some measure μ, then this measure μ can be replaced in the definition by any such finite measure on the torus with respect to which the complex measures $G_{j,j'}$ are absolutely continuous. More explicitly, the following result holds.

Theorem 2.2. *The covariance matrices of a d-dimensional stationary random field $X(p) = (X_1(p), \dots, X_d(p))$, $p \in \mathbb{Z}^\nu$, with expectation zero can be given in the following form. For all $1 \le j, j' \le d$ there exists a complex measure $G_{j,j'}$ with finite total variation on the ν-dimensional torus $[-\pi, \pi)^\nu$ in such a way that for all $1 \le j, j' \le d$ the correlation function $r_{j,j'}(p) = EX_j(0)X_{j'}(p)$, $p \in \mathbb{Z}^\nu$, is given by formula (2.1) with this complex measure $G_{j,j'}$. The $d \times d$ matrix $G = (G_{j,j'})$, $1 \le j, j' \le d$, whose coordinates are the complex measures $G_{j,j'}$ has the following properties. This matrix is Hermitian, i.e. the measures $G_{j,j'}$ satisfy the relation $G_{j',j}(A) = \overline{G_{j,j'}(A)}$ for all pairs of indices $1 \le j, j' \le d$ and measurable sets $A \subset [-\pi, \pi)^\nu$, and the measures $G_{j,j'}$ are even, i.e. $G_{j,j'}(-A) = \overline{G_{j,j'}(A)}$ for all $1 \le j, j' \le d$ and $A \subset [-\pi, \pi)^\nu$. For all pairs (j, j'), $1 \le j, j' \le d$, the function $r_{j,j'}(p)$, $p \in \mathbb{Z}^\nu$, defined by formula (2.1) uniquely determines the complex measure $G_{j,j'}$ with finite total variation. Besides, $G_{j,j'}$ has the following property.*

Let us take a finite measure μ on the torus $[-\pi, \pi)^\nu$ such that all complex measures $G_{j,j'}$ are absolutely continuous with respect to it, (because of the finite total variation of the complex measures $G_{j,j'}$ there exist such measures), and put $g_{j,j'}(x) = g_{j,j',\mu}(x) = \frac{dG_{j,j'}}{d\mu}(x)$. Then the matrix $(g_{j,j'}(x))$, $1 \le j, j' \le d$, is positive semidefinite for almost all $x \in [-\pi, \pi)^\nu$ with respect to the measure μ.

Conversely, if a class of complex measures $G_{j,j'}$ on $[-\pi, \pi)^\nu$, $1 \le j, j' \le d$, have finite total variation, and $(G_{j,j'})$, $1 \le j, j' \le d$, is a positive semidefinite matrix valued, even measure on the torus, then there exists a d-dimensional

stationary Gaussian random field $X(p) = (X_1(p), \dots, X_d(p))$, $p \in \mathbb{Z}^\nu$, *with expectation* $EX_j(p) = 0$ *and covariance* $EX_j(p)X_{j'}(q) = r_{j,j'}(p-q)$, *where the function* $r_{j,j'}(p)$ *is defined in (2.1) with the complex measure* $G_{j,j'}$ *for all parameters* $1 \le j, j' \le d$ *and* $p, q \in \mathbb{Z}^\nu$.

Remark. We shall call the positive semidefinite matrix valued, even measure $(G_{j,j'})$, $1 \le j, j' \le d$, on the torus $[-\pi, \pi)^\nu$ with coordinates $G_{j,j'}$ satisfying relation (2.1) the matrix valued spectral measure of the correlation function $r_{j,j'}(p)$, $1 \le j, j' \le d$, $p \in \mathbb{Z}^\nu$. In general, we shall call an arbitrary positive semidefinite matrix valued, even measure on the torus $[-\pi, \pi)^\nu$ a matrix valued spectral measure on the torus $[-\pi, \pi)^\nu$. (More generally, later we shall call for any $A > 0$ a positive semidefinite matrix valued, even measure on the torus $[-A, A)^\nu$ a matrix valued spectral measure on this torus.) We have the right for such a terminology, since by Theorem 2.2 for an arbitrary positive semidefinite matrix valued, even measure on the torus $[-\pi, \pi)^\nu$ there exists a vector valued stationary Gaussian random field on $\mathbb{Z}^\nu$ such that this positive semidefinite matrix valued, even measure is the spectral measure of its correlation function.

Proof of Theorem 2.2. The statements formulated in the first paragraph of Theorem 2.2 follow from Lemma 2.1. Next we prove that the matrix $(g_{j,j'}(x))$, $1 \le j, j' \le d$, whose elements are defined as the Radon–Nikodym derivatives of the complex measures $G_{j,j'}$ with respect to a measure μ satisfying the conditions of Theorem 2.2 is positive semidefinite for μ almost all x.

We prove this by first showing with the help of Weierstrass' second approximation theorem that

$$\int_{[-\pi,\pi)^\nu} v(x)g(x)v^*(x)\mu(\,dx) \ge 0 \tag{2.2}$$

for any continuous d-dimensional vector valued function $v(x) = (v_1(x), \dots, v_d(x))$ on the ν-dimensional torus $[-\pi, \pi)^\nu$, where $g(x)$ denotes the $d \times d$ matrix $(g_{j,j'}(x))$, $1 \le j, j' \le d$, and $v^*(x)$ is the conjugate of the vector $v(x)$.

To prove (2.2) let us first observe that by Weierstrass' second approximation theorem for all $\varepsilon > 0$ there exists a number $N = N(\varepsilon)$ and d trigonometrical polynomials of order N

$$v_{N,j}(x) = \sum_{\substack{s=(s_1,\dots,s_\nu) \\ -N \le s_k < N,\ 1 \le k \le \nu}} a_{j,s_1,\dots,s_\nu} e^{i(s,x)}, \quad 1 \le j \le d, \quad x \in [-\pi, \pi)^\nu$$

for which

$$\sup_{x \in [-\pi,\pi)^\nu} |v_{N,j}(x) - v_j(x)| \le \varepsilon \quad \text{for all } 1 \le j \le d.$$

Let us also define the random vector $Y_N = (Y_{N,1}, \dots, Y_{N,d})$ with coordinates

$$Y_{N,j} = \sum_{\substack{s=(s_1,\dots,s_\nu) \\ -N \le s_k < N,\ 1 \le k \le \nu}} a_{j,s_1,\dots,s_\nu} X_j(s), \quad 1 \le j \le d,$$

Then we have because of the relation $EX_j(s)X_{j'}(s') = \int e^{i(s-s',x)} g_{j,j'}(x)\mu(\,dx)$

$$0 \le E\left(\sum_{j=1}^{d} Y_{N,j}\right)\overline{\left(\sum_{j=1}^{d} Y_{N,j}\right)} = \sum_{j=1}^{d}\sum_{j'=1}^{d}\int_{[-\pi,\pi)^\nu} g_{j,j'}(x)v_{N,j}(x)\overline{v_{N,j'}(x)}\mu(\,dx).$$

Hence

$$\int_{[-\pi,\pi)^\nu} v_N(x)g(x)v_N^*(x)\mu(\,dx) \ge 0,$$

and we get relation (2.2) from it with the help of the limiting procedure $N \to \infty$.

Let us choose a vector $a = (a_1, \dots, a_d) \in \mathbb{R}^d$ and a non-negative continuous function $u(x)$ on the torus $[-\pi,\pi)^\nu$. Let us apply formula (2.2) with the choice of the function $v(x) = (a_1\sqrt{u(x)}, \dots, a_d\sqrt{u(x)})$. With this choice formula (2.2) yields that

$$0 \le \int_{[-\pi,\pi)^\nu} v(x)g(x)v^*(x)\mu(\,dx) = \int_{[-\pi,\pi)} u(x)h_a(x)\mu(\,dx)$$

with the function $h_a(x) = ag(x)a^*$. Since this inequality holds for all non-negative continuous functions this implies that $h_a(x) \ge 0$ for almost all x with respect to the measure μ. Moreover, since $h_a(x) = ag(x)a^*$ is a continuous function of the parameter a for a fixed number $x \in [-\pi,\pi)^\nu$ this also implies that $g(x)$ is a positive semidefinite matrix for almost all x with respect to the measure μ. We have proved that the covariance matrix of a vector valued stationary field has the properties stated in Theorem 2.2.

Next I show that if we have a class of complex measures $G_{j,j'}$ with finite total variation such that $(G_{j,j'})$ is a positive semidefinite matrix valued even measure on the torus, and the functions $r_{j,j'}(p)$, $p \in \mathbb{Z}^\nu$, are defined by formula (2.1) with these complex measures $G_{j,j'}$, then there exists a vector valued stationary Gaussian field $X(p) = (X_1(p), \dots, X_d(p))$ with expectation zero and covariance function $EX_j(0)X_{j'}(p) = r_{j,j'}(p)$.

First I show that for all $N \ge 1$ there is a set of Gaussian random vectors $X(p) = (X_1(p), \dots, X_d(p))$, with parameters $p = (p_1, \dots, p_\nu)$, $-N \le p_j \le N$ for all $j = 1, \dots, d$, such that $EX_j(p)X_{j'}(q) = r_{j,j'}(p-q))$ for all $1 \le j, j' \le d$ and $p = (p_1, \dots, p_\nu)$, $q = (q_1, \dots, q_\nu)$ with $-N \le p_s, q_s \le N$, $1 \le s \le \nu$.

Let us observe that the covariances $r_{j,j'}(p)$ defined by (2.1) are real-valued, since $G_{j,j'}(A) = \overline{G_{j,j'}(-A)}$. To show that there exists a set of Gaussian random vectors with the desired covariance we have to check that the covariance matrix determined by the coordinates of these random vectors is positive semidefinite. This means that for all sets of complex numbers

$$\mathcal{A}_N = \{a_{j,p} = a_{j,p_1,\dots,p_\nu} \colon\ 1 \le j \le d,\ -N \le p_s \le N,\ \text{for all } 1 \le s \le \nu\}$$

$$I(\mathcal{A}_N) = \sum_{j=1}^{d}\sum_{j'=1}^{d} \sum_{\substack{p=(p_1,\dots,p_\nu)\\ -N\le p_s\le N,\ 1\le s\le\nu}} \sum_{\substack{q=(q_1,\dots,q_\nu)\\ -N\le q_s\le N,\ 1\le s\le\nu}} a_{j,p}\overline{a_{j',q}} r_{j,j'}(p-q) \ge 0.$$

This inequality holds, since

$$I(\mathcal{A}_N) = \int \sum_{j=1}^{d}\sum_{j'=1}^{d} \left(\sum_{\substack{p=(p_1,\dots,p_\nu)\\ -N\le p_s\le N,\ 1\le s\le\nu}} a_{j,p}e^{i(p,x)} \right) g_{j,j'}(x) \Bigg/ \left(\sum_{\substack{p=(p_1,\dots,p_\nu)\\ -N\le p_s\le N,\ 1\le s\le\nu}} a_{j',p}e^{i(p,x)} \right) \mu(\,dx)$$

$$= \int \left(\sum_{j=1}^{d}\sum_{j'=1}^{d} b_j(x) g_{j,j'}(x) \overline{b_{j'}(x)} \right) \mu(\,dx) \ge 0,$$

where $b_j(x) = \sum_{\substack{p=(p_1,\dots,p_\nu)\\ -N\le p_s\le N,\ 1\le s\le\nu}} a_{j,p}e^{i(p,x)}$. This expression is really non-negative, since the matrix $g_{j,j'}(x)$ is positive semidefinite for μ-almost all x, and this implies that the integrand at the right-hand side of this expression is non-negative for μ-almost all x.

Since the distribution of the above sets of Gaussian random vectors are consistent for different parameters N it follows from Kolmogorov's existence theorem for random processes with consistent finite distributions that there exists a Gaussian random field $X(p)$, $p \in \mathbb{Z}^\nu$, with $EZ_p = 0$, $EX_j(p)X_{j'}(q) = r_{j,j'}(p-q)$, where $r_{j,j'}(p)$ is defined by formula (2.1) with our matrix valued spectral measure $G = (G_{j,j'})$, $1 \le j, j' \le d$. In such a way we constructed a stationary Gaussian random field with the desired properties. Theorem 2.2 is proved.

In the next lemma I give a different characterization of positive semidefinite matrix valued, even measures on the torus $[-\pi,\pi)^\nu$.

Lemma 2.3. *Let us have a class of complex measures $G_{j,j'}$, $1 \le j, j' \le d$, with finite total variation on the torus $[-\pi,\pi)^\nu$. Let us define with their help the following σ-additive matrix valued function on the measurable subsets of the torus $[-\pi,\pi)^\nu$. Define for all measurable sets $A \subset [-\pi,\pi)^\nu$ the $d \times d$ matrix $G(A) = (G_{j,j'}(A))$, $1 \le j, j' \le d$. This matrix valued function is a positive semidefinite matrix valued, even measure on the torus $[-\pi,\pi)^\nu$ if and only if the matrix $(G_{j,j'}(A))$, $1 \le j, j' \le d$, is positive semidefinite, and $G_{j,j'}(-A) = \overline{G_{j,j'}(A)}$ for all measurable sets $A \subset [-\pi,\pi)^\nu$ and $1 \le j, j' \le d$.*

Proof of Lemma 2.3. It is clear that if $(G_{j,j'})$ is a positive semidefinite matrix valued, even measure, then the matrix $(G_{j,j'}(A))$ with

$$G_{j,j'}(A) = \int_A g_{j,j'}(x)\mu(\,dx), \quad 1 \le j, j' \le d,$$

is a positive semidefinite matrix, and $G_{j,j'}(-A) = \overline{G_{j,j'}(A)}$ for all measurable sets $A \subset [-\pi,\pi)^\nu$ and $1 \le j, j' \le d$.

On the other hand, it is not difficult to see that if the above properties hold, then $\sum_{j=1}^{d}\sum_{j'=1}^{d}\int v_j(x)\overline{v_{j'}(x)}G_{j,j'}(\,dx) \ge 0$ for all vectors $v(x) = (v_1(x), \dots, v_d(x))$, where $v_j(\cdot)$, $1 \le j \le d$, is a continuous function on the torus $[-\pi, \pi)^\nu$. If μ is a finite measure on $[-\pi, \pi)^\nu$ such that all complex measures $G_{j,j'}$, $1 \le j, j' \le d$, are absolutely continuous with respect to it with Radon–Nikodym derivative $g_{j,j'}(x)$, and we denote the matrix $(g_{j,j'}(x))$, $1 \le j, j' \le d$, by $g(x)$, then the above inequality can be rewritten in the form $\int v(x)g(x)v^*(x)\mu(\,dx) \ge 0$. In the proof of Theorem 2.2 we have seen that this implies that $g(x)$ is a positive semidefinite matrix for μ almost all $x \in [-\pi, \pi)^\nu$. Lemma 2.3 is proved.

Let me also remark that the proof of Lemma 2.3 also implies that if the definition of positive semidefinite matrix valued, even measures holds with some finite measure μ on the torus with the property that each complex measure $G_{j,j'}$, $1 \le j, j' \le d$, is absolutely continuous with respect to it, then the conditions of this definition also hold with any measure μ on the torus with the same properties.

Given a positive semidefinite matrix valued even measure $G = (G_{j,j'})$, $1 \le j, j' \le d$, on the torus $[-\pi, \pi)^\nu$, there is a natural candidate for the choice of the measure μ on the torus $[-\pi, \pi)^\nu$ with respect to which all measures $G_{j,j'}$, $1 \le j, j' \le d$, are absolute continuous. We shall prove an estimate in formula (3.2) which implies that the measure $\mu = \sum_{j=1}^{d} G_{j,j}$, i.e. the trace of the matrix valued measure G has this property. Later this measure will be our choice for the measure μ.

Let me remark that the proof of Lemma 2.3 yields another characterization of positive semidefinite matrix valued measures on the torus. I present it, although I shall not use it later.

A matrix valued measure $G = (G_{j,j'})$, $1 \le j, j' \le d$, on the torus such that $G_{j,j'}(A) = \overline{G_{j',j}(A)}$ for all $1 \le j, j' \le d$ and measurable sets $A \subset [-\pi, \pi)^\nu$ is positive semidefinite if and only if

$$\sum_{j=1}^{d}\sum_{j'=1}^{d}\int_{[-\pi,\pi)^\nu} u_j(x)\overline{u_{j'}(x)}G_{j,j'}(\,dx) \ge 0$$

for all vectors $u(x) = (u_1(x), \dots, u_d(x))$ whose coordinates are continuous functions on the torus $[-\pi, \pi)^\nu$.

3 Random spectral measures in the multi-dimensional case

If $X(p) = (X_1(p), \dots, X_d(p))$, $p \in \mathbb{Z}^\nu$, is a d-dimensional stationary Gaussian random field with expectation zero, then its distribution is determined by its correlation functions $r_{j,j'}(p) = EX_j(0)X_{j'}(p)$, $1 \le j, j' \le d$, $p \in \mathbb{Z}^\nu$. In Theorem 2.2 we described this correlation function as the Fourier transform of a matrix valued spectral measure $G = (G_{j,j'})$, $1 \le j, j' \le d$. In the case of scalar

valued stationary random fields this result has a continuation. A so-called random spectral measure Z_G can be constructed, and the elements of the stationary random field can be represented as an appropriate random integral with respect to it. This result can be interpreted so that the elements of a scalar valued stationary random field can be represented as the Fourier transforms of a random spectral measure. We want to find the multi-dimensional version of this result.

The results about scalar valued stationary random fields also help in the study of vector valued stationary random fields. Indeed, since the j-th coordinates $X_j(p)$, of the random vectors $X(p)$, $p \in \mathbb{Z}^\nu$, define a scalar valued stationary random field we can apply for them the results known in the scalar valued case. This enables us to construct such a random spectral measure $Z_{G.j}$ for all $1 \le j \le d$ for which the identity $X_j(p) = \int_{[-\pi,\pi)^\nu} e^{i(p,x)} Z_{G,j}(\,dx)$ holds for all $p \in \mathbb{Z}^\nu$. The distribution of the random spectral measure $Z_{G,j}$ depends on the coordinate $G_{j,j}$ of the matrix valued spectral measure G, which is the spectral measure of the stationary random field $X_j(p)$, $p \in \mathbb{Z}^\nu$. For a fixed number $1 \le j \le d$ the properties of the random spectral measure $Z_{G,j}$ and the definition of the random integral with respect to it is worked out in the literature. I shall refer to my Lecture Note [9], where I described this theory.

Nevertheless, the results obtained in such a way are not sufficient for us. They describe the distribution of the random spectral measure $Z_{G,j}$ for each $1 \le j \le d$, but we need some additional results about their joint distribution. To get them I recall the results in [9] which led to the construction of the random spectral measures $Z_{G,j}$, and then I extend them in order to get the results we need to describe their joint distribution.

I explain how we define simultaneously all random spectral measures $Z_{G,j}$, $1 \le j \le d$, by recalling the method of [9] with some necessary modifications in the notation to adapt this method to our case.

We construct the random spectral measure $Z_{G,j}$ for all $1 \le j \le d$ in the following way. First we introduce two Hilbert spaces $\mathcal{K}^c_{1,j}$ and $\mathcal{H}^c_{1,j}$, and define an appropriate norm-preserving invertible linear transformation T_j from $\mathcal{K}^c_{1,j}$ to $\mathcal{H}^c_{1,j}$. (Here, and in the subsequent discussion I apply the superscript ^{c} in the notation to emphasize that we are working in a complex, and not in a real Hilbert space.) The Hilbert space $\mathcal{K}^c_{1,j}$ consists of those complex valued functions $u(x)$ on the torus $[-\pi,\pi)^\nu$ for which $\int_{[-\pi,\pi)^\nu} |u(x)|^2 G_{j,j}(\,dx) < \infty$, and the norm is defined in this space by the formula $\|u\|^2_{0,j} = \int_{[-\pi,\pi)^\nu} |u(x)|^2 G_{j,j}(\,dx)$. The Hilbert space $\mathcal{H}^c_{1,j}$ is defined as the closure of the linear space consisting of the linear combinations $\sum c_{p_s} X_j(p_s)$ with some (complex valued) coefficients c_{p_s} and parameters $p_s \in \mathbb{Z}^\nu$ in the Hilbert space $\mathcal{H}^c$. The Hilbert space $\mathcal{H}^c$ consists of the complex valued random variables with finite second moment, measurable with respect to the σ-algebra generated by the random variables $X_j(p)$, $1 \le j \le d$, $p \in \mathbb{Z}^\nu$, and the norm $\|\cdot\|_{1,j}$ in it is determined by the scalar product defined by the formula $\langle \xi, \eta \rangle = E\xi\bar{\eta}$, $\xi, \eta \in \mathcal{H}^c$. First we define the transformation T_j only for finite trigonometrical sums in $\mathcal{K}^c_{1,j}$. We define it by the formula $T_j(\sum c_{p_s} e^{i(p_s,x)}) = \sum c_{p_s} X_j(p_s)$. We showed in [9] that we have defined in such a way a norm-preserving linear transformation from an

everywhere dense subspace of $\mathcal{K}_{1,j}^c$ to an everywhere dense subspace of $\mathcal{H}_{1,j}^c$. This can be extended to a norm-preserving invertible linear transformation T_j from $\mathcal{K}_{1,j}^c$ to $\mathcal{H}_{1,j}^c$ in a unique way. We define the random spectral measure $Z_{G,j}(A)$ for a measurable set $A \subset [-\pi,\pi)^\nu$ by the formula $Z_{G,j}(A) = T_j(\mathbb{I}_A(\cdot))$, where $\mathbb{I}_A(\cdot)$ denotes the indicator function of the set A.

It follows from the results of [9] that for any $1 \le j \le d$ the measure $G_{j,j}$ determines the distribution of the random spectral measure $Z_{G,j}$, (i.e. the joint distribution of the random variables $Z_{G,j}(A_1), \dots Z_{G,j}(A_N)$ for all $N \ge 1$ and measurable sets $A_k \subset [-\pi,\pi)^\nu$, $1 \le k \le N$). Next we shall study the joint distribution of the random fields $Z_{G,j}$ for all $1 \le j \le d$, i.e. the joint distribution of the random variables $Z_{G,j}(A_1), \dots Z_{G,j}(A_N)$ for all $N \ge 1$, measurable sets $A_k \subset [-\pi,\pi)^\nu$, $1 \le k \le N$ and $1 \le j \le d$. In particular, we shall show that the joint distribution of the random fields $Z_{G,j}$, $1 \le j \le d$, are determined by the matrix valued spectral measure $G = (G_{j,j'})$, $1 \le j, j' \le d$. The joint distribution of these random fields are determined by the matrix valued measure G, and not only by their diagonal elements $G_{j,j}$, $1 \le j \le d$.

To investigate the joint behaviour of the random spectral measures $Z_{G,j}$, $1 \le j \le d$, first we define two Hilbert spaces $\mathcal{K}_1^c$ and $\mathcal{H}_1^c$ together with a norm-preserving and invertible transformation between them. The elements of the Hilbert space $\mathcal{K}_1^c$ are the vectors $u = (u_1(x), \dots, u_d(x))$ with $u_j(x) \in \mathcal{K}_{1,j}^c$, $1 \le j \le d$. To define the (semi)-norm in $\mathcal{K}_1^c$ we introduce a positive semidefinite bilinear form $\langle \cdot, \cdot \rangle_0$ on it. To make some subsequent discussions simpler I make the following convention in the rest of the paper. Given a positive semidefinite matrix valued measure $(G_{j,j'})$, $1 \le j, j' \le d$, on the torus $[-\pi,\pi)^\nu$, I fix a finite and even measure μ on $[-\pi,\pi)^\nu$ such that all complex measures $G_{j,j'}$ are absolutely continuous with respect to it, and I denote by $g_{j,j'}(x)$ their Radon–Nikodym derivative with respect to μ. With the help of this notation we define $\langle \cdot, \cdot \rangle_0$ in the following way. If $u(x) = (u_1(x), \dots, u_d(x)) \in \mathcal{K}_1^c$ and $v(x) = (v_1(x), \dots, v_d(x)) \in \mathcal{K}_1^c$, then

$$\begin{aligned} \langle u(x), v(x) \rangle_0 &= \sum_{j=1}^d \sum_{j'=1}^d \int u_j(x) \overline{v_{j'}(x)} G_{j,j'}(\,dx) \qquad (3.1) \\ &= \sum_{j=1}^d \sum_{j'=1}^d \int g_{j,j'}(x) u_j(x) \overline{v_{j'}(x)} \mu(\,dx) \\ &= \int_{[-\pi,\pi)^\nu} u(x) g(x) v(x)^* \mu(\,dx) \end{aligned}$$

with the matrix $g(x) = (g_{j,j'}(x))$, $1 \le j, j' \le d$, where $v^*(x)$ denotes the column vector whose elements are the functions $\overline{v_k(x)}$, $1 \le k \le d$.

To show that the integral in the definition of $\langle u(x), v(x) \rangle_0$ is convergent let us observe that

$$|g_{j,j'}(x)|^2 \le g_{j,j}(x) g_{j',j'}(x) \text{ for almost all } x \text{ with respect to the measure } \mu \qquad (3.2)$$

for all $1 \le j, j' \le d$, because $g(x)$ is a positive semidefinite matrix for almost all x. This fact together with the Schwarz inequality imply that

$$\begin{aligned}
&\left| \int_{[-\pi,\pi)^\nu} u_j(x) g_{j,j'}(x) \overline{v_{j'}(x)} \mu(\,dx) \right| \\
&\quad \le \int_{[-\pi,\pi)^\nu} |u_j(x)| \sqrt{g_{j,j}(x) g_{j',j'}(x)} |v_{j'}(x)| \mu(\,dx) \\
&\quad \le \left(\int_{[-\pi,\pi)^\nu} |u_j(x)|^2 g_{j,j}(x) \mu(\,dx) \right)^{1/2} \left(\int_{[-\pi,\pi)^\nu} |v_{j'}(x)|^2 g_{j',j'}(x) \mu(\,dx) \right)^{1/2} \\
&\quad < \infty
\end{aligned}$$

for all pairs $1 \le j, j' \le d$ and $u_j \in \mathcal{K}^c_{1,j}$ and $v_{j'} \in \mathcal{K}^c_{1,k}$. This implies that the integral in (3.1) is finite. Moreover, the last inequality implies that

$$\begin{aligned}
\langle u(x), u(x) \rangle_0 &\le \left(\sum_{j=1}^d \left(\int_{[-\pi,\pi)^\nu} |u_j(x)|^2 G_{j,j}(\,dx) \right)^{1/2} \right)^2 \\
&\le d \sum_{j=1}^d \int_{[-\pi,\pi)^\nu} |u_j(x)|^2 G_{j,j}(\,dx) = d \sum_{j=1}^d \|u_j\|^2_{0,j} \qquad (3.3)
\end{aligned}$$

for all $u(x) = (u_1(x), \dots, u_d(x)) \in \mathcal{K}^c_1$.

Observe that $\langle u(x), u(x) \rangle_0 \ge 0$, because $g(x)$ is a positive semidefinite matrix, which implies that $u(x) g(x) u^*(x) \ge 0$ for almost all x with respect to the measure μ. In such a way we can define the norm $\| \cdot \|_0$ in $\mathcal{K}^c_1$ by the formula $\|u\|_0 = \langle u(x), u(x) \rangle_0$. We identify two elements u and v in $\mathcal{K}^c_1$ if $\|u - v\|_0 = 0$.

Next we define the Hilbert space $\mathcal{H}^c_1$ with the norm $\| \cdot \|_1$ on it. The elements of $\mathcal{H}^c_1$ are the vectors $\xi = (\xi_1, \dots, \xi_d)$, where $\xi_j \in \mathcal{H}^c_{1,j}$, $1 \le j \le d$, and we define the norm on it by the formula $\|\xi\|^2_1 = E \left| \sum_{j=1}^d \xi_j \right|^2$ if $\xi = (\xi_1, \dots, \xi_d) \in \mathcal{H}^c_1$. It is the norm induced by the scalar product $\langle \xi, \eta \rangle_1 = E \left(\sum_{j=1}^d \xi_j \right) \overline{\left(\sum_{j=1}^d \eta_j \right)}$ for $\xi = (\xi_1, \dots, \xi_d) \in \mathcal{H}^c_1$ and $\eta = (\eta_1, \dots, \eta_d) \in \mathcal{H}^c_1$. We identify two elements $\xi \in \mathcal{H}^c_1$ and $\eta \in \mathcal{H}^c_1$ if $\|\xi - \eta\|_1 = 0$.

Observe that

$$\begin{aligned}
\|\xi\|^2_1 &= E \left(\sum_{j=1}^d \xi_j \right) \overline{\left(\sum_{j'=1}^d \xi_{j'} \right)} \le \sum_{j=1}^d \sum_{j'=1}^d (E|\xi_j|^2)^{1/2} (E|\xi_{j'}|^2)^{1/2} \qquad (3.4) \\
&= \left(\sum_{j=1}^d (E(|\xi_j|^2)^{1/2} \right) \left(\sum_{j'=1}^d (E(|\xi|^2_{j'})^2)^{1/2} \right) \le d \sum_{j=1}^d E|\xi|^2_j = d \sum_{j=1}^k \|\xi_j\|^2_{1,j}
\end{aligned}$$

for a vector $\xi = (\xi_1, \dots, \xi_d) \in \mathcal{H}^c_1$

We define the operator T mapping from $\mathcal{K}^c_1$ to $\mathcal{H}^c_1$ by the formula

$$Tu = T(u_1, \dots, u_d) = (T_1 u_1, \dots, T_d u_d)$$

for $u = (u_1, \dots, u_d)$, $u_j \in \mathcal{K}_{1,j}^c$, with the help of the already defined operators T_j, $1 \le j \le d$. We show that $Tu = T(u_1, \dots, u_d) = (T_1 u_1, \dots, T_d u_d)$ for $u = (u_1, \dots, u_d) \in \mathcal{K}_1^c$ is a norm preserving and invertible transformation from $\mathcal{K}_1^c$ to $\mathcal{H}_1^c$. To prove this let us first observe that because of inequality (3.3) and Weierstrass' second approximation theorem the finite linear combinations

$$\left(\sum_{p \in A_N} c_{1,p} e^{i(p,x)}, \dots, \sum_{p \in A_N} c_{d,p} e^{i(p,x)} \right),$$

where $A_N = \{p = (p_1, \dots, p_\nu) \colon \ -N \le p_s \le N, \text{ for all } 1 \le s \le \nu\}$, constitute an everywhere dense linear subspace in $\mathcal{K}_1^c$, and because of the inequality (3.4) the finite linear combinations

$$\begin{aligned} &\left(\sum_{p \in A_N} c_{1,p} X_1(p), \dots, \sum_{p \in A_N} c_{d,p} X_d(p) \right) \\ &\qquad = T \left(\sum_{p \in A_N} c_{1,p} e^{i(p,x)}, \dots, \sum_{p \in A_N} c_{d,p} e^{i(p,x)} \right) \end{aligned} \tag{3.5}$$

constitute an everywhere dense linear subspace in $\mathcal{H}_1^c$ if $N = 1, 2, \dots$, and the coefficients $c_{j,p}$, $1 \le j \le d$, $p \in A_N$, are arbitrary complex numbers. Hence the following calculation implies that T is a norm preserving and invertible transformation from $\mathcal{K}_1^c$ to $\mathcal{H}_1^c$.

If

$$u(x) = \left(\sum_{p \in A_N} c_{1,p} e^{i(p,x)}, \dots, \sum_{p \in A_N} c_{d,p} e^{i(p,x)} \right)$$

and

$$v(x) = \left(\sum_{p \in A_N} c'_{1,p} e^{i(p,x)}, \dots, \sum_{p \in A_N}^{N} c'_{d,p} e^{i(p,x)} \right),$$

then

$$\begin{aligned} \langle u(x), v(x) \rangle_0 &= \left\langle \left(\sum_{p \in A_N} c_{1,p} e^{i(p,x)}, \dots, \sum_{p \in A_N} c_{d,p} e^{i(p,x)} \right), \right. \\ &\qquad \left. \left(\sum_{p \in A_N} c'_{1,p} e^{-i(p,x)}, \dots, \sum_{p \in A_N} c'_{d,p} e^{-i(p,x)} \right) \right\rangle_0 \\ &= \sum_{j=1}^{d} \sum_{j'=1}^{d} \sum_{s \in A_N} \sum_{t \in A_N} c_{j,s} \bar{c'}_{j',t} \int_{[-\pi,\pi)^\pi} g_{j,j'}(x) e^{i(s-t,x)} \mu(\, dx) \\ &= E \left(\sum_{j=1}^{d} \sum_{s \in A_N} c_{j,s} X_j(s) \right) \overline{\left(\sum_{j'=1}^{d} \sum_{t \in A_N} c'_{j',t} X_{j'}(t) \right)} = \langle Tu(x), Tv(x) \rangle_1. \end{aligned}$$

We shall define the random variables $Z_{G,j}(A)$ for all indices $1 \le j \le d$ and measurable sets $A \subset [-\pi,\pi)^\nu$, by the formula $Z_{G,j}(A) = T_j(\mathbb{I}_A(x))$ with the above defined operators T_j, $1 \le j \le d$, where $\mathbb{I}_A(\cdot)$ denotes the indicator function of the set $A \subset [-\pi,\pi)^\nu$. Next I formulate some properties of this class of random variables. These properties will appear in the definition of random spectral measures. All sets appearing in the next statements are measurable subsets of the torus $[-\pi,\pi)^\nu$.

(i) The random variables $Z_{G,j}(A)$ are complex valued, and their real and imaginary parts are jointly Gaussian, i.e. for any positive integer N and sets A_s, $1 \le s \le N$, the random variables $\mathrm{Re}\, Z_{G,j}(A_s)$, $\mathrm{Im}\, Z_{G,j}(A_s)$, $1 \le s \le N$, $1 \le j \le d$, are jointly Gaussian.

(ii) $EZ_{G,j}(A) = 0$ for all $1 \le j \le d$ and A,

(iii) $EZ_{G,j}(A)\overline{Z_{G,j'}(B)} = G_{j,j'}(A \cap B)$ for all $1 \le j, j' \le d$ and sets A, B.

(iv) $\sum_{s=1}^{n} Z_{G,j}(A_s) = Z_{G,j}\left(\bigcup_{s=1}^{n} A_s\right)$ if $A_1, \dots, A_n$ are disjoint sets, $1 \le j \le d$.

(v) $Z_{G,j}(A) = \overline{Z_{G,j}(-A)}$ for all $1 \le j \le d$ and sets A.

Properties (i)–(v) were proved in the one-dimensional case e.g. in [9]. The only difference in checking its several dimensional version is that we have to apply the multi-dimensional operator T from $\mathcal{K}_1^c$ to $\mathcal{H}_1^c$ to prove property (i), and to apply the same mapping T in proving Property (iii). Here we exploit that $\langle u, v\rangle_0 = \langle Tu, Tv\rangle_1$. We apply this identity with the vector $u \in \mathcal{K}_1^c$ whose j-th coordinate is $\mathbb{I}_A(x)$, and the other coordinates are zero and the vector $v \in \mathcal{K}_1^c$ whose k-th coordinate is $\mathbb{I}_B(x)$ and the other coordinates are zero. Property (v) can be proved as the special case of the following more general relation.

(v′) $T_j(u) = \overline{T_j(u_-)}$ for all $1 \le j \le d$ and $u \in \mathcal{K}_j^c$, where $u_-(x) = \overline{u(-x)}$.

Property (v′) can be proved by first proving it in the special case when $u(x)$ is a trigonometrical polynomial, and then applying a limiting procedure.

Next we define the vector valued random spectral measures corresponding to a matrix valued spectral measure.

Definition of vector valued random spectral measures on the torus. *Let a matrix valued spectral measure $G = (G_{j,j'})$, $1 \le j, j' \le d$, be given on the torus $[-\pi,\pi)^\nu$ together with a set of complex valued random variables indexed by pairs (j, A), where $1 \le j \le d$, and A is an element of the σ-algebra $\mathcal{A}$*

$$\mathcal{A} = \{A\colon\ A \subset [-\pi,\pi)^\nu \text{ is a Borel measurable set}\}$$

of the Borel measurable sets of the torus whose joint distribution depends on the matrix valued spectral measure G. To recall this dependence we denote the random variable indexed by a pair (j, A), $1 \le j \le d$, $A \in \mathcal{A}$, by $Z_{G,j}(A)$. We call

the set of random variables $Z_{G,j}(A)$, $1 \le j \le d$, $A \in \mathcal{A}$, a d-dimensional vector valued random spectral measure corresponding to the matrix valued spectral measure G on the torus $[-\pi,\pi)^\nu$ if this set of random variables satisfies properties (i)–(v) defined above. Given a fixed parameter $1 \le j \le d$ we call the set of random variables $Z_{G,j}(A)$, $A \in \mathcal{A}$, the j-th coordinate of this d-dimensional vector valued random spectral measure, and we denote it by $Z_{G,j}$. We denote the vector valued random spectral measure $Z_{G,j}(A)$, $1 \le j \le d$, $A \in \mathcal{A}$, by $Z_G = (Z_{G,1}, \dots, Z_{G,d})$.

More generally, if a matrix valued spectral measure G is given on the torus $[-B,B)^\nu$ with some number $B > 0$ together with a set of complex valued random variables $Z_{G,j}(A)$, where $1 \le j \le d$, and A is a Borel measurable set on the torus $[-B,B)^\nu$ which satisfies properties (i)–(v) defined above, then we call this set of random variables a d-dimensional vector valued random spectral measure corresponding to the spectral measure G. We call the set of random variables $Z_{G,j}(A)$, $A \in \mathcal{A}$, for a fixed $1 \le j \le d$ the j-th coordinate of this vector valued spectral measure, and denote it by $Z_{G,j}$. We denote the vector valued spectral measure by $Z_G = (Z_{G,1}, \dots, Z_{G,d})$.

Remark: If $G = (G_{j,j'})$, $1 \le j, j' \le d$, is a matrix valued spectral measure, $Z_G = (Z_{G,1}, \dots, Z_{G,d})$ is a vector valued spectral measure corresponding to it, then $G_{j,j}$ is a scalar valued spectral measure for any $1 \le j \le d$, and $Z_{G,j}$ is a scalar valued random spectral measure corresponding to it. As we shall see in Lemma 3.3 the spectral measure G determines the distribution of the random spectral measure Z_G.

It follows from the above considerations that for any d-dimensional matrix valued spectral measure there exists a d-dimensional vector valued random spectral measure corresponding to it. We can define the random integral with respect to it by means of the method applied in the scalar valued case.

We shall define the random integrals of the functions $f \in \mathcal{K}_{1,j}^c$ with respect to the random spectral measure $Z_{G,j}$, $1 \le j \le d$. First we define these integrals for elementary functions. They are finite sums of the form $\sum_{s=1}^N c_s \mathbb{I}_{A_s}(x)$, where $A_1, \dots, A_N$ are disjoint sets in $[-\pi,\pi)^\nu$, and c_s, $1 \le s \le N$, are arbitrary complex numbers. Their integrals with respect to the random spectral measure $Z_{G,j}$, $1 \le j \le d$, are defined as

$$\int \left(\sum_{s=1}^N c_s \mathbb{I}_{A_s}(x) \right) Z_{G,j}(\,dx) = \sum_{s=1}^N c_s Z_{G,j}(A_s).$$

As it is remarked in [9], property (iv) implies that this definition is meaningful, the integral of an elementary function does not depend on its representation. Then a simple calculation with the help of (iii) shows that for two elementary functions u and v

$$E\left(\int u(x) Z_{G,j}(\,dx) \overline{\int v(x) Z_{G,j}(\,dx)} \right) = \int u(x) \overline{v(x)} G_{j,j}(\,dx), \quad 1 \le j \le d. \tag{3.6}$$

This implies that the integral of the elementary functions with respect to the random spectral measure $Z_{G,j}$ define a norm preserving transformation from an everywhere dense subspace of the Hilbert space of $\mathcal{K}_{1,j}^c$ to an everywhere dense subspace of the Hilbert space of $\mathcal{H}_{1,j}^c$. This can be extended to a unitary transformation from $\mathcal{K}_{1,j}^c$ to $\mathcal{H}_{1,j}^c$ in a unique way, and this extension defines the integral of a function $u \in \mathcal{K}_{1,j}^c$. It is clear that relation (3.6) remains valid for general functions $u, v \in \mathcal{K}_{1,j}^c$. Moreover, it is not difficult to see with the help of (iii) that it can be generalized to the formula

$$E\left(\int u(x)Z_{G,j}(\,dx)\overline{\int v(x)Z_{G,j'}(\,dx)}\right) = \int u(x)\overline{v(x)}G_{j,j'}(\,dx) \tag{3.7}$$

if $u \in \mathcal{K}_{1,j}^c$ and $v \in \mathcal{K}_{1,j'}^c$, $1 \le j, j' \le d$.

It is clear that

$$E\int u(x)Z_{G,j}(\,dx) = 0 \quad \text{for all } u \in \mathcal{K}_{1,j}, \quad 1 \le j \le d. \tag{3.8}$$

Another important property of the random integrals with respect to $Z_{G,j}$ is that for all $1 \le j \le d$

$$\int u(x)Z_{G,j}(\,dx) \quad \text{is real valued if } u(-x) = \overline{u(x)} \text{ for } \mu \text{ almost all } x \in [-\pi,\pi)^\nu. \tag{3.9}$$

This relation holds, since $\int u(x)Z_{G,j}(\,dx) = \overline{\int u(x)Z_{G,j}(\,dx)}$ if $u(-x) = \overline{u(x)}$. We get this identity by means of the change of variables $x \to -x$ with the help of relation (v).

In the next Theorem I formulate the results we have about random spectral measures and random integrals with respect to them.

Theorem 3.1 *Given a positive semidefinite matrix valued, even measure $G = (G_{j,j'})$, $1 \le j, j' \le d$, on the torus $[-\pi,\pi)^\nu$ there exists a vector valued random spectral measure $Z_G = (Z_{G,1}, \dots, Z_{G,d})$ corresponding to it. We have defined the random integrals $\int u(x)Z_{G,j}(\,dx)$ for all $1 \le j \le d$ and $u \in \mathcal{K}_{1,j}^c$. This is a linear operator which satisfies relations (3.7), (3.8), (3.9), and the formula*

$$X_j(p) = \int_{[-\pi,\pi)^\nu} e^{i(p,x)}Z_{G,j}(\,dx), \quad 1 \le j \le d, \;\; p \in \mathbb{Z}^\nu, \tag{3.10}$$

defines a d-dimensional vector valued Gaussian stationary field whose matrix valued spectral measure is $G = (G_{j,j'})$, $1 \le j, j' \le d$. Moreover, if a d-dimensional vector valued Gaussian stationary random field is given with this matrix valued spectral measure, then the random integrals in formula (3.10) taken with respect to the random spectral measure that we have constructed with its help through an operator T in this section equals this vector valued Gaussian stationary random field.

Proof of Theorem 3.1. We have already proved the existence of the vector valued random spectral measure, and we constructed the random integral with respect

to it. It satisfies formulas (3.7) and (3.8). The random variables $X_j(p)$ defined in (3.10) are real valued by (3.9) and Gaussian with expectation zero. Hence we can show that they define a Gaussian stationary sequence with spectral measure $G = (G_{j.j'})$, $1 \le j, j' \le d$, by calculating their correlation function. We get by formula (3.7) that $EX_j(p)X_{j'}(q) = \int_{[-\pi,\pi)^\nu} e^{i(p-q,x)} G_{j,j'}(\,dx)$, and this had to be checked. If the random spectral measure is constructed in the way as we have done in this section, then a comparison of the random integral we have defined with its help and of the operator T shows that $\int u(x) Z_{G,j}(\,dx) = T_j(u(x))$ for all $u \in \mathcal{K}_{1,j}^c$. In particular, $\int_{-[\pi,\pi)^\nu} e^{i(p,x)} Z_{G,j}(\,dx) = T_j(e^{i(p,x)}) = X_j(p)$. This identity implies the last statement of Theorem 3.1. Theorem 3.1 is proved.

Formula (3.9) and Theorem 3.1 make possible to define for all $1 \le j \le d$ a real Hilbert space $\mathcal{K}_{1,j}$ consisting of appropriate elements of $\mathcal{K}_{1,j}^c$ for which the operator T_j is a norm preserving invertible transformation from $\mathcal{K}_{1,}j$ to the real Hilbert space $\mathcal{H}_{1,j}$ consisting of the real valued functions of the Hilbert space $\mathcal{H}_{1,j}^c$. More precisely, the following statement holds.

Lemma 3.2. *Let $(G_{j,j'})$, $1 \le j, j' \le d$, be a matrix valued spectral measure on the torus $[-\pi,\pi)^\nu$, and let $(Z_{G,1}, \dots, Z_{G,d})$ be a vector valued spectral measure corresponding to it. Define the d-dimensional vector valued Gaussian stationary field $(X_1(p), \dots, X_p(d))$ by formula (3.10) with the help of this vector valued random spectral measure. Define for all $1 \le j \le d$ the set of complex valued functions $\mathcal{K}_{1,j}$ on the torus $[-\pi,\pi)^\nu$ as*

$$\mathcal{K}_{1,j} = \left\{ u\colon \int |u(x)|^2 G_{j,j}(\,dx) < \infty, \quad u(-x) = \overline{u(x)} \text{ for all } x \in [-\pi,\pi)^\nu \right\}.$$

Then $\mathcal{K}_{1,j}$ is a real Hilbert space with the scalar product

$$\langle u, v \rangle = \int u(x)\overline{v(x)} G_{j,j}(\,dx), \quad u, v \in \mathcal{K}_{1,j}.$$

Let $\mathcal{H}_{1,j}$ be the real Hilbert space consisting of the closure of the finite linear combinations $\sum_{k=1}^N c_k X_j(p_k)$, $p_k \in \mathbb{Z}^\nu$, with real coefficients c_k in the Hilbert space $\mathcal{H}$ of random variables with finite second moments in the probability space where the random spectral measures $Z_{G,j}$ exists. (We define the scalar product in $\mathcal{H}$ in the usual way.) Then the map $T_j(u) = \int u(x) Z_{G,j}(\,dx)$, $u \in \mathcal{K}_{1,j}$, is a norm preserving, invertible linear transformation from the real Hilbert space $\mathcal{K}_{1,j}$ to the real Hilbert space $\mathcal{H}_{1,j}$.

Proof of Lemma 3.2. The space $\mathcal{K}_{1,j}$ is a real Hilbert space, since the change of variable $x \to -x$ in the integral $\langle u, v \rangle = \int u(x)\overline{v(x)} G_{j,j}(\,dx)$ implies that $\langle u, v \rangle = \overline{\langle u, v \rangle}$ for all $u, v \in \mathcal{K}_{1,j}$ because of the evenness of the measure $G_{j,j}$. Clearly $e^{i(p,x)} \in \mathcal{K}_{1,j}$ for all $p \in \mathbb{Z}^\nu$. The class of functions $\mathcal{K}_{1,j}$ agrees with the class of functions which have the form $u(x) = \frac{v(x)+\overline{v(-x)}}{2}$ with some $v \in \mathcal{K}_{1,j}^c$. As a consequence the set of finite trigonometrical polynomials $\sum c_k e^{i(p_k,x)}$, $p_k \in \mathbb{Z}^\nu$, with real valued coefficients c_k is an everywhere dense subspace of

$\mathcal{K}_{1,j}$. Since $T_j(\sum c_k e^{i(p_k,x)}) = \sum c_k X_j(p_k)$, the transformation T_j maps an everywhere dense subspace of $\mathcal{K}_{1,j}$ to an everywhere dense subspace of $\mathcal{H}_{1,j}$. Because of formulas (3.7) and (3.9) T_j is a norm preserving transformation in $\mathcal{K}_{1,j}$. Hence T_j is an invertible, norm preserving transformation from $\mathcal{K}_{1,j}$ to $\mathcal{H}_{1,j}$. Lemma 3.2 is proved.

I would remark that the transformation T_j on $\mathcal{K}_{1,j}$ defined in Lemma 3.2 is the restriction of the previously defined transformation T_j on $\mathcal{K}_{1,j}^c$ to its subset $\mathcal{K}_{1,j}$. I make also the following remark.

Lemma 3.3. *The positive semidefinite matrix valued, even measure $G(A) = (G_{j,j'}(A))$, $1 \le j, j' \le d$, $A \in [-\pi, \pi)^\nu$, determines the distribution of a vector valued spectral random measure $Z_{G,j}$, $1 \le j \le d$, corresponding to it.*

To prove this lemma we have to show that for any collection of measurable sets $A_1, \dots, A_N$, the matrix valued measure $G(A)$ determines the joint distribution of the random vector consisting of the elements $\mathrm{Re}\, Z_{G,j}(A_s)$, $\mathrm{Im}\, Z_{G,j}(A_s)$, $1 \le s \le N$, $1 \le j \le d$. Since this is a Gaussian random vector with expectation zero, it is enough to check that the covariance of these random variables can be expressed by means of the matrix valued measure $G(A)$. Since $\mathrm{Re} Z_{G,j}(A) = \frac{Z_{G,j}(A)+\overline{Z_{G,j}(A)}}{2}$ and $\mathrm{Im}\, Z_{G,j}(A) = \frac{Z_{G,j}(A)-\overline{Z_{G,j}(A)}}{2i}$ we can calculate these covariances with the help of properties (iii) and (v) of vector valued random spectral measures.

Finally I prove an additional property of the vector valued random spectral measures which will be useful in Section 5, in the study of multiple Wiener–Itô integrals.

(vi) The random variables of the form $Z_{G,j}(A \cup (-A))$ are real valued. Let a set $A \cup (-A)$ be disjoint from some sets $B_1 \cup (-B_1), \dots, B_n \cup (-B_n)$. Then for any indices $1 \le j, j' \le d$ the (complex valued) random vector $(Z_{G,j}(A), Z_{G,j'}(A))$, is independent of the random vector consisting of the elements $Z_{G,k}(B_s)$, $1 \le s \le n$, $1 \le k \le d$.

Proof of property (vi). It follows from property (v) that $Z_{G,j}(A \cup (-A)) = \overline{Z_{G,j}(A \cup (-A))}$, hence $Z_{G,j}(A \cup (-A))$ is real valued. To prove the second statement of (vi) it is enough to check that under its conditions the (real valued) random variables $\mathrm{Re}\, Z_{G,j}(A)$ and $\mathrm{Im}\, Z_{G,j}(A)$ are uncorrelated to all random variables $\mathrm{Re}\, Z_{G,k}(B_s)$, $\mathrm{Im}\, Z_{G,k}(B_s)$, $1 \le s \le n$, $1 \le k \le d$. This relation holds, since by the conditions of (vi) $(\pm A) \cap (\pm B_s) = \emptyset$, hence relation (iii) implies that $EZ_{G,j}(\pm A)\overline{Z_{G,j'}(\pm B_s)} = 0$ for all sets B_s, $1 \le s \le n$, and indices $1 \le j, j' \le d$. On the other hand, all covariances can be expressed as a linear combination of such expressions, since by relation (v) $\mathrm{Re}\, Z_{G,j}(\pm A) = \frac{Z_{G,j}(\pm A)+\overline{Z_{G,j}(\pm A)}}{2} = \frac{Z_{G,j}(\pm A)+Z_{G,j}(\mp A)}{2}$, and a similar relation holds also for $\mathrm{Im}\, Z_{G,j}(\pm A)$, $\mathrm{Re}\, Z_{G,j'}(\pm B_s)$ and $\mathrm{Im}\, Z_{G,j'}(\pm B_s)$, $1 \le s \le n$, $1 \le j' \le d$.

4 Spectral representation of vector valued stationary generalized random fields

In Sections 2 and 3 we discussed the properties of vector valued Gaussian stationary random fields with discrete parameters, which means a class of Gaussian random vectors $X(p)$, $p \in \mathbb{Z}^\nu$, with some nice properties. Similarly, we could have defined and investigated vector valued Gaussian stationary random fields with continuous parameters, where we consider a set of random vectors $X(t)$ indexed by $t \in \mathbb{R}^\nu$ which have some nice properties. But we do not discuss this topic here. Here we define and investigate instead so-called vector valued Gaussian stationary generalized random fields $X(\varphi) = (X_1(\varphi), \dots, X_d(\varphi))$, parametrized with a nice linear space of functions φ.

Actually I am interested here in the vector valued Gaussian stationary generalized random fields not for their own sake. We shall construct a class of vector valued Gaussian stationary generalized random fields. We shall show that their distribution can be described by means of a matrix valued spectral measure. We can also construct a vector valued random spectral measure in such a way that the elements of our vector valued generalized random field can be expressed in a form that can be considered as the Fourier transform of this random spectral measure. These matrix valued spectral measures and vector valued random spectral measures slightly differ from those defined in Sections 2 and 3, but since they are very similar to the corresponding objects defined for stationary random fields with discrete parameters it is natural to give them the same name.

The results that we shall prove are very similar to the results we got about vector valued random fields with discrete parameters. The main difference is that we can construct a larger class of matrix valued spectral measures and vector valued random spectral measures by means of generalized random fields. We shall need them, because in our later investigations we shall deal with such limit theorems where we can express the limit by means of these new, more general objects. On the other hand, these new vector valued random spectral measures behave similarly to the previous ones. In particular, the later results of this paper about multiple Wiener–Itô integrals also hold for this more general class of vector valued random spectral measures. Let me remark that we met a similar picture in the study of scalar valued Gaussian random fields in [9], so that here we actually generalize the results in that work to the multi-dimensional case.

In the definition of vector valued generalized random fields we shall choose the functions of the Schwartz space for the class of parameter set. So to define the vector valued generalized random fields first I recall the definition of the Schwartz space, (see [6]).

We define the Schwartz space $\mathcal{S}$ of real valued functions on $\mathbb{R}^\nu$ together with its version $\mathcal{S}^c$ consisting of complex valued functions on $\mathbb{R}^\nu$. The space $\mathcal{S}^c = (\mathcal{S}^\nu)^c$ consists of those complex valued functions of ν arguments which decrease at infinity, together with their derivatives, faster than any polynomial

degree. More explicitly, $\varphi \in \mathcal{S}^c$ for a complex valued function φ defined on $\mathbb{R}^\nu$ if

$$\left| x_1^{k_1} \cdots x_\nu^{k_\nu} \frac{\partial^{q_1+\cdots+q_\nu}}{\partial x_1^{q_1} \dots \partial x_\nu^{q_\nu}} \varphi(x_1, \dots, x_\nu) \right| \le C(k_1, \dots, k_\nu, q_1, \dots, q_\nu)$$

for all points $x = (x_1, \dots, x_\nu) \in \mathbb{R}^\nu$ and vectors $(k_1, \dots, k_\nu)$, $(q_1, \dots, q_\nu)$ with non-negative integer coordinates with some constant $C(k_1, \dots, k_\nu, q_1, \dots, q_\nu)$ which may depend on the function φ. The elements of the space $\mathcal{S}$ are defined similarly, with the only difference that they are real valued functions.

To complete the definition of the spaces $\mathcal{S}$ and $\mathcal{S}^c$ we still have to define the topology in them. We introduce the following topology in these spaces.

Let a basis of neighbourhoods of the origin consist of the sets

$$U(k, p, \varepsilon) = \left\{ \varphi \colon \ \varphi \in \mathcal{S}, \max_{\substack{q=(q_1,\dots,q_\nu) \\ 0 \le q_s \le p, \text{ for all } 1 \le s \le \nu}} \sup_x (1 + |x|^2)^k |D^q \varphi(x)| < \varepsilon \right\}$$

with $k = 0, 1, 2, \dots$, $p = 1.2, \dots$ and $\varepsilon > 0$, where $|x|^2 = x_1^2 + \cdots + x_\nu^2$, and $D^q = \frac{\partial^{q_1+\cdots+q_\nu}}{\partial x_1^{q_1} \dots \partial x_\nu^{q_\nu}}$ for $q = (q_1, \dots, q_\nu)$. A basis of neighbourhoods of an arbitrary function $\varphi \in \mathcal{S}^c$ (or $\varphi \in \mathcal{S}$) consists of sets of the form $\varphi + U(k, q, \varepsilon)$, where the class of sets $U(k, q, \varepsilon)$ is a basis of neighbourhood of the origin. Actually we shall use only the following property of this topology. A sequence of functions $\varphi_n \in \mathcal{S}^c$ (or $\varphi_n \in \mathcal{S}$) converges to a function φ in this topology if and only if

$$\lim_{n \to \infty} \sup_{x \in \mathbb{R}^\nu} (1 + |x|^2)^k |D^q \varphi_n(x) - D^q \varphi(x)| = 0.$$

for all $k = 1, 2, \dots$ and $q = (q_1, \dots, q_\nu)$. The limit function φ is also in the space $\mathcal{S}^c$ (or in the space $\mathcal{S}$).

I shall define the notion of vector valued generalized random fields together with some related notions with the help of the notion of Schwartz spaces. A d-dimensional generalized random field is a random field whose elements are d-dimensional random vectors

$$(X_1(\varphi), \dots, X_d(\varphi)) = (X_1(\varphi, \omega), \dots, X_d(\varphi, \omega))$$

defined for all functions $\varphi \in \mathcal{S}$, where $\mathcal{S} = \mathcal{S}^\nu$ is the Schwartz space. Before defining vector valued generalized random fields I write down briefly the idea of their definition. This is explained in [9] and [10] in more detail.

Given a vector valued Gaussian stationary random field

$$X(t) = (X_1(t), \dots, X_d(t)), \quad t \in \mathbb{R}^\nu,$$

we can define with its help the random field $X(\varphi) = (X_1(\varphi), \dots, X_d(\varphi))$, $\varphi \in \mathcal{S}^\nu$, $X_j(\varphi) = \int \varphi(t) X_j(t)\, dt$, $1 \le j \le d$, indexed by the elements of the Schwartz space, and this determines the original random field. We define generalized random fields with elements indexed by $\varphi \in \mathcal{S}$ as such random fields which behave similarly to the random fields defined by means of such integrals.

Definition of vector valued generalized random fields. *We say that the set of random vectors* $(X_1(\varphi), \dots, X_d(\varphi))$, $\varphi \in \mathcal{S}$, *is a* d*-dimensional vector valued generalized random field over the Schwartz space* $\mathcal{S} = \mathcal{S}^\nu$ *of rapidly decreasing smooth functions if:*

(a) $X_j(a_1\varphi + a_2\psi) = a_1X_j(\varphi) + a_2X_j(\psi)$ *with probability 1 for the* j*-th coordinate of the random vectors* $(X_1(\varphi), \dots, X_d(\varphi))$ *and* $(X_1(\psi), \dots, X_d(\psi))$. *This relation holds for each coordinate* $1 \le j \le d$, *all real numbers* a_1 *and* a_2, *and pair of functions* φ, ψ *from the Schwartz space* $\mathcal{S}$. *(The exceptional set of probability 0 where this identity does not hold may depend on* a_1, a_2, φ *and* ψ.*)*

(b) $X_j(\varphi_n) \Rightarrow X_j(\varphi)$ *stochastically for any* $1 \le j \le d$ *if* $\varphi_n \to \varphi$ *in the topology of* $\mathcal{S}$.

We also introduce the following definition. In its formulation we use the notation $\stackrel{\Delta}{=}$ for equality in distribution.

Definition of stationarity and Gaussian property for a vector valued generalized random field. *The* d*-dimensional vector valued generalized random field* $X = \{(X_1(\varphi) \dots, X_d(\varphi)),\ \varphi \in \mathcal{S}\}$ *is stationary if*

$$(X_1(\varphi) \dots, X_d(\varphi)) \stackrel{\Delta}{=} (X_1(T_t\varphi) \dots, X_d(T_t\varphi))$$

for all $\varphi \in \mathcal{S}$ *and* $t \in \mathbb{R}^\nu$, *where* $T_t\varphi(x) = \varphi(x - t)$. *It is Gaussian if* $(X_1(\varphi), \dots, X_d(\varphi))$ *is a Gaussian random vector for all* $\varphi \in \mathcal{S}$. *We call a vector valued generalized random field a vector valued generalized random field with zero expectation if* $EX_j(\varphi) = 0$ *for all* $\varphi \in \mathcal{S}$ *and coordinates* $1 \le j \le d$.

In the definition of stationarity and Gaussian property we imposed a condition for a single random vector. But because of the linearity property of generalized random fields formulated in property (a) of their definition and the fact that if we have N random vectors $\xi_1, \dots, \xi_N$ and $\eta_1, \dots, \eta_N$ such that the linear combinations $\sum_{k=1}^N a_k\xi_k$ and $\sum_{k=1}^N a_k\eta_k$ have the same distribution for any coefficients a_k, $1 \le k \le N$, then the joint distribution of the random vectors $\xi_1, \dots, \xi_N$ and $\eta_1, \dots, \eta_N$ agree imply that an analogous statement holds about the properties of the joint distribution of several random vectors in a vector valued stationary random field. Indeed, if we take N random vectors $(X_1(\varphi_k), \dots, X_d(\varphi_k))$, $1 \le k \le N$, then their joint distribution agrees with the joint distribution of their shifts $(X_1(T_t\varphi_k), \dots, X_d(T_t\varphi_k))$, $1 \le k \le N$, for any $t \in \mathbb{R}^\nu$. This follows from the fact that

$$\sum_{k=1}^N a_k(X_1(\varphi_k), \dots, X_d(\varphi_k)) \stackrel{\Delta}{=} \sum_{k=1}^N a_k(X_1(T_t\varphi_k), \dots, X_d(T_t\varphi_k))$$

for all $t \in \mathbb{R}^\nu$ and coefficients a_k, $1 \le k \le N$, for a d-dimensional vector valued stationary generalized random field because of the linearity property of the

generalized random fields and the properties of the operator T_t. A similar argument shows that the joint distribution of some vectors $(X_1(\varphi_k), \dots, X_d(\varphi_k))$, $1 \le k \le N$, in a vector valued Gaussian generalized random field is Gaussian.

I shall construct a large class of d-dimensional vector valued Gaussian stationary generalized random fields with expectation zero. I shall construct them with the help of positive semidefinite matrix valued even measures on $\mathbb{R}^\nu$. In the next step I write down this definition. The main difference between the definition of this notion and its counterpart defined on the torus $[-\pi, \pi)^\nu$ is that now we consider such complex measures which may have non-finite total variation. We impose instead a less restrictive condition. We shall work with complex measures on $\mathbb{R}^\nu$ which have locally finite total variation. For the sake of completeness I give their definition.

Definition of complex measures on $\mathbb{R}^\nu$ with locally finite total variation. The definition of their evenness property. *A complex measure on $\mathbb{R}^\nu$ with locally finite total variation is such a complex valued function on the bounded, Borel measurable subsets of $\mathbb{R}^\nu$ whose restrictions to the measurable subsets of a cube $[-T, T]^\nu$ are complex measures with finite total variation for all $T > 0$. We say that a complex measure G on $\mathbb{R}^\nu$ with locally finite total variation is even, if $G(-A) = \overline{G(A)}$ for all bounded and measurable sets $A \subset \mathbb{R}^\nu$.*

Let me remark that not all complex measures with locally finite total variation can be extended to a complex measure on all measurable subsets of $\mathbb{R}^\nu$. On the other hand, this can be done if we are working with a (real, positive number valued) measure. Next I formulate the definition we need in our discussion.

Definition of positive semidefinite matrix valued measures on $\mathbb{R}^\nu$ with moderately increasing distribution at infinity. The definition of their evenness property. *A Hermitian matrix valued measure on $\mathbb{R}^\nu$ is a class of such Hermitian matrices $(G_{j,j'}(A))$, $1 \le j, j' \le d$, defined for all bounded, measurable sets $A \subset \mathbb{R}^\nu$ for which all coordinates $G_{j,j'}(\cdot)$, $1 \le j, j' \le d$, are complex measures on $\mathbb{R}^\nu$ with locally finite total variation. We call a Hermitian matrix valued measure $(G_{j,j'}(\cdot))$, $1 \le j, j' \le d$, on $\mathbb{R}^\nu$ positive semidefinite if there exists a (σ-finite) positive measure μ on $\mathbb{R}^\nu$ such that for all numbers $T > 0$ and indices $1 \le j, j' \le d$ the restriction of the complex measures $G_{j,j'}$ to the cube $[-T, T]^\nu$ is absolutely continuous with respect to μ, and the matrices $(g_{j,j'}(x))$, $1 \le j, j' \le d$, defined with the help of the Radon–Nikodym derivatives $g_{j,j'}(x) = \frac{dG_{j,j'}}{d\mu}(x)$, $1 \le j, j' \le d$, are Hermitian, positive semidefinite matrices for almost all $x \in \mathbb{R}^\nu$ with respect to the measure μ. We call this Hermitian matrix valued measure $(G_{j,j'}(\cdot))$, $1 \le j, j' \le d$, on $\mathbb{R}^\nu$ even if the complex measures $G_{j,j'}$ with locally finite variation are even for all $1 \le j, j' \le d$.*

We shall say that the distribution of a positive semidefinite matrix valued measure $(G_{j,j'}(\cdot))$, $1 \le j, j' \le d$, on $\mathbb{R}^\nu$ is moderately increasing at infinity if

$$\int (1+|x|)^{-r} G_{j,j}(\,dx) < \infty \quad \text{for all } 1 \le j \le d \text{ with some number } r > 0. \tag{4.1}$$

Remark. We can give, similarly to Lemma 2.3, a different characterization of positive semidefinite matrix valued, even measures on $\mathbb{R}^\nu$. Let us have some complex measures $G_{j,j'}$, $1 \le j, j' \le d$, on the σ-algebra of the Borel measurable sets of $\mathbb{R}^\nu$ such that their restrictions to any cube $[-T,T]^\nu$, $T > 0$, have finite total variation. Let us consider the matrix valued measure $(G_{j,j'}(A))$, $1 \le j, j' \le d$ on $\mathbb{R}^\nu$ for all bounded, measurable sets $A \subset \mathbb{R}^\nu$. This matrix valued measure is positive semidefinite and even if and only if it satisfies the following two conditions.

(i.) The $d \times d$ matrix $(G_{j,j'}(A))$, $1 \le j, j' \le d$, is Hermitian, positive semidefinite for all bounded, measurable sets $A \subset \mathbb{R}^\nu$.

(ii.) $G_{j,j'}(-A) = \overline{G_{j,j'}(A)}$, for all $1 \le j, j' \le d$ and bounded, measurable sets $A \subset \mathbb{R}^\nu$.

This statement has almost the same proof as Lemma 2.3. The only difference in the proof is that now we have to work with such vectors $v(x) = (v_1(x), \dots, v_d(x))$ whose coordinates $v_j(x)$ are continuous functions on $\mathbb{R}^\nu$ with bounded support, $1 \le j \le d$. Let me also remark that the following statement also follows from this proof. If a matrix valued measure $(G_{j,j'}(A))$, $1 \le j, j' \le d$, on $\mathbb{R}^\nu$ satisfies the conditions in the definition of positive semidefinite matrices with some σ-finite measure μ on $\mathbb{R}^\nu$ with respect to which all complex measures $G_{j,j}$ are absolutely continuous, then it satisfies these conditions with any σ-finite measure μ on $\mathbb{R}^\nu$ with the same property.

Before constructing a large class of vector valued Gaussian stationary generalized random fields I recall an important property of the Fourier transform of the functions in the Schwartz spaces $\mathcal{S}$ and $\mathcal{S}^c$, (see e.g. [6]). Actually this property of the Schwartz spaces made useful their choice in the definition of generalized fields.

The Fourier transform $f \to \tilde{f}$ is a bicontinuous map from $\mathcal{S}^c$ to $\mathcal{S}^c$. (This means that this transformation is invertible, and both the Fourier transform and its inverse are continuous maps from $\mathcal{S}^c$ to $\mathcal{S}^c$.) (The restriction of the Fourier transform to the space $\mathcal{S}$ of real valued functions is a bicontinuous map from $\mathcal{S}$ to the subspace of $\mathcal{S}^c$ consisting of those functions $f \in \mathcal{S}^c$ for which $f(-x) = \overline{f(x)}$ for all $x \in \mathbb{R}^\nu$.)

Next I formulate the following result.

Theorem 4.1 about the construction of vector valued Gaussian stationary generalized random fields with zero expectation. *Let $(G_{j,j'})$, $1 \le j, j' \le d$, be a positive semidefinite matrix valued even measure on $\mathbb{R}^\nu$ whose distribution is moderately increasing at infinity.*

Then there exists a vector valued Gaussian stationary generalized random field $(X_1(\varphi), \dots, X_d(\varphi))$, $\varphi \in \mathcal{S}$, such that $EX_j(\varphi) = 0$ for all $\varphi \in \mathcal{S}$, and given two Shwartz functions $\varphi \in \mathcal{S}$ and $\psi \in \mathcal{S}$, the covariance function $r_{j,j'}(\varphi, \psi) = EX_j(\varphi)X_{j'}(\psi)$ is given by the formula

$$r_{j,j'}(\varphi, \psi) = EX_j(\varphi)X_{j'}(\psi) = \int \tilde{\varphi}(x)\bar{\tilde{\psi}}(x)G_{j,j'}(\,dx) \quad \text{for all } \varphi, \psi \in \mathcal{S}, \quad (4.2)$$

where ˜ denotes Fourier transform, and ¯ is complex conjugate.

Formula (4.2) and the identity $EX_j(\varphi) = 0$ *for all* $\varphi \in \mathcal{S}$ *determine the distribution of the vector valued, Gaussian stationary random field* $(X_1(\varphi), \dots, X_d(\varphi))$.

Contrariwise, for all $1 \le j, j' \le d$ *the covariance function* $EX_j(\varphi)X_{j'}(\psi)$, $\varphi, \psi \in \mathcal{S}$, *determines the coordinate* $G_{j,j'}$ *of the positive semidefinite, even matrix* $(G_{j,j'})$. $1 \le j, j' \le d$, *with moderately increasing distribution at infinity for which identity (4.2) holds.*

Let me remark that the moderate decrease of the distribution of the positive semidefinite matrix $(G_{j,j'})$, $1 \le j, j' \le d$, together with inequality (3.2) and the fast decrease of the functions $\varphi \in \mathcal{S}$ at infinity guarantee that the integral in (4.2) is convergent.

Condition (4.1) which we wrote in the definition of moderately increasing positive semidefinite matrix valued measures appears in the theory of generalized functions in a natural way. Such a condition characterizes those measures which are generalized functions, i.e. continuous linear maps in the Schwartz space.

In [9] we have proved with the help of some important results of Laurent Schwartz about generalized functions that in the case of scalar valued models, i.e. if $d = 1$ the covariance function of every Gaussian stationary generalized random field with expectation zero agrees with the covariance function of a Gaussian stationary generalized random field constructed in the same way as we have done in Theorem 4.1. (In the case $d = 1$ the formulation of this result is simpler.) It seems very likely that a refinement of that argument would give the proof of an analogous statement in the general case. I did not investigate this question, because in the present paper we do not need such a result.

Remark. Similarly to the case of vector valued stationary fields with discrete parameter we shall introduce the following terminology. If $(G_{j,j'})$, $1 \le j, j' \le d$, is a positive semidefinite, matrix valued even measure with moderately increasing distribution at infinity, and there is a stationary generalized random field $(X_1(\varphi), \dots, X_d(\varphi))$, $\varphi \in \mathcal{S}$, whose covariance function

$$r_{j,j'}(\varphi, \psi) = EX_j(\varphi)X_{j'}(\psi), \quad 1 \le j, j' \le d, \quad \varphi, \psi \in \mathcal{S},$$

satisfies relation (4.2) with this matrix valued measure G, then we call G the matrix valued spectral measure of this covariance function $r_{j,j'}(\varphi, \psi)$. In general, we shall call a positive semidefinite matrix valued even measure on $\mathbb{R}^\nu$ with moderately increasing distribution at infinity a matrix valued spectral measure on $\mathbb{R}^\nu$. We have the right for such a terminology, because by Theorem 4.1 for any such matrix valued measure there exists a Gaussian stationary generalized random field such that this matrix valued measure is the matrix valued spectral measure of its covariance function.

Let me remark that the diagonal elements $G_{j,j}$ of the matrix valued spectral measure of the correlation function $r_{j,j'}(\varphi, \psi)$ of a vector valued stationary random field may be non finite measures on $\mathbb{R}^\nu$, they have to satisfy only relation (4.1). As a consequence, we can find a much richer class of matrix valued

spectral measures by working with generalized random fields than by working only with classical stationary random fields. As we shall see, also vector valued random spectral measures corresponding to these matrix valued spectral measures can be constructed. Actually we discussed vector valued stationary generalized random fields in this paper in order to construct this larger class of matrix valued spectral and vector valued random spectral measures.

Proof of Theorem 4.1. Let us observe that the function $r_{j,j'}(\varphi,\psi)$ defined in (4.2) is real valued. This can be seen by applying the change of variables $x \to -x$ in this integral and by exploiting that $G_{j,j'}(-A) = \overline{G_{j,j'}(A)}$, and $\tilde{\varphi}(-x) = \bar{\tilde{\varphi}}(x)$, $\tilde{\psi}(-x) = \bar{\tilde{\psi}}(x)$, since this calculation yields that $r_{j,j'}(\varphi,\psi) = \overline{r_{j,j'}(\varphi,\psi)}$. Let us also remark that $r_{j,j'}(\varphi,\psi) = r_{j',j}(\psi,\varphi)$, since by formula (4.2) and the property $G_{j,j'}(A) = \overline{G_{j',j}(A)}$ of the matrix $(\overline{G_{j,j'}(A)})$, $1 \le j,j' \le d$, for all measurable sets $A \subset \mathbb{R}^\nu$ we have $r_{j,j'}(\varphi,\psi) = \overline{r_{j',j}(\psi,\varphi)}$, and we know that both side of this identity is real valued.

First we show that for all positive integers N and functions $\varphi_k \in \mathcal{S}$, $1 \le k \le N$, there are some Gaussian random vectors $(X_1(\varphi_k),\dots,X_d(\varphi_k))$, $1 \le k \le N$, with expectation zero and covariances $EX_j(\varphi_k)X_{j'}(\varphi_{k'}) = r_{j,j'}(\varphi_k,\varphi_{k'})$ for all $1 \le j,j' \le d$, $1 \le k,k' \le N$, on an appropriate probability space, where $r_{j,j'}(\varphi_k,\varphi_{k'})$ is defined at the right-hand side of formula (4.2) with our matrix valued measure $(G_{j,j'})$, $1 \le j,j' \le d$, and with the choice $\varphi = \varphi_k$, $\psi = \varphi_{k'}$.

We prove this statement if we show that the matrix with elements

$$d_{(j,k),(j',k')} = r_{j,j'}(\varphi_k,\varphi_{k'}), \quad 1 \le j,j' \le d, \;\; 1 \le k,k' \le N,$$

is positive semidefinite. To prove this result take any vector $(a_{j,k},\ 1 \le j \le d, 1 \le k \le N)$, and observe that

$$\begin{aligned}
&\sum_{j=1}^{d}\sum_{j'=1}^{d}\sum_{k=1}^{N}\sum_{k'=1}^{N} a_{j,k}\overline{a_{j',k'}} r_{j,j'}(\varphi_k,\varphi_{k'}) \\
&\quad = \sum_{j=1}^{d}\sum_{j'=1}^{d}\sum_{k=1}^{N}\sum_{k'=1}^{N} \int (a_{j,k}\tilde{\varphi}_k(x))(\overline{a_{j',k'}\tilde{\varphi}_{k'}(x)}) g_{j,j'}(x)\mu(\,dx) \\
&\quad = \sum_{j=1}^{d}\sum_{j'=1}^{d} \int \psi_j(x)\overline{\psi_{j'}(x)} g_{j,j'}(x)\mu(\,dx) = \int \psi(x) g(x) \overline{\psi(x)}\,\mu(\,dx) \ge 0,
\end{aligned}$$

where $\psi_j(x) = \sum_{k=1}^{N} a_{j,k}\tilde{\varphi}_k(x)$, $1 \le j \le d$, $\psi(x) = (\psi_1(x),\dots,\psi_d(x))$, and $g(x)$ denotes the matrix $(g_{j,j'}(x))$, $1 \le j,j' \le d$. In this calculation we applied formula (4.2), the representation $G_{j,j'}(\,dx) = g_{j,j'}(x)\mu(\,dx)$ and finally the fact that $g(x)$ is a semidefinite matrix for μ almost all x.

Then it follows from Kolmogorov's existence theorem for random processes with consistent finite distributions that there is a Gaussian random field

$$(X_1(\varphi),\dots,X_d(\varphi)), \quad \varphi \in \mathcal{S},$$

with zero expectation such that $EX_j(\varphi)X_{j'}(\psi) = r_{j,j'}(\varphi,\psi)$ for all functions $\varphi \in \mathcal{S}$, ($\psi \in \mathcal{S}$ and $1 \le j, j' \le d$. Besides, the finite dimensional distributions of this random field are determined because of the Gaussian property. Next we show that this random field is a vector valued generalized random field.

Property (a) of the vector valued generalized random fields follows from the following calculation.

$$\begin{aligned} &E[a_1X_j(\varphi) + a_2X_j(\psi) - X_j(a_1\varphi + a_2\psi)]^2 \\ &\quad = \int \Big(a_1\tilde{\varphi}(x) + a_2\tilde{\psi}(x) - \widetilde{(a_1\varphi + a_2\psi)}(x)\Big) \\ &\qquad\qquad \Big(\overline{a_1\tilde{\varphi}(x)} + \overline{a_2\tilde{\psi}(x)} - \overline{\widetilde{(a_1\varphi + a_2\psi)}(x)}\Big) G_{j,j}(\,dx) = 0 \end{aligned}$$

by formula (4.2) for all real numbers a_1, a_2, $1 \le j \le d$ and $\varphi, \psi \in \mathcal{S}$.

Property (b) of the vector valued generalized random fields also holds for this model. Actually it is proved in [9] that if $\varphi_n \to \varphi$ in the topology of the space $\mathcal{S}$, then $E[X_j(\varphi_n) - X_j(\varphi)]^2 = \int |\tilde{\varphi}_n(x) - \tilde{\varphi}(x)|^2 G_{j,j}(\,dx) \to 0$ as $n \to \infty$, hence property (b) also holds. (The proof is not difficult. It exploits that for a sequence of functions $\varphi_n \in \mathcal{S}^c$, $n = 0, 1, 2, \dots$, $\varphi_n \to \varphi_0$ as $n \to \infty$ in the topology of $\mathcal{S}^c$ if and only if $\tilde{\varphi}_n \to \tilde{\varphi}_0$ in the same topology. Besides, the measure $G_{j,j}$ satisfies inequality (4.1).)

It is also clear that the Gaussian random field constructed in such a way is stationary.

It remained to show that the covariance function $r_{j,j'}(\varphi,\psi) = EX_j(\varphi)X_{j'}(\psi)$ determines the complex measure $G_{j,j'}$. To show this we have to observe that inequality (3.2) holds also in this case, hence the Schwarz inequality implies that

$$\int (1 + |x|)^{-r} |g_{j,j'}(x)| \mu(\,dx) < \infty \quad \text{for all } 1 \le j, j' \le d$$

for a positive semidefinite matrix valued measure with moderately increasing distribution, i.e. this inequality holds not only for $j = j'$. Then it follows from the standard theory of Schwartz spaces that the class of Schwartz functions is sufficiently rich to guarantee that the function $r_{j,j'}(\varphi,\psi)$ determines the complex measure $G_{j,j'}$. Theorem 4.1 is proved.

Next we construct a vector valued random spectral measure corresponding to a matrix valued spectral measure $(G_{j,j'})$, $1 \le j, j' \le d$, on $\mathbb{R}^\nu$. We argue similarly to Section 3, where the vector valued random spectral measures corresponding to matrix valued spectral measures on $[-\pi,\pi)^\nu$ were considered. In the construction we shall also refer to some results in [9].

Let us have a vector valued Gaussian stationary generalized random field $X = (X_1(\varphi), \dots, X_d(\varphi))$, $\varphi \in \mathcal{S}$, $1 \le j \le d$, with a matrix valued spectral measure $(G_{j,j'})$, $1 \le j, j' \le d$. First we define for all $1 \le j \le d$ some (complex) Hilbert spaces $\mathcal{K}^c_{1,j}$, $\mathcal{H}^c_{1,j}$ and a norm preserving, invertible linear transformation T_j between them in the following way. $\mathcal{K}^c_{1,j}$ consists of those complex valued functions $u(x)$ on $\mathbb{R}^\nu$ for which $\int |u(x)|^2 G_{j,j}(\,dx) < \infty$ with the scalar product

$\langle u(x), v(x)\rangle = \int u(x)\overline{v(x)}G_{j,j}(\,dx)$. To define the Hilbert space $\mathcal{H}^c_{1,j}$ let us first introduce the Hilbert space $\mathcal{H} = \mathcal{H}^c$ of (complex valued) random variables with finite second moment on the probability space $(\Omega, \mathcal{A}, \mathcal{P})$ where our stationary generalized random field is defined. We define the Hilbert space $\mathcal{H}^c$ in the space consisting of these random variables with the usual scalar product $\langle \xi, \eta\rangle = E\xi\bar{\eta}$ in $\mathcal{H}^c$. The Hilbert space $\mathcal{H}^c_{1,j}$ is defined as the closure of the linear subspace of $\mathcal{H}^c$ consisting of the complex valued random variables $X_j(\varphi)+iX_j(\psi)$, $\varphi, \psi \in \mathcal{S}$.

First we define the operator T_j for functions of the form $\widetilde{\varphi + i\psi}$, $\varphi, \psi \in \mathcal{S}$. We define it by the formula

$$T_j(\widetilde{\varphi + i\psi}) = X_j(\varphi) + iX_j(\psi), \quad \varphi,\, \psi \in \mathcal{S}. \tag{4.3}$$

Some calculation which was actually carried out in [9] shows that the set of functions $\widetilde{\varphi + i\psi}$, $\varphi, \psi \in \mathcal{S}$, is dense in $\mathcal{K}^c_{1,j}$, and the transformation T_j, defined in (4.3) can be extended to a norm preserving, invertible linear transformation from $\mathcal{K}^c_{1,j}$ to $\mathcal{H}^c_{1,j}$. (In the calculation leading to this statement we apply formula (4.2) with the choice $j' = j$.)

Then we can define the random spectral measure $Z_{G,j}(A)$, similarly to the case discussed in Section 3, by the formula $Z_{G,j}(A) = T_j\mathbb{I}_A(\cdot))$ for all bounded measurable sets $A \subset \mathbb{R}^\nu$. To determine the joint distribution of the spectral measures $Z_{G,j}$ we make the following version of the corresponding argument in Section 3.

We define the following two Hilbert spaces $\mathcal{K}^c_1$ and $\mathcal{H}^c_1$ together with a norm preserving linear transformation T between them.

The elements of the Hilbert space $\mathcal{K}^c_1$ are the vectors $u = (u_1(x), \dots, u_d(x))$ with $u_j(x) \in \mathcal{K}^c_{1,j}$, $1 \le j \le d$. We define the scalar product on $\mathcal{K}^c_1$ with the help of the following positive semidefinite bilinear form $\langle \cdot, \cdot\rangle_0$. If $u(x) = (u_1(x), \dots, u_d(x)) \in \mathcal{K}^c_1$ and $v(x) = (v_1(x), \dots, v_d(x)) \in \mathcal{K}^c_1$, then

$$\begin{aligned}\langle u(x), v(x)\rangle_0 &= \sum_{j=1}^d \sum_{j'=1}^d \int u_j(x)\overline{v_{j'}(x)}G_{j,j'}(\,dx) \\ &= \sum_{j=1}^d \sum_{j'=1}^d \int g_{j,j'}(x)u_j(x)\overline{v_{j'}(x)}\mu(\,dx) = \int u(x)g(x)v(x)^*\mu(\,dx)\end{aligned}$$

with the matrix $g(x) = (g_{j,j'}(x))$, $1 \le j, j' \le d$, where $v^*(x)$ denotes the column vector whose elements are the functions $\overline{v_{j'}(x)}$, $1 \le j' \le d$. Actually, here we simply copied the corresponding definition in Section 3 for the discrete time model, and we can also prove that $\mathcal{K}^c_1$ is a Hilbert space with the scalar $\langle \cdot, \cdot\rangle_0$ in the same way as it was done in Section 3.

The construction $\mathcal{H}^c_1$, and the proof of its properties is again a simple copying of argument made in Section 3. The elements of $\mathcal{H}^c_1$ are the vectors $\xi = (\xi_1, \dots, \xi_d)$, where $\xi_j \in \mathcal{H}^c_{1,j}$, $1 \le j \le d$, and we define the norm on it by means of the scalar product $\langle \xi, \eta\rangle_1 = E\left(\sum_{j=1}^d \xi_j\right)\overline{\left(\sum_{j=1}^d \eta_j\right)}$ for

$\xi = (\xi_1, \dots, \xi_d) \in \mathcal{H}_1^c$ and $\eta = (\eta_1, \dots, \eta_d) \in \mathcal{H}_1^c$. We identify two elements $\xi \in \mathcal{H}_1^c$ and $\eta \in \mathcal{H}_1^c$ if $\|\xi - \eta\|_1 = 0$. Then the argument of Section 3 yields that $\mathcal{H}_1^c$ is a Hilbert space with the scalar product $\langle \cdot, \cdot \rangle_1$.

We define the operator T from $\mathcal{K}_1^c$ to $\mathcal{H}_1^c$ again in the same way as in Section 3. We define it by the formula

$$Tu = T(u_1, \dots, u_d) = (T_1 u_1, \dots, T_d u_d)$$

for $u = (u_1, \dots, u_d)$, $u_j \in \mathcal{K}_{1,j}^c$, with the help of the already defined operators T_j, $1 \le j \le d$. We want to show that it is a norm preserving and invertible transformation from $\mathcal{K}_1^c$ to $\mathcal{H}_1^c$. Here again we apply a similar, but sightly different argument from that in Section 3. We exploit that if we take the class of vectors

$$W = \{w = (u_1 + iv_1, \dots, u_d + iv_d) \colon\ u_j \in \mathcal{S},\ v_j \in \mathcal{S} \text{ for all } 1 \le j \le d\}$$

then the class of vectors

$$\tilde{W} = \{(\widetilde{u_1 + iv_1}, \dots, \widetilde{u_d + iv_d}) \colon\ (u_1 + iv_1, \dots, u_d + iv_d) \in W\}$$

is an everywhere dense subspace of $\mathcal{K}_1^c$. and the class of vectors

$$W(X) = \{((X_1(u_1 + iv_1), \dots, X_d(u_d + iv_d)) \colon\ (u_1 + iv_1, \dots, u_d + iv_d) \in W\}$$

is an everywhere dense subspace of $\mathcal{H}_1^c$. (Here again the sign $\tilde{}$ denotes Fourier transform.)

Take two vectors $(u_{1,1}+iv_{1,1}, \dots, u_{d,1}+iv_{d,1}) \in W$ and $(u_{1,2}+iv_{1,2}, \dots, u_{d,2}+iv_{d,2}) \in W$. The desired property of the operator T will follow from the following calculation.

$$\begin{aligned}
&\langle (u_{1,1} \widetilde{+ iv}_{1,1}, \dots, u_{d,1} \widetilde{+ v}_{d,1}), (u_{1,2} \widetilde{+ iv}_{1,2}, \dots, u_{d,2} \widetilde{+ v}_{d,2}) \rangle_0 \\
&\quad = \sum_{j=1}^{d} \sum_{j'=1}^{d} \int (u_{j,1}(\widetilde{x) + iv}_{j,1}(x)) \overline{(u_{j',2}(\widetilde{x) + iv}_{j',2}(x))} G_{j,j'}(\,dx) \\
&\quad = \sum_{j=1}^{d} \sum_{j'=1}^{d} E[X_j(u_{j,1}) + iX_j(v_{j,1})][X_{j'}(u_{j',2}) - iX_j(u_{j',2})] \\
&\quad = \langle (X_1(u_{1,1}) + iX_1(v_{1,1}), \dots, X_d(u_{d,1}) + iX_d(v_{d,1})), \\
&\qquad\qquad (X_1(u_{1,2}) + iX_1(v_{1,2}), \dots, X_d(u_{d,2}) + iX_d(v_{d,2})) \rangle_1,
\end{aligned}$$

i.e.

$$\begin{aligned}
&\langle (u_{1,1} \widetilde{+ iv}_{1,1}, \dots, u_{d,1} \widetilde{+ v}_{d,1}), (u_{1,2} \widetilde{+ iv}_{1,2}, \dots, u_{d,2} \widetilde{+ v}_{d,2}) \rangle_0 \\
&\quad = \langle (T_1(u_{1,1} + iv_{1,1}), \dots, T_d(u_{d,1} + iv_{d,1})), \\
&\qquad\qquad (T_1(u_{1,2} + iv_{1,2})), \dots, T_d(u_{d,2} + iv_{d,2})) \rangle_1.
\end{aligned}$$

This means that the operator T maps the everywhere dense subspace $\tilde{W}$ of $\mathcal{K}_1^c$ to the everywhere dense subspace $W(X)$ of $\mathcal{H}_1^c$ in a norm preserving form. This implies that T is a norm preserving, invertible transformation from $\mathcal{K}_1^c$ to $\mathcal{H}_1^c$.

Now we turn to the definition of the vector valued random spectral measures corresponding to a matrix valued spectral measure on $\mathbb{R}^\nu$.

Let a vector valued, Gaussian stationary generalized random field

$$X(\varphi) = (X_1(\varphi), \dots, X_d(\varphi)), \quad \varphi \in \mathcal{S},$$

be given with a matrix valued spectral measure $(G_{j,j'})$, $1 \le j, j' \le d$, on $\mathbb{R}^\nu$. (We take such generalized, stationary random fields which were constructed in Theorem 4.1.) Let us consider the operators T_j, $1 \le j \le d$, and T constructed above with the help of these quantities. We define, similarly to the case of Gaussian stationary random fields with discrete parameters discussed in Section 3 the random variables $Z_{G,j}(A) = T_j(\mathbb{I}_A(x))$ for all $1 \le j \le d$ and bounded, measurable sets $A \subset \mathbb{R}^\nu$. (These functions $\mathbb{I}_A(\cdot)$ are clearly elements of the Hilbert space $\mathcal{K}_{1,j}^c$ for all $\le j \le d$). It can be proved with the help of the properties of the operator T that these random functions satisfy properties (i)–(v) formulated in the definition of random spectral measures on the torus, considered in Section 3. The argument applied in Section 3 holds also in in this case. In particular, property (v) can be proved with the help of property (v′). Property (v′) can be proved with some work, and actually this was done in [9]. We prove (v′) by checking it first for functions $u \in \mathcal{S}^c$.

The above result makes natural the following definition of vector valued random spectral measures corresponding to a matrix valued spectral measure on $\mathbb{R}^\nu$. This is very similar to the definition of vector valued random spectral measures on the torus.

Definition of vector valued random spectral measures on $\mathbb{R}^\nu$. *Let $G = (G_{j,j'})$, $1 \le j, j' \le d$, be a matrix valued spectral measure on $\mathbb{R}^\nu$. We call a set of complex valued random variables $Z_{G,j}(A)$ depending on pairs (j, A), where $1 \le j \le d$, $A \in \mathcal{A}$, and $\mathcal{A}$ is the algebra*

$$\mathcal{A} = \{A\colon\ A \text{ is a bounded Borel measurable set in } \mathbb{R}^\nu\},$$

a d-dimensional vector valued random spectral measure corresponding to the matrix valued spectral measure G on $\mathbb{R}^\nu$ if this set of random variables $Z_{G,j}(A)$, $1 \le j \le d$, $A \in \mathcal{A}$, satisfies properties (i)–(v) introduced in Section 3 in the definition of vector valued random spectral measures on the torus. Given a fixed index $1 \le j \le d$, we call the set of random variables $Z_{G,j}(A)$, $A \in \mathcal{A}$, with this index j the j-th coordinate of this matrix valued spectral measure, and we denote it by $Z_{G,j}$. We denote a d-dimensional vector valued random spectral measure corresponding to the matrix valued spectral measure G by $Z_G = (Z_{G,1}, \dots, Z_{G,d})$.

We can show with the help of the arguments applied in Section 3 that for any d-dimensional matrix valued spectral measure on $\mathbb{R}^\nu$ there exists a d-dimensional vector valued random spectral measure corresponding to it.

We can define the random integral $\int f(x) Z_{G,j}(\,dx)$ of the functions $f \in \mathcal{K}_{1,j}^c$ with respect to the random spectral measure $Z_{G,j}$, $1 \le j \le d$, corresponding to

the matrix valued spectral measure $(G_{j,j'})$, $1 \le j, j' \le d$, of a Gaussian stationary generalized field in the same way as we defined these random integrals with respect to random spectral measures corresponding to a spectral measures on the torus $[-\pi, \pi)^\nu$ in Section 3. First we define these integrals for elementary functions which are defined in the same way as it was done in Section 3. Then following the calculation of that section we can define these integrals for a general function $f \in \mathcal{K}_{1,j}^c$, and it can be seen that formulas (3.7), (3.8) and (3.9) remain valid for them. In particular, the random integrals $\int \tilde{\varphi}(x) Z_{G,j}(\,dx)$ are (meaningful and) real valued random variables for all $\varphi \in \mathcal{S}$, and

$$E\left(\int \tilde{\varphi}(x) Z_{G,j}(\,dx) \int \bar{\tilde{\psi}}(x) Z_{G,j'}(\,dx)\right) = \int \tilde{\varphi}(x) \bar{\tilde{\psi}}(x) G_{j,j'}(\,dx)$$

for all $\varphi, \psi \in \mathcal{S}$ and $1 \le j, j' \le d$. This identity together with relation (3.7) and the fact that the above considered random integrals are linear operators imply that the set of random variables

$$X_j(\varphi) = \int \tilde{\varphi}(x) Z_{G,j}(\,dx), \quad \varphi \in \mathcal{S}, \;\; 1 \le j \le d, \tag{4.4}$$

constitute a vector valued Gaussian, stationary generalized random field with spectral measure $(G_{j,j'})$, $1 \le j, j' \le d$.

This implies that the natural version of Theorem 3.1 remains valid if we consider a matrix valued spectral measure $(G_{j,j'})$, $1 \le j, j' \le d$, on $\mathbb{R}^\nu$. Then there exists a random spectral measure $Z_G = (Z_{G,1}, \dots, Z_{G,d})$ corresponding to it, and we have defined the random integrals $\int u(x) Z_{G,j}(\,dx)$, $1 \le j \le d$, with respect to it for all $u \in \mathcal{K}_{1,j}^c$. The class of random variables, $X_j(\varphi)$, $\varphi \in \mathcal{S}$, $1 \le j \le d$, defined in (4.4) constitute a vector valued, Gaussian stationary generalized random field with matrix valued spectral measure $(G_{j,j'})$, $1 \le j, j' \le d$. Moreover, if a d-dimensional vector valued Gaussian stationary random field is given with spectral measure $(G_{j,j'})$, $1 \le j, j' \le d$, then we can consider the random spectral measure $(Z_{G,1}, \dots, Z_{G,d})$ constructed in this section with the help of this random field. This random spectral measure has the property that the random field given by the random integrals defined in formula (4.4) with their help agrees with the original vector valued Gaussian stationary generalized random field.

We can formulate a natural version of Lemma 3.2 where we consider a matrix valued spectral measure $(G_{j,j'})$, $1 \le j, j' \le d$, on $\mathbb{R}^\nu$ instead of a matrix valued spectral measure on the torus $[-\pi, \pi)^\nu$. In this version of Lemma 3.2 we define $\mathcal{K}_{1,j}$ as

$$\mathcal{K}_{1,j} = \left\{ u \colon \int |u(x)|^2 G_{j,j}(\,dx) < \infty, \quad u(-x) = \overline{u(x)} \text{ for all } x \in \mathbb{R}^\nu \right\},$$

with the scalar product $\langle u, v \rangle = \int u(x) \overline{v(x)} G_{j,j}(\,dx)$, $u, v \in \mathcal{K}_{1,j}$, and $\mathcal{H}_{1,j}$ as the closure of the linear space consisting of the finite linear combination of the random variables $X_j(\varphi)$, $\varphi \in \mathcal{S}$, with real coefficients in the Hilbert space $\mathcal{H}$.

This version of Lemma 3.2 states that $\mathcal{K}_{1,j}$ and $\mathcal{H}_{1,j}$ are real Hilbert spaces, and $T_j(u) = \int u(x) Z_{G,j}(\,dx)$ is a norm preserving and invertible transformation from $\mathcal{K}_{1,j}$ to $\mathcal{H}_{1,j}$.

The proof of this version of Lemma 3.2 is very similar to the proof of the original lemma. The main difference is that now we show that the class of functions $\tilde{\varphi}$ with $\varphi \in \mathcal{S}$ is a dense linear subspace of $\mathcal{K}_{1,j}$, and the transformation $T_j(\tilde{\varphi}) = \int \tilde{\varphi}(x) Z_{G,j}(\,dx) = X_j(\varphi)$, $\varphi \in \mathcal{S}$, is a norm preserving transformation from an everywhere dense subspace of $\mathcal{K}_{1,j}$ to an everywhere dense subspace of $\mathcal{H}_{1,j}$.

The natural version of Lemma 3.3 also holds. It states that a matrix valued spectral measure $(G_{j,j'})$, $1 \le j, j' \le d$, on $\mathbb{R}^\nu$ determines the distribution of a vector valued random spectral measure $Z_{G,j}$, $1 \le j \le d$, corresponding to it. The proof of this version is the same as the proof of the original lemma. The only difference is that now we consider the random spectral measure $Z_{G,j}(A)$ for all measurable, bounded sets $A \subset \mathbb{R}^\nu$.

Finally I would remark that property (vi) of the random spectral measures also remains valid for this new class of random spectral measures, because its proof applies only properties (i)–(v) of random spectral measures.

5 Multiple Wiener–Itô integrals with respect to vector valued random spectral measures

Next we want to rewrite the random variables with finite second moments which are measurable with respect the σ-algebra generated by the elements of a vector valued Gaussian stationary random field in an appropriate form which enables us to rewrite also the random sums defined in (1.1) in a form that helps in the study of their limit behaviour. In the scalar valued case, i.e. when $d = 1$ we could do this with the help of multiple Wiener–Itô integrals. We could rewrite the random sums (1.1) with their help in such a form that provided great help in the study of the limit theorems we were interested in. Next we show that a similar method can be applied also in the case of vector valued Gaussian stationary random fields. To do this first we have to define the multiple Wiener–Itô integrals also in the vector valued case. We start the definition of multiple Wiener–itô integrals in this case with the introduction of the following notation.

Let $X(p) = (X_1(p), \dots, X_d(p))$, $EX(p) = 0$, $p \in \mathbb{Z}^\nu$, be a vector valued stationary Gaussian random field with some matrix valued spectral measure $G = (G_{j,j'})$, $1 \le j, j' \le d$. Let $Z_G = (Z_{G,1}, \dots, Z_{G,d})$ be a vector valued random spectral measure corresponding to it which is chosen in such a way that $X_j(p) = \int e^{i(p,x)} Z_{G,j}(\,dx)$ for all $p \in \mathbb{Z}^\nu$ and $1 \le j \le d$. Let us consider the (real) Hilbert space $\mathcal{H}$ of square integrable random variables measurable with respect to the σ-algebra generated by the random vectors $X(p)$, $p \in \mathbb{Z}^\nu$. More generally, let us consider a (possibly generalized) matrix valued spectral measure $G = (G_{j,j'})$, $1 \le j, j' \le d$, and a vector valued random spectral measure $Z_G = (Z_{G,1}, \dots, Z_{G,d})$ corresponding to it, where the matrix valued spectral

measures $G_{j,j'}$ and vector valued random spectral measures $Z_{G,j}$ are defined either on the torus $[-\pi,\pi)^\nu$ or on $\mathbb{R}^\nu$, and consider the (real) Hilbert space $\mathcal{H}$ of the square integrable (real valued) random variables, measurable with respect to the σ-algebra generated by the random variables of the vector valued random spectral measures Z_G with the usual scalar product in this space. We would like to write the elements of the Hilbert space $\mathcal{H}$ in the form of a sum of multiple Wiener–Itô integrals with respect to the vector valued random spectral measure Z_G. I shall construct these Wiener–Itô integrals in this section, and I prove some of their important properties.

As a discussion in Section 2 of [11] will show we cannot write all elements of $\mathcal{H}$ in the form of a sum of Wiener–Itô integrals, but we can do this for the elements of an everywhere dense subspace of $\mathcal{H}$. In particular, if we consider finitely many random variables $X_j(p)$, $1 \le j \le d$, $p \in \mathbb{Z}^\nu$ of a discrete or $X_j(\varphi)$, $1 \le j \le d$, $\varphi \in \mathcal{S}^\nu$, of a generalized vector valued stationary Gaussian random field, then all polynomials of these random variables can be written as the sum of Wiener–Itô integrals. Such a result will be sufficient for our purposes. In the subsequent discussion I impose a technical condition about the properties of the matrix valued spectral measure $G = (G_{j,j'})$ I shall be working with. I assume that it is non-atomic. More precisely, I assume that we are working with such a dominating measure μ for the coordinates of the matrix valued spectral measures $G_{j,j'}$ for which $\mu(\{x\}) = 0$ for all $x \in \mathbb{R}^\nu$.

First I define for all $n = 1, 2, \dots$ and $1 \le j_s \le d$ for the indices $1 \le s \le n$ the n-fold multiple Wiener–Itô integral

$$I_n(f|j_1,\dots,j_n) = \int f(x_1,\dots,x_n) Z_{G,j_1}(\,dx_1)\dots Z_{G,j_n}(d\,x_n)$$

with respect to the coordinates of a vector valued random spectral measure $Z_G = (Z_{G,1},\dots,Z_{G,d})$, corresponding to a matrix valued spectral measure $G = (G_{j,j'})$, $1 \le j, j' \le d$. I shall define these Wiener–Itô integrals with kernel functions $f \in \mathcal{K}_{n,j_1,\dots,j_n}$ in a (real) Hilbert space $\mathcal{K}_{n,j_1,\dots,j_n} = \mathcal{K}_{n,j_1,\dots,j_n}(G_{j_1,j_1},\dots,G_{j_n,j_n})$ defined below.

We define $\mathcal{K}_{n,j_1,\dots,j_n} = \mathcal{K}_{n,j_1,\dots,j_n}(G_{j_1,j_1}\dots,G_{j_n,j_n})$ as the Hilbert space consisting of those complex valued functions $f(x_1,\dots,x_n)$ on $\mathbb{R}^{n\nu}$ which satisfy the following relations (a) and (b):

(a) $f(-x_1,\dots,-x_n) = \overline{f(x_1,\dots,x_n)}$ for all $(x_1,\dots,x_n) \in \mathbb{R}^{n\nu}$,

(b) $\|f\|^2 = \int |f(x_1,\dots,x_n)|^2 G_{j_1,j_1}(\,dx_1)\dots G_{j_n,j_n}(\,dx_n) < \infty$.

We define the scalar product in $\mathcal{K}_{n,j_1,\dots,j_n}$ in the following way. If $f, g \in \mathcal{K}_{n,j_1,\dots,j_n}$, then

$$\begin{aligned}\langle f, g\rangle &= \int f(x_1,\dots,x_n)\overline{g(x_1,\dots,x_n)} G_{j_1,j_1}(\,dx_1)\dots G_{j_n,j_n}(\,dx_n)\\ &= \int f(x_1,\dots,x_n) g(-x_1,\dots,-x_n) G_{j_1,j_1}(\,dx_1)\dots G_{j_n,j_n}(\,dx_n).\end{aligned}$$

Because of the symmetry $G_{j_s,j_s}(A) = G_{j_s,j_s}(-A)$ of the spectral measure $\langle f, g\rangle = \overline{\langle f, g\rangle}$, i.e. the scalar product $\langle f, g\rangle$ is a real number for all $f, g \in \mathcal{K}_{n,j_1,\dots,j_n}$. This means that $\mathcal{K}_{n,j_1,\dots,j_n}$ is a real Hilbert space, as I claimed. We also define the real Hilbert space $\mathcal{K}_0$ for $n = 0$ as the space of real constants with the norm $\|c\| = |c|$.

Remark. In the case $n = 1$ the above defined real Hilbert space $\mathcal{K}_{1,j}$ agrees with the real Hilbert space $\mathcal{K}_{1,j}$ introduced in Lemma 3.2.

Similarly to the scalar valued case, first we introduce so-called simple functions and define the multiple integrals for them. We prove some properties of this integral which enable us to extend its definition by means of an L_2 extension for all functions $f \in \mathcal{K}_{j_1,\dots,j_n}$. We define the class of simple functions together with the notion of regular systems.

Definition of regular systems and of the class of simple functions. *Let*

$$\mathcal{D} = \{\Delta_k,\ k = \pm1, \pm2, \dots, \pm N\}$$

be a finite collection of bounded, measurable sets in $\mathbb{R}^\nu$ indexed by the integers $\pm1,\dots, \pm N$ with some positive integer N. We say that $\mathcal{D}$ is a regular system if $\Delta_k = -\Delta_{-k}$, and $\Delta_k \cap \Delta_l = \emptyset$ if $k \neq l$ for all $k, l = \pm1, \pm2, \dots, \pm N$. A function $f \in \mathcal{K}_{n,j_1,\dots,j_n}$ is adapted to this system $\mathcal{D}$ if $f(x_1, \dots, x_n)$ is constant on the sets $\Delta_{k_1} \times \Delta_{k_2} \times \cdots \times \Delta_{k_n}$, $k_l = \pm1, \dots, \pm N$, $l = 1, 2, \dots, n$, it vanishes outside these sets, and it also vanishes on those sets of the above form for which $k_l = \pm k_{l'}$ for some $l \neq l'$.

A function $f \in \mathcal{K}_{n,j_1,\dots,j_n}$ is in the class $\hat{\mathcal{K}}_{n,j_1,\dots,j_n}$ of simple functions if it is adapted to some regular system $\mathcal{D} = \{\Delta_k,\ k = \pm1, \dots, \pm N\}$.

Definition of Wiener–Itô integrals of simple functions. *Let a simple function $f \in \hat{\mathcal{K}}_{n,j_1,\dots,j_n}$ be adapted to some regular system*

$$\mathcal{D} = \{\Delta_k,\ k = \pm1, \dots, \pm N\}.$$

Its n-fold Wiener–Itô integral with respect to $Z_G = (Z_{G,1}, \dots, Z_{G,d})$ with parameters $j_1, \dots, j_n$, $1 \le j_k \le d$ for all $1 \le k \le n$, is defined as

$$\begin{aligned}\int f(x_1, \dots, x_n) Z_{G,j_1}(dx_1) \dots Z_{G,j_n}(dx_n) &= I_n(f|j_1, \dots, j_n) \\ &= \sum_{\substack{k_l = \pm1,\dots,\pm N \\ l=1,2,\dots,n}} f(u_{k_1}, \dots, u_{k_n}) Z_{G,j_1}(\Delta_{k_1}) \cdots Z_{G,j_n}(\Delta_{k_n}),\end{aligned} \tag{5.1}$$

where $u_k \in \Delta_k$, $k = \pm1, \dots, \pm N$.

Although the regular system $\mathcal{D}$ to which f is adapted is not uniquely determined (e.g. the elements of $\mathcal{D}$ can be divided to smaller sets), the integral defined in (5.1) is meaningful, i.e. its value does not depend on the choice of $\mathcal{D}$. This can be proved with the help of property (iv) of vector valued random spectral

measures defined in Section 3 in the same way as it was done in the scalar valued case in [9]. (Let me also remark that here I defined the random integral $I_n(f|j_1,\dots,j_n)$ with a normalization different from the normalization of the corresponding expression $I_G(f)$ introduced in [9]. Here I omitted the norming term $\frac{1}{n!}$.)

Because of the definition of simple functions the sum in (5.1) does not change if we allow in it summation only for such sequences $k_1,\dots,k_n$ for which $k_l \neq \pm k_{l'}$ if $l \neq l'$. This fact will be exploited in the subsequent considerations.

Next I formulate some important properties about the Wiener–Itô integrals of simple functions. Later we shall see that these properties remain valid in the general case.

$$I_n(f|j_1,\dots,j_n) \text{ is a real valued random variable for all } f \in \hat{\mathcal{K}}_{n,j_1,\dots,j_n}. \quad (5.2)$$

Indeed, $I_n(f|j_1,\dots,j_n) = \overline{I_n(f|j_1,\dots,j_n))}$ by Property (a) of the functions in $\mathcal{K}_{n,j_1,\dots,j_n}$ and property (v) of the random spectral measures defined in Section 3, hence (5.2) holds. It is also clear that $\hat{\mathcal{K}}_{n,j_1,\dots,j_n}$ is a linear space, and the mapping $f \to I_n(f|j_1,\dots,j_n)$ is a linear transformation on it.

The relation

$$EI_n(f|j_1,\dots,j_n) = 0 \quad \text{for } f \in \hat{\mathcal{K}}_{n,j_1,\dots,j_k} \quad \text{if } n \neq 0 \quad (5.3)$$

also holds. (In the non-zero terms of the sum in (5.1) we have the product of independent random variables with expectation zero by property (vi) of the random spectral measures described also in Section 3.) Next I express the covariance between random variables of the form $I_n(f|j_1,\dots,j_n)$. To do this first I introduce the following notation. Let $\Pi(n)$ denote the set of all permutations of the set $\{1,\dots,n\}$, and let $\pi = (\pi(1),\dots,\pi(n))$ denote one of its element.

Let us have a positive integer $n \geq 1$, and two sequences $j_1,\dots,j_n$ and $j'_1,\dots,j'_n$, $1 \leq j_s, j'_s \leq d$ for all $1 \leq s \leq d$. Let $f \in \hat{\mathcal{K}}_{n,j_1,\dots,j_n}$ and $h \in \hat{\mathcal{K}}_{n,j'_1,\dots,j'_n}$. I shall show that

$$\begin{aligned} EI_n(f|j_1,\dots,j_n)I_n(h(|j'_1,\dots,j'_n) \quad (5.4)\\ = \sum_{\pi\in\Pi(n)} \int f(x_1,\dots x_n)\overline{h(x_{\pi(1)},\dots,x_{\pi(n)})} \\ G_{j_1,j'_{\pi^{-1}(1)}}(\,dx_1)\dots G_{j_n,j'_{\pi^{-1}(n)}}(\,dx_n). \end{aligned}$$

On the other hand, if $n \neq n'$, and $f \in \hat{\mathcal{K}}_{n,j_1,\dots,j_n}$, $h \in \hat{\mathcal{K}}_{n',j'_1,\dots,j'_{n'}}$, then

$$EI_n(f|j_1,\dots,j_n)I_{n'}(h(|j'_1,\dots,j'_{n'}) = 0. \quad (5.5)$$

Next I show the following inequality with the help of formula (5.4).

$$\begin{aligned} E|I_n(f|j_1,\dots,j_n)|^2 &\leq n! \int |f(x_1,\dots x_n)|^2 G_{j_1,j_1}(\,dx_1)\dots G_{j_n,j_n}(\,dx_n) \\ &= n!\|f_{n,j_1,\dots,j_n}\|^2 \quad (5.6) \end{aligned}$$

for all $f \in \hat{\mathcal{K}}_{n,j_1,\dots,j_n}$.

Indeed we get by applying (5.4) for $f = h \in \hat{\mathcal{K}}_{n,j_1,\dots,j_n}$ together with relation (3.2) that

$$E|I_n(f|j_1,\dots,j_n)|^2 \le \sum_{\pi\in\Pi(n)} \int |f(x_1,\dots x_n)||f(x_{\pi(1)},\dots,x_{\pi(n)})| \quad (5.7)$$
$$\prod_{s=1}^{n} \left(g_{j_s,j_s}(x_s) g_{j_{\pi^{-1}(s)},j_{\pi^{-1}(s)}}(x_s)\right)^{1/2} \mu(\,dx_1)\dots\mu(\,dx_n).$$

On the other hand, we get with the help of the Schwarz inequality that

$$\int |f(x_1,\dots x_n)||f(x_{\pi(1)},\dots,x_{\pi(n)})| \prod_{s=1}^{n} \left(g_{j_s,j_s}(x_s) g_{j_{\pi^{-1}(s)},j_{\pi^{-1}(s)}}(x_s)\right)^{1/2}$$
$$\mu(\,dx_1)\dots\mu(\,dx_n) \quad (5.8)$$
$$\le \left(\int |f(x_1,\dots x_n)|^2 \prod_{s=1}^{n} g_{j_s,j_s}(x_s)\mu(\,dx_1)\dots\mu(\,dx_n)\right)^{1/2}$$
$$\left(\int |f(x_{\pi(1)},\dots,x_{\pi(n)})|^2 \prod_{s=1}^{n} g_{j_{\pi^{-1}(s)},j_{\pi^{-1}(s)}}(x_s)\mu(\,dx_1)\dots\mu(\,dx_n)\right)^{1/2}$$

for all $\pi \in \Pi(n)$. Let us also observe that the map T from $\mathbb{R}^{n\nu}$ to $\mathbb{R}^{n\nu}$, defined as

$$T(x_1,\dots,x_n) = (x_{\pi(1)},\dots,x_{\pi(n)})$$

is a bijection, and it is a measure preserving transformation from

$$(\mathbb{R}^{n\nu}, G_{j_1,j_1} \times \cdots \times G_{j_n,j_n}) = (\mathbb{R}^{n\nu}, g_{j_1,j_1}(x_1)\cdots g_{j_n,j_n}(x_n)\mu(\,dx_1)\dots\mu(\,dx_n)\,)$$

to

$$(\mathbb{R}^{n\nu}, G_{j_{\pi^{-1}(1)},j_{\pi^{-1}(1)}} \times \cdots \times G_{j_{\pi^{-1}(n)},j_{\pi^{-1}(n)}})$$
$$= (\mathbb{R}^{n\nu}, g_{j_{\pi^{-1}(1)},j_{\pi^{-1}(1)}}(x_1)\cdots g_{j_{\pi^{-1}(n)},j_{\pi^{-1}(n)}}(x_n)\mu(\,dx_1)\dots\mu(\,dx_n)\,).$$

To see this it is enough to check that if $A = A_1 \times \cdots \times A_n$, then

$$(G_{1,1} \times \cdots \times G_{n,n})(A) = \prod_{l=1}^{n} G_{l,l}(A_l),$$

$TA = A_{\pi^{-1}(1)} \times \cdots \times A_{\pi^{-1}(n)}$,

$$(G_{j_{\pi^{-1}(1)},j_{\pi^{-1}(1)}} \times \cdots \times G_{j_{\pi^{-1}(n)},j_{\pi^{-1}(n)}})(TA)$$
$$= \prod_{l=1}^{n} G_{j_{\pi^{-1}(l)},j_{\pi^{-1}(l)}}(A_{\pi^{-1}(l)}) = (G_{1,1} \times \cdots \times G_{n,n})(A).$$

The last identity together with the bijective property of T imply that it is measure preserving.

Because of the measure preserving property of the operator T we can write that

$$\int |f(x_1,\dots x_n)|^2 \prod_{s=1}^n g_{j_s,j_s}(x_s)\mu(\,dx_1)\dots\mu(\,dx_n) \tag{5.9}$$
$$= \int |f(x_{\pi(1)},\dots,x_{\pi(n)})|^2 \prod_{s=1}^n g_{j_{\pi^{-1}(s)},j_{\pi^{-1}(s)}}(x_s)\mu(\,dx_1)\dots\mu(\,dx_n).$$

Relation (5.6) follows from relations (5.7), (5.8) and (5.9).

To prove formulas (5.4) and (5.5) first we prove the following relations. Let a regular system $\mathcal{D} = \{\Delta_k,\ k = \pm1, \pm2, \dots, \pm N\}$ be given, choose an integer $n \geq 1$, some numbers $j_1, \dots, j_n$ and $j'_1 \dots, j'_n$ such that $1 \leq j_s, j'_s \leq d$, $1 \leq s \leq d$, together with two sequences of numbers $k_1, \dots, k_n$ and $l_1, \dots, l_n$ such that $k_s, l_s \in \{\pm1, \dots, \pm N\}$ for all $1 \leq s \leq n$, and they also satisfy the relation $k_s \neq \pm k_{s'}$, and $l_s \neq \pm l_{s'}$ if $s \neq s'$. I claim that under these conditions

$$EZ_{G,j_1}(\Delta_{k_1}) \cdots Z_{G,j_n}(\Delta_{k_n})\overline{Z_{G,j'_1}(\Delta_{l_1}) \cdots Z_{G,j'_n}(\Delta_{l_n})} = 0 \tag{5.10}$$

if $\{k_1, \dots, k_n\} \neq \{l_1, \dots, l_n\}$. On the other hand, if

$$l_p = k_{\pi(p)} \ \text{ for all } 1 \leq p \leq n \tag{5.11}$$

with some permutation $\pi \in \Pi(n)$, then

$$EZ_{G,j_1}(\Delta_{k_1}) \cdots Z_{G,j_n}(\Delta_{k_n})\overline{Z_{G,j'_1}(\Delta_{l_1}) \cdots Z_{G,j'_n}(\Delta_{l_n})}$$
$$= G_{j_1,j'_{\pi^{-1}(1)}}(\Delta_{k_1}) \cdots G_{j_n,j'_{\pi^{-1}(n)}}(\Delta_{k_n}). \tag{5.12}$$

Let me remark that there cannot be two different permutations $\pi \in \Pi(n)$ satisfying relation (5.11), since by our assumption also elements of the set $\{k_1, \dots, k_n\}$ are different, and the same relation holds for the set $\{1_1, \dots, l_n\}$.

To prove (5.10) we show that under its conditions the product

$$Z_{G,j_1}(\Delta_{k_1}) \cdots Z_{G,j_n}(\Delta_{k_n})\overline{Z_{G,j'_1}(\Delta_{l_1}) \cdots Z_{G,j'_n}(\Delta_{l_n})}$$

can be written in the form of a product of two independent terms in such a way that one of them has expectation zero.

Indeed, since $\{k_1, \dots, k_n\} \neq \{l_1, \dots, l_n\}$, there is such an element k_s for which $k_s \neq l_t$ for all $1 \leq t \leq n$, and also the relation $k_s \neq \pm k_t$ if $s \neq t$, holds. If the relation $k_s \neq \pm l_t$ also holds for all $1 \leq t \leq n$, then $Z_{G,j_s}(\Delta_{k_s})$ is independent of the product of the product of the remaining terms in this product because of property (vi) of vector valued random spectral measures given in Section 3, and $EZ_{G,j_s}(\Delta_{k_s}) = 0$. Hence relation (5.10) holds in this case.

In the other case, there is an index s' such that $l_{s'} = -k_s$. In this case the vector

$$\begin{aligned}(Z_{G,j_s}(\Delta_{k_s}), \overline{Z_{G,j_{s'}}(\Delta_{l_{s'}})}) &= (Z_{G,j_s}(\Delta_{k_s}), Z_{G,j_{s'}}(-\Delta_{l_{s'}})) \\ &= (Z_{G,j_s}(\Delta_{k_s}), Z_{G,j_{s'}}(\Delta_{k_s}))\end{aligned}$$

is independent of the remaining terms, (because of property (vi) of the vector valued random spectral measures). In last the relation we exploited that $-\Delta_{l_{s'}} = \Delta_{k_s}$). Hence

$$EZ_{G,j_s}(\Delta_{k_s})\overline{Z_{G,j_{s'}}(\Delta_{l_{s'}})} = EZ_{G,j_s}(\Delta_{k_s})\overline{Z_{G,j_{s'}}(-\Delta_{k_s})} = 0,$$

and relation (5.10) holds in this case, too.

To prove (5.12) let us observe that under its condition the investigated product can be written in the form

$$Z_{G,j_1}(\Delta_{k_1}) \cdots Z_{G,j_n}(\Delta_{k_n})\overline{Z_{G,j'_1}(\Delta_{l_1}) \cdots Z_{G,j'_n}(\Delta_{l_n})}$$
$$= \prod_{p=1}^{n} Z_{G,j_p}(\Delta_{k_p})\overline{Z_{G,j'_{\pi^{-1}(p)}}(\Delta_{k_p})}.$$

The terms in the product at the right-hand side are independent for different indices s, and $EZ_{G,j_p}(\Delta_{k_p})\overline{Z_{G,j'_{\pi^{-1}(p)}}(\Delta_{k_p})} = G_{j_p,j'_{\pi^{-1}(p}}(\Delta_{k_p})$. Formula (5.12) follows from these relations and the independence between the terms in the last product. (Here we use again property (vi) of the random spectral measures.)

To prove formula (5.4) let us take a regular system

$$\mathcal{D} = \{\Delta_k,\ k = \pm 1, \dots, \pm N\}$$

such that both functions f and h are adapted to it. This can be done by means of a possible refinement of the original regular systems corresponding to the functions f and h. Then we can write, by exploiting (5.2) and (5.10) that

$$EI_n(f|j_1, \dots, j_n)I_n(h(|j'_1, \dots, j'_n) = EI_n(f|j_1, \dots, j_n)\overline{I_n(h(|j'_1, \dots, j'_n)}$$
$$= \sum_{\pi \in \Pi(n)} \sum_{\substack{(k_1,\dots k_n),\ (l_1,\dots l_n) \\ k_p = \pm 1,\dots,\pm N,\ p=1,\dots,n \\ l_p = k_{\pi(p)}\ p=1,\dots,n}} f(u_{k_1}, \dots u_{k_n})\overline{h(u_{k_{\pi(1)}}, \dots, u_{k_{\pi(n)}})}$$
$$EZ_{G,j_1}(\Delta_{k_1}) \cdots Z_{G,j_n}(\Delta_{k_n})\overline{Z_{G,j'_1}(\Delta_{l_1}) \cdots Z_{G,j'_n}(\Delta_{l_n})},$$

where $u_k \in \Delta_k$ for all $k = \pm 1, \dots, \pm N$.

The expected value of the product at the right-hand side of this identity can be calculated with the help of (5.12), and this yields that

$$EI_n(f|j_1, \dots, j_n)I_n(h(|j'_1, \dots, j'_n)$$
$$= \sum_{\pi \in \Pi(n)} \sum_{\substack{(k_1,\dots k_n),\ (l_1,\dots l_n) \\ k_p = \pm 1,\dots,\pm N,\ p=1,\dots,n \\ l_p = k_{\pi(p)},\ p=1,\dots,n}} f(u_{k_1}, \dots u_{k_n})\overline{h(u_{l_1}, \dots, u_{l_n})}$$
$$G_{j_1,j'_{\pi^{-1}(1)}}(\Delta_{k_1}) \cdots G_{j_n,j'_{\pi^{-1}(n)}}(\Delta_{k_n})$$
$$= \sum_{\pi \in \Pi(n)} \int f(x_1, \dots x_n)\overline{h(x_{\pi(1)}, \dots, x_{\pi(n)})}$$
$$G_{j_1,j'_{\pi^{-1}(1)}}(\,dx_1) \dots G_{j_n,j'_{\pi^{-1}(n)}}(\,dx_n).$$

Formula (5.4) is proved.

The proof of (5.5) is based on a similar idea, but it is considerably simpler. It can be proved similarly to relation (5.10) that for $n \neq n'$

$$EZ_{G,j_1}(\Delta_{k_1}) \cdots Z_{G,j_n}(\Delta_{k_n}) \overline{Z_{G,j'_1}(\Delta_{l_1}) \cdots Z_{G,j'_{n'}}(\Delta_{l_{n'}})} = 0 \tag{5.13}$$

if we define this expression by means a regular system

$$\mathcal{D} = \{\Delta_k,\ k = \pm 1, \pm 2, \dots, \pm N\},$$

some numbers $j_1, \dots, j_n$ and $j'_1 \dots, j'_{n'}$, all of them between 1 and d, together with two sequences of numbers $k_1, \dots, k_n$ and $l_1, \dots, l_{n'}$ such that $k_s, l_s \in \{\pm 1, \dots, \pm N\}$ for all these numbers, and they satisfy the relation $k_s \neq \pm k_{s'}$, and $l_s \neq \pm l_{s'}$ if $s \neq s'$. Then, if we express

$$EI_n(f|j_1, \dots, j_n) I_{n'}(h(|j'_1, \dots, j'_{n'}) = EI_n(f|j_1, \dots, j_n) \overline{I_{n'}(h(|j'_1, \dots, j'_{n'})}$$

similarly as we have done in the proof of (5.12) we get such a sum where all terms equal zero because of (5.13). This implies relation (5.5).

To define the Wiener–Itô integral for all functions $f \in \mathcal{K}_{n,j_1,\dots,j_n}$ we still need the following result.

Lemma 5.1. *The class of simple functions $\hat{\mathcal{K}}_{n,j_1,\dots,j_n}$ is a dense linear subspace of the (real) Hilbert space $\mathcal{K}_{n,j_1,\dots,j_n}$.*

Lemma 5.1 is the multivariate version of Lemma 4.1 in [9]. (A more transparent proof of this result was given in the Appendix of [10].) Actually, we do not have to prove Lemma 5.1, because it simply follows from Lemma 4.1 of [9]. By applying this result for $G = \sum_{j=1}^{n} G_{j,j}$ we get that all bounded functions of $\mathcal{K}_{n,j_1,\dots,j_n}$ are in the closure of $\hat{\mathcal{K}}_{n,j_1,\dots,j_n}$. But this implies that all functions of $\mathcal{K}_{n,j_1,\dots,j_n}$ are in this closure.

Let us take the L_2 norm in the Hilbert space $\mathcal{H}$. Then we have for all $f \in \hat{\mathcal{K}}_{n,j_1,\dots,j_n}$ $I_n(f|j_1, \dots, j_n) \in \mathcal{H}$, and by formula (5.6)

$$\|I_n(f|j_1, \dots, j_n)\| = \left[E(I_n(f|j_1, \dots, j_n)^2)\right]^{1/2} \leq \sqrt{n!} \|f_{n,j_1,\dots,j_n}\|.$$

Hence Lemma 5.1 enables us to extend the Wiener–Itô integral $I_n(f|j_1, \dots, j_n)$ for all $f \in \mathcal{K}_{n,j_1,\dots,j_n}$. Moreover, relations (5.2)—(5.6) remain valid in the Hilbert space $\mathcal{K}_{n,j_1,\dots,j_n}$ after this extension.

Remark. In (5.6) we have given an upper bound for the second moment of a multiple Wiener–Itô integral, but we cannot write equality in this formula. In the scalar-valued case we had an identity in the corresponding relation. At least this was the case if we took the Wiener–Itô integral of a symmetric function. On the other hand, working only with Wiener–Itô integrals of symmetric functions did not mean a serious restriction. This relative weakness of formula (5.6) (the lack of identity) is the reason why we cannot represent such a large class of random variables in the form of a sum of Wiener–Itô integrals as in the scalar valued case. (This problem will be discussed in Section 2 of [11].)

I would mention that there is a slightly stronger version of Lemma 5.1 which is useful in the study in the second part of this paper, in [11], when we are interested in the question under what conditions we can state that a sequence of Wiener–Itô integrals converges to a Wiener–Itô integral. Here is this result.

Lemma 5.2. *For all functions $f \in \mathcal{K}_{n,j_1,\dots,j_n}$ and numbers $\varepsilon > 0$ there is such a simple function $g \in \hat{\mathcal{K}}_{n,j_1,\dots,j_n}$ for which $\|f-g\| \le \varepsilon$ in the norm of the Hilbert space $\mathcal{K}_{n,j_1,\dots,j_n}$, and there is a regular system $\mathcal{D} = \{\Delta_k,\ k = \pm1, \pm2, \dots, \pm N\}$ to which the function g is adapted, and the boundary of all sets $\Delta_k \in \mathcal{D}$ has zero μ-probability with the measure μ we chose as the dominating measure for the complex measures $G_{j,j'}$ in our considerations.*

Lemma 5.2 also follows from the results of [9] or [10].

Finally, I make the following remark. If we define a new function by reindexing the variables of a function of $h \in \mathcal{K}_{n,j_1,\dots,j_n}$ by means of a permutation of the indices, and we change the indices of the spectral measure Z_{G,j_s} in the Wiener-Itô integral $I_n(h|j_1,\dots,j_n)$ in an appropriate way, then we get a new Wiener–Itô integral whose value agrees with the original integral $I_n(h|j_1,\dots,j_n)$. More explicitly, the following result holds.

Lemma 5.3. *Given a function $h \in \mathcal{K}_{n,j_1,\dots,j_n}$ and a permutation $\pi \in \Pi(n)$ define the function $h_\pi(x_1,\dots,x_n) = h(x_{\pi(1)},\dots,x_{\pi(n)})$. The following identity holds.*

$$\int h(x_1,\dots,x_n) Z_{G,j_1}(\,dx_1)\dots Z_{G,j_n}(\,dx_n) = \int h_\pi(x_1,\dots,x_n) Z_{G,j_{\pi(1)}}(\,dx_1)\dots Z_{G,j_{\pi(n)}}(\,dx_n). \tag{5.14}$$

(In particular, $h_\pi \in \mathcal{K}_{n,j_{\pi(1)},\dots,j_{\pi(n)}}$, thus the integrals on both sides of the identity are meaningful.)

Proof of Lemma 5.3. This identity can be simply checked if h is a simple function. It is enough to observe that if $h(x_1,\dots,x_n) = h_1(x_1)\cdots h_n(x_n)$ with some $x_l \in \Delta_{k_l}$, $g(_l(\cdot)$ is some function on $\mathbb{R}^\nu$, $1 \le l \le n$, then

$$\int h(x_1,\dots,x_n) Z_{G,j_1}(\,dx_1)\dots Z_{G,j_n}(\,dx_n) = \prod_{l=1}^n h_l(x_l) Z_{G,j_l}(\Delta_{k_l}),$$

$h_\pi(x_1,\dots,x_l) = h_1(x_{\pi_1})\cdots h_n(x_{\pi_n})$,

$$\int h_\pi(x_1,\dots,x_n) Z_{G,j_{\pi(1)}}(\,dx_1)\dots Z_{G,j_{\pi(n)}}(\,dx_n) = \prod_{l=1}^n h(x_{\pi_l}) Z_{G,j_{\pi_l}}(\Delta_{k_{\pi(l)}}),$$

and the last two Wiener–Itô integrals equal. Then a simple limiting procedure implies it in the general case. Lemma 5.3 is proved.

We saw in [9] that in the scalar valued case the value of a Wiener–Itô integral $\int f(x_1,\dots,x_n) Z_G(\,dx_1)\dots Z_G(\,dx_n)$ does not change if we replace the kernel

function f by the function we get by permuting its variables $x_1, \dots, x_n$ in an arbitrary way. Lemma 5.3 is the generalization of this result to the case when we integrate with respect to the coordinates of a vector valued random spectral measure.

Remark. A consequence of the result of Lemma 5.3 shows an essential difference between the behaviour of multiple Wiener–Itô integrals with respect to scalar and vector valued random spectral measures. It follows from the scalar valued version of Lemma 5.3 that in the scalar valued case the Wiener–Itô integral of a kernel function agrees with the Wiener–itô integral of the symmetrization of this kernel function. This has the consequence that in the scalar valued case we can restrict our attention to the Wiener–Itô integrals of symmetrical functions which do not change their values by any permutation of their variables. It can be seen that any random variable which can be written as the sum of Wiener–Itô integrals can be written in a unique form as a sum of Wiener–Itô integrals of different multiplicity with symmetric kernel functions. The analogous result does not hold in the vector valued case. Indeed, if there is some linear dependence among the coordinates of the underlying vectors in a vector valued stationary random field, then such functions f_j can be found for which $\sum_{j=1}^d \int f_j(x) Z_{G,j}(\,dx) \equiv 0$, and not all kernel functions f_j disappear in the above sum. This shows that the unique representation of the random variables by means of a sum of Wiener–Itô integrals may not hold in vector valued models.

6 The diagram formula for the product of multiple Wiener–Itô integrals

Let us consider a vector valued random spectral measure $(Z_{G,1}, \dots, Z_{G,d})$ corresponding to the matrix valued spectral measure $(G_{j,j'})$, $1 \le j, j' \le d$, of a vector valued stationary Gaussian random field with expectation zero (either to a discrete random field $X(p) = (X_1(p), \dots, X_d(p))$, $p \in \mathbb{Z}^\nu$, or to a generalized one $X(\varphi) = (X_1(\varphi), \dots, X_d(\varphi))$, $\varphi \in \mathcal{S}^\nu$). Let us assume that the spectral measure $G_{j,j'}$, $1 \le j, j' \le d$, is non-atomic, and take two Wiener–Itô integrals

$$I_n(h_1|j_1, \dots, j_n) = \int h_1(x_1, \dots, x_n) Z_{G,j_1}(\,dx_1) \dots Z_{G,j_n}(dx_n) \tag{6.1}$$

and

$$I_m h_2|j'_1, \dots, j'_m) = \int h_2(x_1, \dots, x_m) Z_{G,j'_1}(\,dx_1) \dots Z_{G,j'_m}(dx_m) \tag{6.2}$$

with some kernel functions $h_1 \in \mathcal{K}_{n,j_1,\dots,j_n}$ and $h_2 \in \mathcal{K}_{m,j'_1,\dots,j'_m}$, where $j_s, j'_t \in \{1, \dots, d\}$ for all $1 \le s \le n$ and $1 \le t \le m$.

Actually we formulate our problems in a slightly different form which is more appropriate for our discussion. We take two functions $h_1(x_1, \dots, x_n)$ and $h_2(x_{n+1}, \dots, x_{n+m})$ in the space $\mathbb{R}^{(n+m)\nu}$, and define the function

$h_2^{(0)}(x_1, \dots, x_m)$ by the identity

$$h_2^{(0)}(x_1, \dots, x_m) = h_2(x'_{n+1}, \dots, x'_{n+m})) \text{ if } (x_1, \dots, x_m) = (x'_{n+1}, \dots, x'_{n+m}).$$

We assume that $h_1 \in \mathcal{K}_{n,j_1,\dots,j_n}$, $h_2^{(0)} \in \mathcal{K}_{m,j'_1,\dots,j'_m}$. Then we define the Wiener–Itô integrals (6.1) and (6.2) with the kernel functions h_1 and $h_2^{(0)}$. In formula (6.2) we should have written the function $h_2^{(0)}$, but we omitted the superscript $^{(0)}$.

I shall present a result in which we express the product of these two Wiener–Itô integrals as a sum of Wiener–Itô integrals. This result is called the diagram formula, since the kernel functions of the Wiener–Itô integrals appearing in this sum are expressed by means of some diagrams. This result is a multivariate version of the diagram formula proved in Chapter 5 of [9]. In that work also the product of more than two Wiener–Itô integrals is expressed in the form of a sum of Wiener–integrals. But actually the main point of the proof is to show the validity of the diagram formula for the product of two Wiener–Itô integrals, and we shall need only this result. So I restrict my attention to this case. Actually we need the diagram formula only in a special case. The result in this special case will be given in a corollary.

To express the product of the two Wiener–Itô integrals in formulas (6.1) and (6.2) as a sum of Wiener–Itô integrals first I introduce a class of coloured diagrams $\Gamma = \Gamma(n, m)$ that will be used in the definition of the Wiener–Itô integrals we shall be working with. A coloured diagram $\gamma \in \Gamma$ is a graph whose vertices are the pairs of integers $(1, s)$, $1 \le s \le n$, and $(2, t)$, $1 \le t \le m$. Each vertex is coloured with one of the numbers $1, \dots, d$. The colour of the vertex $(1, s)$ is j_s, $1 \le s \le n$, and the colour of the vertex $(2, t)$ is j'_t, $1 \le t \le m$. The set of vertices of the form $(1, s)$ will be called the first row and the set of vertices of the form $(2, t)$ will be called the second row of a diagram $\gamma \in \Gamma$. The coloured diagrams $\gamma \in \Gamma$ are those undirected graphs with the above coloured vertices for which edges can go only between vertices of the first and second row, and from each vertex there starts zero or one edge. Given a coloured diagram $\gamma \in \Gamma$ we shall denote the number of its edges by $|\gamma|$.

I shall define for all coloured diagrams $\gamma \in \Gamma$ a multiple Wiener–Itô integral depending on γ. The diagram formula states that the product of the Wiener–Itô integrals in (6.1) and (6.2) equals the sum of these Wiener–Itô integrals.

In the formulation of the diagram formula I shall work with the functions $h_1(x_1, \dots, x_n)$ and $h_2(x_{n+1}, \dots, x_{n+m})$ in $\mathbb{R}^{n+m}$. The function $h_2(x_{n+1}, \dots, x_{n+m})$ is the function which corresponds to the kernel function $h_2^{(0)}(x_1, \dots, x_m)$ in the definition of the Wiener–Itô integral in (6.2). We define with their help the function

$$H(x_1, \dots, x_{n+m}) = h_1(x_1, \dots, x_n) h_2(x_{n+1}, \dots, x_{n+m}). \tag{6.3}$$

We shall define the kernel functions appearing in the Wiener–itô integrals in the diagram formula with the help of the functions $H(x_1, \dots, x_{n+m})$. In the definition of these kernel functions I shall apply the following natural bijection S

between the coordinates of the vectors in $\mathbb{R}^{n+m}$, i.e. the set $\{1, \dots, n+m\}$ and the vertices of the diagrams of $\gamma \in \Gamma$.

$$S((1,k)) = k \text{ for } 1 \le k \le n, \quad \text{and} \quad S((2,k)) = n+k \text{ for } 1 \le k \le m. \tag{6.4}$$

To simplify the formulation of the diagram formula I shall introduce the following notation with the help of the colours of the diagrams.

$$J(1,k) = j_k, \ \ 1 \le k \le n \quad \text{and} \quad J(2,l) = j'_l, \ \ 1 \le l \le m. \tag{6.5}$$

First I give the formal definition of the Wiener–Itô integrals that appear in the diagram formula. These Wiener-Itô integrals correspond to the diagrams $\gamma \in \Gamma$ introduced before. Then I describe the diagram formula with the help of these Wiener–Itô integrals. The definition of the Wiener–Itô integrals we need in the diagram formula applies a rather complicated notation, but its informal explanation given after formula (6.16) may help to understand it. For the sake of a better comprehension of the calculations in the diagram formula I shall present an example after the formulation of this result, where the product of two Wiener–Itô integrals is considered, and I show how to calculate a typical term in the sum of Wiener–Itô integrals which appears in the diagram formula for this product.

Let us fix some diagram $\gamma \in \Gamma$. I explain how to define the Wiener–Itô integral corresponding to γ in the diagram formula. First I define a function $H_\gamma(x_1, \dots, x_{n+m})$ which we get by means of an appropriate permutation of the indices of the function H defined in (6.3). This permutation of the indices depends on the diagram γ.

To define this permutation of the indices first I define a map T_γ which maps the set $\{1, \dots, n+m\}$ to the elements in the rows of the diagrams. This map depends on the diagram γ.

To define this map first I introduce the following sets depending on the diagram γ:

$$\begin{aligned} A_1 = A_1(\gamma) \quad = \quad & \{r_1, \dots, r_{n-|\gamma|} \colon \ 1 \le r_1 < r_2 < \dots < r_{n-|\gamma|} \le n \\ & \text{no edge of } \gamma \text{ starts from } (1, r_k), \quad 1 \le k \le n - |\gamma|\}, \end{aligned} \tag{6.6}$$

$$\begin{aligned} A_2 = A_2(\gamma) \quad = \quad & \{t_1, \dots, t_{m-|\gamma|} \colon \ 1 \le t_1 < t_2 < \dots < t_{m-|\gamma|} \le m \\ & \text{no edge of } \gamma \text{ starts from } (2, t_k), \quad 1 \le k \le m - |\gamma|\} \end{aligned} \tag{6.7}$$

and

$$\begin{aligned} B = B(\gamma) \quad = \quad & \{(v_1, w_1), \dots, (v_{|\gamma|}, w_{|\gamma|})) \colon \ 1 \le v_1 < v_2 < \dots v_{|\gamma|} \le n \\ & ((1, v_k), (2, w_k)) \text{ is an edge of } |\gamma|, \ \ 1 \le k \le |\gamma|\}. \end{aligned} \tag{6.8}$$

Let us also define with the help of the set B the sets

$$B_1 = B_1(\gamma) = \{v_1, \dots, v_{|\gamma|}\}, \quad B_2 = B_2(\gamma) = \{w_1, \dots, w_{|\gamma|}\} \tag{6.9}$$

with the numbers v_k and w_l appearing in the set

$$B = B(\gamma) = \{(v_1, w_1)), \dots, (v_{|\gamma|}, w_{|\gamma|}))\}.$$

Now, I define the map T_γ in the following way.

$$\begin{aligned} &T_\gamma(k) = (1, r_k) \text{ for } 1 \le k \le n - |\gamma|, \\ &T_\gamma(n - |\gamma| + k) = (2, t_k) \text{ for } 1 \le k \le m - |\gamma|, \\ &T_\gamma(n + m - 2|\gamma| + k) = (1, v_k) \text{ for } 1 \le k \le |\gamma|, \\ &T_\gamma(n + m - |\gamma| + k) = (2, w_k) \text{ for } 1 \le k \le |\gamma|. \end{aligned} \tag{6.10}$$

In formula (6.10) we worked with the numbers r_k, t_k, v_k and w_k defined in (6.6)—(6.9). It has the following meaning. We listed the vertices of the diagram γ in the form $T_\gamma(s)$, $1 \le s \le n + m$. If the vertex $T_\gamma(s)$ gets the index s, then the first $n - |\gamma|$ indices are given in increasing order to the vertices from the first row from which no edge starts. The vertices of the second row from which no edge starts get the next $m - |\gamma|$ indices also in increasing order. Then the $|\gamma|$ vertices from the first row from which an edge starts get the subsequent $|\gamma|$ indices in increasing order. The remaining $|\gamma|$ vertices from the second row from which an edge starts get the indices between $n + m - |\gamma| + 1$ and $n + m$. They are indexed in such a way that if two vertices $(1, v_k)$ and $(2, w_k)$ are connected by en edge then the index of $(2, w_k)$ is obtained if we add $|\gamma|$ to the index of $(1, v_k)$.

I define with the help of the function T_γ and the map $S(\cdot)$ defined in (6.4) the permutation

$$\pi_\gamma(k) = S(T_\gamma(k)), \quad 1 \le k \le n + m \tag{6.11}$$

of the set $\{1, \dots, n + m\}$. Next I introduce the Euclidean space $\mathbb{R}^{n+m}_\gamma$ with elements $x(\gamma) = (x(\gamma)_1, \dots, x(\gamma)_{n+m})$ by reindexing the arguments of the Euclidean space $\mathbb{R}^{n+m}$, where the functions $h_1(x_1, \dots, x_n)$ and $h_2(x_{n+1}, \dots, x_{n+m})$ are defined in the following way.

$$(x(\gamma)_1, \dots, x(\gamma)_{n+m}) = (x_{\pi_\gamma(1)}, \dots, x_{\pi_\gamma(n+m)})$$

with $(x(\gamma)_1, \dots, x(\gamma)_{n+m}) \in \mathbb{R}^{n+m}_\gamma$ and $(x_1, \dots, x_{n+m}) \in \mathbb{R}^{n+m}$. It will be simpler to define the quantities needed in the definition of the Wiener–Itô integral corresponding to the diagram γ as functions defined in the space R^{n+n}_γ. First we define the function H_γ as

$$\begin{aligned} &H_\gamma(x(\gamma)_1, \dots, x(\gamma)_{n+m}) \\ &\quad = H(x(\gamma)_1, \dots, x(\gamma)_{n-|\gamma|}, x(\gamma)_{n+m-2|\gamma|+1}, \dots, x(\gamma)_{n+m-|\gamma|}, \\ &\qquad\qquad x(\gamma)_{n-|\gamma|+1}, \dots, x(\gamma)_{n+m-2|\gamma|+1}, x(\gamma)_{(n+m-|\gamma|+1}, \dots, x(\gamma)_{n+m}) \\ &\quad = h_1(x(\gamma)_1, \dots, x(\gamma)_{n-|\gamma|}, x(\gamma)_{\pi_\gamma(n+m-2|\gamma|+1)}, \dots, x(\gamma)_{n+m-|\gamma|}) \\ &\qquad h_2(x(\gamma)_{n-|\gamma|+1}, \dots, x(\gamma)_{n+m-2|\gamma|+1}, x(\gamma)_{n+m-|\gamma|+1}, \dots, x(\gamma)_{n+m}). \end{aligned} \tag{6.12}$$

Next I define the function $\bar{h}_\gamma(x(\gamma)_1, \dots, x(\gamma)_{n+m-|\gamma|}))$ (with $n + m - |\gamma|$ arguments) which we get by replacing the arguments $x(\gamma)_{n+m-|\gamma|+k}$ by

$-x(\gamma)_{n+m-2|\gamma|+k})$ in the function H_γ defined in formula (6.12) for all $1 \le k \le \gamma$, i.e. I define

$$\begin{aligned}
&\bar{h}_\gamma(x(\gamma)_1, \dots, x(\gamma)_{n+m-|\gamma|}) \qquad (6.13)\\
&\quad = H_\gamma(x(\gamma)_1, \dots, x(\gamma)_{n+m-|\gamma|}, -x(\gamma)_{n+m-2|\gamma|+1}, \dots, -x(\gamma)_{n+m-|\gamma|})\\
&\quad = H(x(\gamma)_1, \dots, x(\gamma)_{n-|\gamma|}, x(\gamma)_{n+m-2|\gamma|+1}, \dots, x(\gamma)_{n+m-|\gamma|},\\
&\qquad\qquad x(\gamma)_{n-|\gamma|+1}, \dots, x(\gamma)_{n+m-2|\gamma|+1},\\
&\qquad\qquad\qquad -x(\gamma)_{n+m-2|\gamma|+1}, \dots, -x(\gamma)_{n+m-|\gamma|})\\
&\quad = h_1(x(\gamma)_1, \dots, x(\gamma)_{n-|\gamma|}, x(\gamma)_{n+m-2|\gamma|+1}, \dots, x(\gamma)_{n+m-|\gamma|})\\
&\qquad h_2(x(\gamma)_{n-|\gamma|+1}, \dots, x(\gamma)_{n+m-2|\gamma|+1},\\
&\qquad\qquad -x(\gamma)_{n+m-2|\gamma|+1}, \dots, -x(\gamma)_{n+m)-|\gamma|}).
\end{aligned}$$

In the next step I define the function $\bar{\bar{h}}_\gamma(x(\gamma)_1, \dots, x(\gamma)_{n+m-2|\gamma|})$. This will be the kernel function of the Wiener–Itô integral which corresponds to the diagram γ in the diagram formula if we express it as a Wiener–Itô integral with respect to the variables $x(\gamma)_1, \dots, x(\gamma)_{n+m-2|\gamma|}$.

$$\begin{aligned}
\bar{\bar{h}}_\gamma(x_\gamma)_1, \dots, x(\gamma)_{n+m-2|\gamma|}) &= \int \bar{h}_\gamma(x(\gamma)_1, \dots, x(\gamma)_{n+m-|\gamma|}) \qquad (6.14)\\
&\qquad \prod_{k=1}^{|\gamma|} G_{J(S^{-1}(n+m-2|\gamma|+k)), J(S^{-1}(n+m-|\gamma|+k))}(\, dx(\gamma)_{n+m-2|\gamma|+k})\\
&= \int \bar{h}_\gamma(x(\gamma)_1, \dots, x(\gamma)_{n+m-|\gamma|}) \prod_{k=1}^{|\gamma|} G_{j_{v_k}, j'_{w_k}}(\, dx(\gamma)_{n+m-2|\gamma|+k})
\end{aligned}$$

with the function $J(\cdot)$ defined in (6.5), the indices v_k and w_k defined in (6.8) and the function T_γ defined in (6.10).

I shall show that the Wiener–Itô integrals

$$\begin{aligned}
&I_{n+m-2|\gamma|}(\bar{\bar{h}}_\gamma | j_{r_1}, \dots, j_{r_{n-|\gamma|}}, j'_{t_1}, \dots, j'_{t_{m-|\gamma|}}) \qquad (6.15)\\
&\quad = \int \bar{\bar{h}}_\gamma(x(\gamma)_1, \dots, x(\gamma)_{n+m-2|\gamma|}) \prod_{k=1}^{n+m-2|\gamma|} Z_{G, J(S^{-1}(k))}(dx(\gamma)_k)\\
&\quad = \int \bar{\bar{h}}_\gamma(x(\gamma)_1, \dots, x(\gamma)_{n+m-2|\gamma|})\\
&\qquad\qquad \prod_{k=1}^{n-|\gamma|} Z_{G, j_{r_k}}(\, dx(\gamma)_k) \prod_{l=1}^{m-|\gamma|} Z_{G, j'_{t_l}}(\, dx(\gamma)_{l+n-|\gamma|})
\end{aligned}$$

exist for all $\gamma \in \Gamma$, and these Wiener–Itô integrals appear in the diagram formula. The numbers r_k and t_l in this formula were defined in (6.6) and (6.7).

In formula (6.15) we integrated with respect to the coordinates $x(\gamma)_s$, $1 \le s \le n+m$, of the vectors in the Euclidean space $\mathbb{R}^{n+m}_\gamma$. If we replace the

variables $x(\gamma)_s$ by x_s in (6.15), then we get a Wiener–itô integral in the space $\mathbb{R}^{n+m}$ which has the same value. This means that the following relation holds.

$$\begin{aligned} I_{n+m-2|\gamma|}(\bar{\bar{h}}_\gamma | j_{r_1}, \dots, j_{r_{n-|\gamma|}}, j'_{t_1}, \dots, j'_{t_{m-|\gamma|}}) \\ = I_{n+m-2|\gamma|}(h_\gamma | j_{r_1}, \dots, j_{r_{n-|\gamma|}}, j'_{t_1}, \dots, j'_{t_{m-|\gamma|}}) \\ = \int h_\gamma(x_1, \dots, x_{n+m-2|\gamma|}) \\ \prod_{k=1}^{n-|\gamma|} Z_{G,j_{r_k}}(\, dx_k) \prod_{l=1}^{m-|\gamma|} Z_{G,j'_{t_l}}(\, dx_{l+n-|\gamma|}) \end{aligned} \tag{6.16}$$

with

$$\begin{aligned} h_\gamma(x_1, \dots, x_{n+m-2|\gamma|}) &= \bar{\bar{h}}_\gamma(x(\gamma)_1, \dots, x(\gamma)_{n+m-2|\gamma|}) \\ &= \bar{\bar{h}}_\gamma(x_{\pi_\gamma(1)}, \dots, x_{\pi_\gamma(n+m-2|\gamma|)}). \end{aligned}$$

Before describing the diagram formula I explain the content of the above defined formulas.

Let us fix a diagram $\gamma \in \Gamma$, and let us call a vertex of it from which no edge starts open, and a vertex from which an edge starts closed. We listed the open vertices from the first row in increasing order as $(1, r_1), \dots, (1, r_{n-|\gamma|})$, and the open vertices from the second row as $(2, t_1), \dots, (2, t_{m-|\gamma|})$. We listed the closed vertices from the first row in increasing order as $(1, v_1), \dots, (1, v_\gamma)$. Finally we listed the closed vertices from the second row as $(2, w_1), \dots, (2, w_\gamma)$, and we indexed them in such a way that the vertices $(1, v_k)$ and $(2, w_k)$ are connected by an edge for all $1 \le k \le \gamma$.

In formula (6.10) we defined the map T_γ from the set $\{1, \dots, n+m\}$ to the set of vertices of the diagram γ with the help of the above listing of the vertices. First we considered the open vertices from the first row, then the open vertices from the second row, and then we finished with the closed vertices first from the first and then from the second row. We defined in (6.11) the permutation π_γ of the set $\{1, \dots, n+m\}$ by applying first the map the map T_γ and then the map S defined (6.4). We defined the function H_γ in (6.13) with the help of this permutation. We have introduced a Euclidean space $\mathbb{R}^{n+m}_\gamma$ whose elements we get by rearranging the indices of the coordinates of the Euclidean space $\mathbb{R}^{n+m}$ where we are working with the help of the permutation π_γ, and we have defined our functions in this space.

We defined the function H_γ on the space $\mathbb{R}^{n+m}_\gamma$ as the product of the functions h_1 and h_2 with reindexed variables. In the function h_1 first we took the variables $x(\gamma)_s = x_{\pi_\gamma(s)}$ with those indices $\pi_\gamma(s)$ which correspond to the open vertices of the first row, and then the variables with indices corresponding to the closed vertices of the first row. We defined the reindexation of the variables in the second row similarly. First we took those variables whose indices correspond to the open vertices and then the variables whose indices correspond to the closed vertices of the second row.

The variables

$$x(\gamma)_{n+m-2|\gamma|+k} = x_{\pi_\gamma(n+m-2|\gamma|+k)} \text{ and } x(\gamma)_{n+m-|\gamma|+k} = x_{\pi_\gamma(n+m-|\gamma|+k)}$$

in the function H_γ are variables with indices corresponding to vertices connected by an edge. So in the definition of the function $\bar{h}_\gamma$ in (6.14) I replaced in H_γ the variable corresponding to the endpoint of an edge from the second row of the diagram γ by the variable corresponding to the other endpoint of this edge, and multiplied this variable by -1. Thus the variables $x(\gamma)_{n+m-2|\gamma|+k} = x_{\pi_\gamma(n+m-2|\gamma|+k)}$, $1 \le k \le |\gamma|$, of the function $\bar{h}_\gamma$ correspond to the edges of the diagram γ. I defined the function $\bar{\bar{h}}_\gamma$ by integrating the function $\bar{h}_\gamma$ by these variables. The variable $x(\gamma)_{n+m-2|\gamma|+k} = x_{\pi_\gamma(n+m-2|\gamma|+k)}$ corresponds to the k-th edge of the diagram, and we integrate this variable with respect to the measure $G_{j_{v_k},j'_{w_k}}$, that is with respect to the measure $G_{u,v}$ whose coordinates are the colours of the endpoints of the k-th edge.

Finally we define the Wiener–Itô integral corresponding to the diagram γ with kernel function $\bar{\bar{h}}_\gamma$. We integrate the argument $x(\gamma)_k$ with respect to that random spectral measure $Z_{G,j}$ whose parameter agrees with the colour of the vertex corresponding to this variable. Thus we choose $Z_{G,j_{r_k}}(\,dx(\gamma)_k)$ for $1 \le k \le n - |\gamma|$ and $Z_{G_{j'_{t_{k-n+|\gamma}}}}(\,dx(\gamma)_k)$ if $n - |\gamma| + 1 \le k \le n + m - 2|\gamma|$. We can replace this Wiener–Itô integral defined in (6.15) with kernel function $\bar{\bar{h}}_\gamma$ by the Wiener–Itô integral defined in (6.16) with kernel function h_γ.

Next I formulate the diagram formula.

Theorem 6.1. The diagram formula. *Let us consider the Wiener–Itô integrals $I_n(h_1|j_1,\dots,j_n)$ and $I_m(h_2|j'_1,\dots,j'_m)$ introduced in formulas (6.1) and (6.2). The following results hold.*

(A) *The function $\bar{\bar{h}}_\gamma$ defined in (6.14) satisfies the relations*

$$\bar{\bar{h}}_\gamma \in \mathcal{K}_{n+m-2|\gamma|,j_{r_1},\dots,j_{r_{n-|\gamma|}},j'_{t_1},\dots,j'_{t_{m-|\gamma|}}},$$

and $\|\bar{\bar{h}}_\gamma\| \le \|h_1\|\|h_2\|$ for all $\gamma \in \Gamma$. Here the norm of the function h_1 in $\mathcal{K}_{n,j_1,\dots,j_n}$, the norm of $\bar{\bar{h}}_2$ in $\mathcal{K}_{m,j'_1,\dots,j'_m}$, and the norm of $\bar{\bar{h}}_\gamma$ in $\mathcal{K}_{n+m-2|\gamma|,j_{r_1},\dots,j_{r_{n-|\gamma|}},j'_{t_1},\dots,j'_{t_{m-|\gamma|}}}$ is taken.

(B)

$$I_n(h_1|j_1,\dots,j_n)I_m(h_2|j'_1,\dots,j'_m) = \sum_{\gamma\in\Gamma} I_{n+m-2|\gamma|}(\bar{\bar{h}}_\gamma|j_{r_1},\dots,j_{r_{n-|\gamma|}},j'_{t_1},\dots,j'_{t_{m-|\gamma|}}). \tag{6.17}$$

The terms in the sum at the right-hand side of formula (6.17) were defined in formulas (6.12)—(6.15). The Wiener–Itô integral

$$I_{n+m-2|\gamma|}(\bar{\bar{h}}_\gamma|j_{r_1},\dots,j_{r_{n-|\gamma|}},j'_{t_1},\dots,j'_{t_{m-|\gamma|}})$$

in formula (6.17) can be replaced by the Wiener–Itô integral

$$I_{n+m-2|\gamma|}(h_\gamma | j_{r_1}, \dots, j_{r_{n-|\gamma|}}, j'_{t_1}, \dots, j'_{t_{m-|\gamma|}})$$

defined in (6.16).

To understand the formulation of the diagram formula better let us consider the following example. We take a five dimensional stationary Gaussian random field with some spectral measure $(G_{j,j'}(x))$, $1 \le j, j' \le 5$, and random spectral measure $Z_{G,j}(\,dx)$, $1 \le j \le 5$, corresponding to it. Let us understand how we define the Wiener–Itô integral corresponding to a typical diagram when we apply the diagram formula in the following example. Take the product of two Wiener–Itô integrals of the following form:

$$I_3(h_1|2,3,5) = \int h_1(x_1,x_2,x_3) Z_{G,2}(\,dx_1) Z_{G,3}(\,dx_2) Z_{G,5}(\,dx_3)$$

and

$$\begin{aligned} I_4(h_2|1,5,4,1) \quad &= \quad \int h_2(x_1,x_2,x_3,x_4) \\ &\qquad Z_{G.1}(\,dx_1) Z_{G,5}(\,dx_2) Z_{G,4}(\,dx_3) Z_{G,2}(\,dx_4), \end{aligned}$$

and let us write it in the form of a sum of Wiener–Itô integrals with the help of the diagram formula.

First I give the vertices of the coloured diagrams we shall be working with together with their colours.

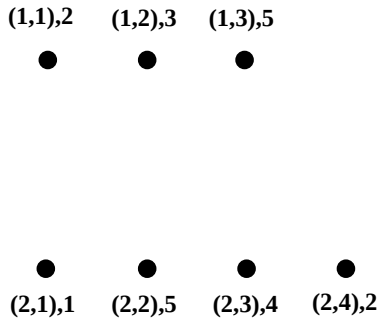

Figure 1: the vertices of the diagrams together with their colours

Next I consider a diagram γ which yields one of the terms in the sum expressing the product of these two Wiener–Itô integrals. I take the diagram which has two edges, one edge connecting the vertices $(1,2)$ and $(2,4)$, and another edge connecting the vertices $(1,3)$ and $(2,1)$. Let us calculate which Wiener–Itô integral corresponds to this diagram γ.

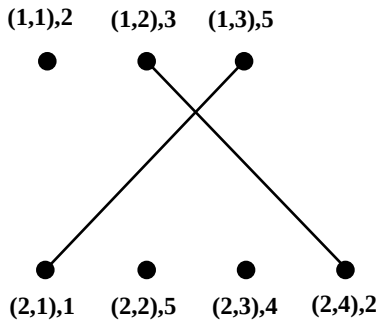

Figure 2: a typical diagram

In the next step I take this diagram γ, and I show not only the indices and colours of its vertices, but for each vertex I also tell which value $T_\gamma(k)$ it equals. Here $T_\gamma(k)$ is the function defined in formula (6.10).

To define the Wiener–Itô integral corresponding to this diagram let us first consider the function

$$H(x_1, \dots, x_7) = h_1(x_1, x_2, x_3)h_2(x_4, x_5, x_6, x_7)$$

defined in (6.3). Simple calculation shows that the function $\pi_\gamma(\cdot) = S(T_\gamma(\cdot))$ has the following form in this example. $\pi_\gamma(1) = 1$, $\pi_\gamma(2) = 5$, $\pi_\gamma(3) = 6$, $\pi_\gamma(4) = 2$, $\pi_\gamma(5) = 3$, $\pi_\gamma(6) = 7$, $\pi_\gamma(7) = 4$. This also means that the coordinates of the vectors in the Euclidean space $\mathbb{R}^7_\gamma$ which we get by reindexing the coordinates of the vectors in $\mathbb{R}^7$ have the form

$$(x(\gamma)_1, x(\gamma)_2, x(\gamma)_3, x(\gamma)_4, x(\gamma)_5, x(\gamma)_6, x(\gamma)_7) = (x_1, x_5, x_6, x_2, x_3, x_7, x_4).$$

Then we can write the function $\bar{H}_\gamma$ and $\bar{h}_\gamma$ defined in (6.12) and (6.13) as

$$H_\gamma(x(\gamma)_1, \dots, x(\gamma)_7) = h_1(x(\gamma)_1, x(\gamma)_4, x(\gamma)_5)h_2(x(\gamma)_2, x(\gamma)_3, x(\gamma)_6, x(\gamma)_7),$$

and

$$\bar{h}_\gamma(x(\gamma)_1, \dots, x(\gamma)_5) = h_1(x(\gamma)_1, x(\gamma)_4, x(\gamma)_5)h_2(x(\gamma)_2, x(\gamma)_3, -x(\gamma)_4, -x(\gamma)_5).$$

Then we have

$$\bar{\bar{h}}_\gamma(x(\gamma)_1, x(\gamma)_2, x(\gamma)_3) = \int \bar{h}_\gamma(x(\gamma)_1, \dots, x(\gamma)_5)G_{3,2}(\, dx(\gamma)_4)G_{5,1}(\, dx(\gamma)_5),$$

and

$$\begin{aligned} &I_3(\bar{\bar{h}}_\gamma|2, 5, 4) \\ &\quad = \int \bar{\bar{h}}_\gamma(x(\gamma)_1, x(\gamma)_2, x(\gamma)_3)Z_{G,2}(\, dx(\gamma)_1)Z_{G,5}(\, dx(\gamma)_2)Z_{G,4}(\, dx(\gamma)_3) \end{aligned}$$

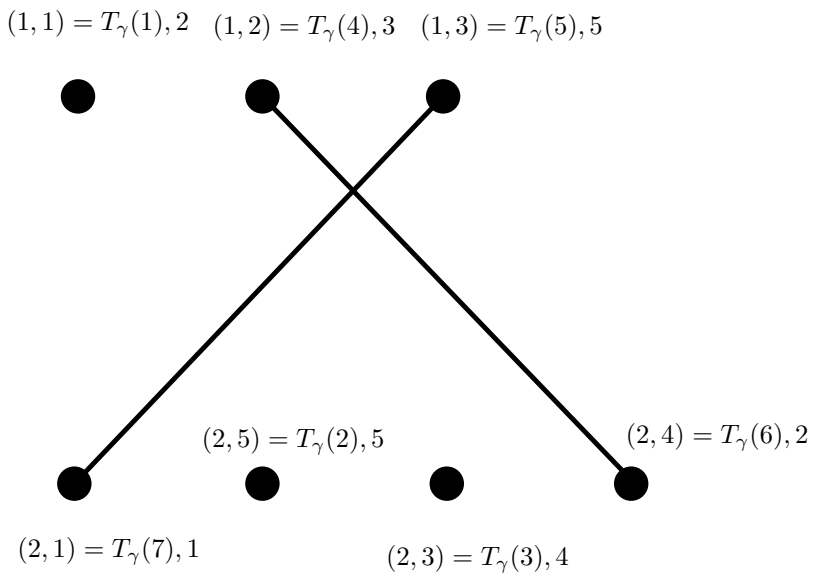

Figure 3: the previous diagram and the enumeration of their vertices with the help of the function T_γ

is the multiple Wiener–Itô integral corresponding to the diagram γ in the diagram formula. To understand the definition of the function $\bar{\bar{h}}_\gamma$ and of the Wiener–Itô integral $I_3(\bar{\bar{h}}_\gamma)$ let us observe that the first edge of the diagram connects the vertices $(1,2)$ and $(2,4)$ with colours 3 and 2, hence in the definition of $\bar{\bar{h}}_\gamma$ we integrate the argument $x(\gamma)_4$ by $G_{3,2}(\,dx(\gamma)_4)$, the second edge connects the vertices $(1,3)$ and $(2,1)$ with colours 5 and 1, hence we integrate the variable $x(\gamma)_5$ by $G_{5,1}(\,dx(\gamma)_5)$. In the definition of the Wiener integral the variable $x(\gamma)_1$ corresponds to the vertex $S^{-1}(\pi_\gamma(1)) = (1,1)$ which has colour 2, hence we integrate the variable $x(\gamma)_1)$ by $Z_{G,2}(\,dx(\gamma)_1)$. Similarly, we define the variable $x(\gamma)_2$ by the measure determined by the colour of $S^{-1}(\pi_\gamma(2)) = (2,2)$ which is 5, i.e. we integrate by $Z_{G,5}(\,dx(\gamma)_2)$. Finally $S^{-1}(\pi_\gamma(3)) = (2,3)$ has colour 4, and we integrate the variable $x(\gamma)_3$ by $Z_{G,4}(\,dx(\gamma)_3)$.

The Wiener–Itô integral $I_3(\bar{\bar{h}}_\gamma|3,1,3)$ can be rewritten with the help of formula (6.16) in the following form.

$$I_3(\bar{\bar{h}}_\gamma|2,5,4) = I_3(h_\gamma|2,5,4) = \int h_\gamma(x_1,x_2,x_3) Z_{G,2}(\,dx_1) Z_{G,5}(\,dx_2) Z_{G,4}(\,dx_3)$$

with

$$h_\gamma(x_1,x_2,x_3) = \int h_1(x_1,x_4,x_5) h_2(x_2,x_3,-x_4,-x_5) G_{3,2}(\,dx_4) G_{5,1}(\,dx_5).$$

This expression can be calculated similarly to $I_3(\bar{\bar{h}}_\gamma|2,5,4)$, only we have to replace $x(\gamma)_s$ everywhere by x_s in the calculation.

I formulate a Corollary of the diagram formula in which I consider that special case of this result when the second Wiener–Itô integral defined in formula (6.2) is a one-fold integral. In this case it has the simpler form

$$I_1(h_2|j_1') = \int h_2(x_1) Z_{G,j_1'}(\,dx_1) \quad \text{with } h_2 \in \mathcal{K}_{1,j_1'}. \tag{6.18}$$

Here again we formulate the problem in the following way. We take a pair of functions $h_1(x_1,\dots,x_n)$ and $h_2(x_{n+1})$ on $\mathbb{R}^{(n+1)\nu}$. Then we define a function $h_2^{(0)}(x_1)$ on $\mathbb{R}^1$ by the formula $h_2^{(0)}(x_1) = h_2(x_{n+1})$ if $x_1 = x_{n+1}$. We integrate the function $h_2^{(0)}(x)$ in formula (6.18), but we omit the superscript $^{(0)}$ in our notation. We assume that $h_1 \in \mathcal{K}_{n,j_1,\dots,j_n}$, and $h_2 \in \mathcal{K}_{1,j_1'}$.

In the next Corollary I express the product of the Wiener–Itô integrals given in (6.1) and (6.18) as a sum of Wiener–Itô integrals. This formula will be needed in the proof of the multivariate version of Itô's formula in paper [11].

The diagram formula in this case has a simpler form, since the second row of the diagrams we are working with consists only of one point $(2,1)$. Hence there are only the diagram $\gamma_0 \in \Gamma$ that contains no edges, and the diagrams $\gamma_p \in \Gamma$, $1 \le p \le n$, which contain one edge that connects the vertices $(1,p)$ and $(2,1)$.

Corollary of Theorem 6.1. *The product of the Wiener–Itô integrals*

$$I_n(h_1|j_1,\dots,j_n) \quad \textit{and} \quad I_1(h_2|j_1')$$

introduced in formulas (6.1) and (6.18) satisfy the identity

$$\begin{aligned} &I_n(h_1|j_1,\dots,j_n) I_1(h_2|j_1') \\ &\quad = \int h_{\gamma_0}(x_1,\dots,x_{n+1}) Z_{G,j_1}(\,dx_1)\cdots Z_{G,j_n}(\,dx_n) Z_{G,j_1'}(\,dx_{n+1}) \\ &\qquad + \sum_{p=1}^{n} \int h_{\gamma_p}(x_1,\dots,x_{n-1}) \prod_{s=1}^{p-1} Z_{G,j_s}(\,dx_s) \prod_{s=p}^{n-1} Z_{G,j_{s+1}}(\,dx_s) \\ &\quad = I_{n+1}(h_{\gamma_0}|j_1,\dots,j_n,j_1') + \sum_{p=1}^{n} I_{n-1}(h_{\gamma_p}|j_1,\dots,j_{p-1},j_{p+1},\dots,j_n), \end{aligned} \tag{6.19}$$

where $h_{\gamma_0}(x_1,\dots,x_{n+1}) = h_1(x_1,\dots,x_n)h_2(x_{n+1})$, *and for* $1 \le p \le n$

$$h_{\gamma_p}(x_1,\dots,x_{n-1}) = \int h_{1,\gamma_p}(x_1,\dots,x_n)\overline{h_2(x_n)} G_{j_p,j_1'}(\,dx_n)$$

with $h_{1,\gamma_p}(x_1,\dots,x_n) = h_1(x_{\pi_p(1)},\dots,x_{\pi_p(n)})$, *where* $\pi_p(k) = k$ *if* $1 \le k \le p-1$, $\pi_p(p) = n$, *and* $\pi_p(k) = k-1$ *if* $p+1 \le k \le n$.

To make the definition of formula (6.19) complete I remark that for $p = 1$ *we put* $\prod_{s=1}^{0} Z_{G,j_s}(\,dx_s) \equiv 1$ *and for* $p = n$ $\prod_{s=n}^{n-1} Z_{G,j_s}(\,dx_s) \equiv 1$.

Proof of the Corollary. We get the result of the corollary by applying Theorem 6.1 in the special case when the second Wiener–Itô integral is defined by

formula (6.18) instead of (6.2). We have to check that in this case the function h_{γ_0} corresponding to the diagram γ_0 agrees with the function h_{γ_0} defined in the corollary, and to calculate the functions h_{γ_p} defined in (6.14) for the remaining diagrams γ_p, $1 \le p \le n$. In this case $\pi_{\gamma_p}(k) = k$ for $1 \le k \le p-1$, $\pi_{\gamma_p}(k) = k+1$ for $p \le k \le n-1$, $\pi_{\gamma_p}(n) = p$, $\pi_{\gamma_p}(n+1) = n+1$, hence

$$(x(\gamma_p)_1, \dots, x(\gamma_p)_{n+1}) = (x_1, \dots, x_{p-1}, x_{p+1}, \dots, x_n, x_p, x_{n+1}),$$

and

$$\bar{h}_{\gamma_p}(x(\gamma_p)_1, \dots, x(\gamma_p)_{n+1}) = h_1(x(\gamma_p)_1, \dots, x(\gamma_p)_n) h_2(-x(\gamma_p)_n)$$

for $1 \le p \le n$. On the other hand, $h_2(-x) = \overline{h_2(x)}$, since $h_2 \in \mathcal{K}_{1,j_1'}$. Thus

$$\begin{aligned}\bar{\bar{h}}_{\gamma_p}&(x(\gamma_p)_1, \dots, x(\gamma_p)_{n-1}) \\ &= \int h_1(x(\gamma_p)_1, \dots, x(\gamma_p)_{n-1}, x(\gamma_p)_n) \overline{h_2(x(\gamma_p)_n)} G_{j_p, j_1'}(\, dx(\gamma_p)_n).\end{aligned}$$

Then simple calculation shows that for $\gamma = \gamma_p$ the kernel function $h_\gamma = h_{\gamma_p}$ in formula (6.16) agrees with the function h_{γ_p} defined in the corollary for all $1 \le p \le n$, and Theorem 6.1 yields identity (6.19) under the conditions of the corollary. The corollary is proved.

The proof of Theorem 6.1 is similar to the proof of the diagram formula (Theorem 5.3 in [9]). It applies the same method, only the notation becomes more complicated than the also rather complicated notation of the original proof, since we have to work with spectral measures of the form $G_{j_s, j_t'}$ and random spectral measures of the form Z_{G, j_s} or $Z_{G, j_t'}$ instead of the spectral measure G and random spectral measure Z_G. Hence I decided not to describe the complete proof, I only concentrate on its main ideas and the formulas that explain why such a result appears in the diagram formula. The interested reader can reconstruct the proof by means of a careful study of the proof of Theorem 5.3 in [9].

A sketch of proof for Theorem 6.1. The proof of Part A is relatively simple. One can check that the function h_γ satisfies relation (a) in the definition of the functions in $\mathcal{K}_{n+m-2|\gamma|, j_{r_1}, \dots, j_{r_{n-|\gamma|}}, j_{t_1}', \dots, j_{t_{m-|\gamma|}}'}$ given in Section 5 by exploiting formula (6.14), the similar property of the functions h_1 and h_2 together with the symmetry property $G_{j,j'}(-A) = \overline{G_{j,j'}(A)}$ for all $1 \le j, j' \le d$ and sets A of the spectral measure G.

To prove the inequality formulated in Part A let us first rewrite the definition of h_γ in (6.14) by replacing all measures of the form $G_{j.j'}(dx)$ by $g_{j,j'}(x)\mu(\, dx) = G_{j,j'}(\, dx)$, where μ is a dominating measure for all complex measures $G_{j,j'}$, $g_{j,j'}$ is the Radon–Nikodym derivative of $G_{j,j'}$ with respect to μ, and observe that

the inequality (3.2) and formula (6.13) and (6.14) imply that

$$
\begin{aligned}
&|\bar{\bar{h}}_\gamma(x(\gamma)_1,\dots,x(\gamma)_{n+m-2|\gamma|})|\\
&\quad\le \int h_1(x_{\pi_\gamma(1)},\dots,x_{\pi_\gamma(n-|\gamma|)},x_{\pi_\gamma(n+m-2|\gamma|+1)},\dots,x_{\pi_\gamma(n+m-|\gamma|+1)})\\
&\qquad h_2(x_{\pi_\gamma(n-|\gamma|+1)},\dots,x_{\pi_\gamma(n+m-2|\gamma|)},\\
&\qquad\quad -x_{\pi_\gamma(n+m-|2\gamma|+1)},\dots,-x_{\pi_\gamma(n+m-|\gamma|)})\\
&\qquad\quad \prod_{k=1}^{|\gamma|}\sqrt{g_{j_{v_k},j_{v_k}}(x_{\pi_\gamma(n+m-2|\gamma|+k)})}\sqrt{g_{j'_{w_k},j'_{w_k}}(x_{\pi_\gamma(n+m-2|\gamma|+k)})}\\
&\qquad\qquad \mu(\,dx_{\pi_\gamma(n+m-2|\gamma|+k)}).
\end{aligned}
$$

We get, by applying the Schwarz inequality the evenness of the measures $G_{j,j}$ and by replacing the measures of the form $g_{j,j}(x)\mu(\,dx)$ or $g_{j',j'}(x)\mu(\,dx)$ by the measures of the form $G_{j,j}(\,dx)$ and $G_{j',j'}(\,dx)$ that

$$
\begin{aligned}
&|\bar{\bar{h}}_\gamma(x(\gamma)_1,\dots,x(\gamma)_{n+m-2|\gamma|})|^2\\
&\quad\le \int |h_1(x_{\pi_\gamma(1)},\dots,x_{\pi_\gamma(n-|\gamma|)},x_{\pi_\gamma(n+m-2|\gamma|+1)},\dots,x_{\pi_\gamma(n+m-|\gamma|+1)})|^2\\
&\qquad\quad \prod_{k=1}^{|\gamma|}G_{j_{v_k},j_{v_k}}(\,dx_{\pi_\gamma(n+m-2|\gamma|+k)})\\
&\qquad \int |h_2(x_{\pi_\gamma(n-|\gamma|+1)},\dots,x_{\pi_\gamma(n+m-2|\gamma|)},\\
&\qquad\qquad\qquad -x_{\pi_\gamma(n+m-|2\gamma|+1)},\dots,-x_{\pi_\gamma(n+m-|\gamma|)})|^2\\
&\qquad\qquad \prod_{k=1}^{|\gamma|}G_{j'_{w_k},w_k}(\,dx_{\pi_\gamma(n+m-2|\gamma|+k)}).
\end{aligned}
$$

Let us integrate the last inequality with respect to the product measure

$$
\begin{aligned}
&\prod_{k=1}^{n-|\gamma|}G_{j_{r_k},j_{r_k}}(\,dx(\gamma)_k)\prod_{l=1}^{m-|\gamma|}G_{j'_{t_l},j'_{t_l}}(\,dx(\gamma)_{n-|\gamma|+l})\\
&\quad=\prod_{k=1}^{n-|\gamma|}G_{j_{r_k},j_{r_k}}(\,dx_{\pi_\gamma(k)})\prod_{l=1}^{m-|\gamma|}G_{j'_{t_l},j'_{t_l}}(\,dx_{\pi_\gamma(n-|\gamma|+l)}).
\end{aligned}
$$

A careful analysis shows that the inequality we get in such a way agrees with the inequality formulated in Part A of Theorem 6.1. Indeed, we get at the left-hand side of this inequality $\|\bar{\bar{h}}_\gamma\|$ with the norm formulated in Part A of Theorem 6.1, and the right-hand side equals the product $\|h_1\|\|h_2\|$. We got the same integrals as the integrals defining these norms, only we integrate by the variables of the functions h_1 and h_2 in a different order. We also have to exploit that the measures $G_{j,j}$ are symmetric, hence the value of the integrals

we are investigating does not change if we replace the coordinate x_k by $-x_k$ in the kernel function for certain coordinates k.

Next I turn to the proof of Part B of Theorem 6.1. First we prove this result, i.e. identity (6.17) in the special case when both h_1 and h_2 are simple functions. We may also assume that they are adapted to the same regular system

$$\mathcal{D} = \{\Delta_p,\ p = \pm 1, \pm 2, \dots, \pm N\},$$

and by a possible further division of the sets Δ_p we may also assume that the elements of $\mathcal{D}$ are very small. More explicitly, first we choose such a measure μ on $\mathbb{R}^\nu$ which has finite value on all compact sets, all complex measures $G_{k,l}$, $1 \le k, l \le d$ are absolutely continuous with respect to μ, and their Radon–Nikodym derivatives satisfy the inequality $|\frac{dG_{k,l}}{d\mu}(x)| \le 1$ for all $x \in \mathbb{R}^\nu$. Fix a small number $\varepsilon > 0$. We may achieve, by splitting up the sets Δ_p into smaller sets if it is necessary, that $\mu(\Delta_p) \le \varepsilon$ for all $\Delta_p \in \mathcal{D}$. Let us fix a number $u_p \in \Delta_p$ in all sets $\Delta_p \in \mathcal{D}$. We can express the product $I_n(h_1|j_1, \dots, j_n) I_m(h_2|j'_1, \dots, j'_m)$ as

$$I = I_n(h_1|j_1, \dots, j_n) I_m(h_2|j'_1, \dots, j'_m) = \sum{}' h_1(u_{p_1}, \dots, u_{p_n}) h_2(u_{q_1}, \dots, u_{q_m})$$
$$Z_{G,j_1}(\Delta_{p_1}) \cdots Z_{G,j_n}(\Delta_{p_n}) Z_{G,j'_1}(\Delta_{q_1}) \cdots Z_{G,j'_m}(\Delta_{q_m}).$$

The summation in the sum $\sum'$ goes through all pairs $((p_1, \dots, p_n), (q_1, \dots, q_m))$ such that $p_k,\ q_l \in \{\pm 1, \dots, \pm N\}$, $k = 1, \dots, n$, $l = 1, \dots, m$, and $p_k \ne \pm p_{\bar{k}}$, if $k \ne \bar{k}$, and $q_l \ne \pm q_{\bar{l}}$ if $l \ne \bar{l}$.

Write

$$I = \sum_{\gamma \in \Gamma} \sum{}^{\gamma} h_1(u_{p_1}, \dots, u_{p_n}) h_2(u_{q_1}, \dots, u_{q_m})$$
$$Z_{G,j_1}(\Delta_{p_1}) \cdots Z_{G,j_n}(\Delta_{p_n}) Z_{G,j'_1}(\Delta_{q_1}) \cdots Z_{G,j'_n}(\Delta_{q_m}).$$

where $\sum^{\gamma}$ contains those terms of $\sum'$ for which $p_k = q_l$ or $p_k = -q_l$ if the vertices $(1,k)$ and $(2,l)$ are connected in γ, and $p_k \ne \pm q_l$ if $(1,k)$ and $(2,l)$ are not connected in γ.

Let us introduce the notation

$$\Sigma^{\gamma} = \sum{}^{\gamma} h_1(u_{p_1}, \dots, u_{p_n}) h_2(u_{q_1}, \dots, u_{q_m})$$
$$Z_{G,j_1}(\Delta_{p_1}) \cdots Z_{G,j_n}(\Delta_{p_n}) Z_{G,j'_1}(\Delta_{q_1}) \cdots Z_{G,j'_n}(\Delta_{q_m}).$$

for all $\gamma \in \Gamma$.

We want to show that for small $\varepsilon > 0$ (where ε is an upper bound for the measure μ of the sets $D_p \in \mathcal{D}$) the expression Σ^{γ} is very close to

$$I_\gamma = I_{n+m-2|\gamma|}(\bar{\bar{h}}_\gamma | j_{v_1}, \dots, j_{v_{(n-|\gamma|}}, j'_{w_1}, \dots, j'_{w_{m-|\gamma|}}) \tag{6.20}$$

for all $\gamma \in \Gamma$. For this goal we make the decomposition $\Sigma^{\gamma} = \Sigma_1^{\gamma} + \Sigma_2^{\gamma}$ of Σ^{γ}

with

$$\Sigma_1^\gamma = \sum\nolimits^\gamma h_1(u_{p_1},\dots,u_{p_n})h_2(u_{q_1},\dots,u_{q_m}) \prod_{k\in A_1} Z_{G,j_k}(\Delta_{p_k}) \prod_{l\in A_2} Z_{G,j'_l}(\Delta_{q_l}) \cdot \prod_{(k,l)\in B} E\left(Z_{G,j_k}(\Delta_{p_k})Z_{G,j'_l}(\Delta_{q_l})\right)$$

and

$$\Sigma_2^\gamma = \Sigma^\gamma - \Sigma_1^\gamma,$$

where the sets A_1, A_2 and B were defined in formulas (6.6), (6.7) and (6.8).

It is not difficult to check that both Σ_1^γ and Σ_2^γ are real valued random variables. We want to show that Σ_1^γ is close to the random variable I_γ introduced in (6.20), while Σ_2^γ is a small error term. To understand the behaviour of Σ_1^γ observe that

$$E(Z_{G,j_k}(\Delta_{p_k})Z_{G,j'_l}(\Delta_{q_l}) = E(Z_{G,j_k}(\Delta_{p_k})\overline{Z_{G,j'_l}(-\Delta_{q_l})} = 0$$

if $\Delta_{p_k} = \Delta_{q_l}$ (and as a consequence if $\Delta_{p_k} \cap (-\Delta_{q_l}) = \emptyset$), and

$$E(Z_{G,j_k}(\Delta_{p_k})Z_{G,j'_l}(\Delta_{q_l}) = E(Z_{G,j_k}(\Delta_{p_k})\overline{Z_{G,j'_l}(-\Delta_{q_l})} = G_{j_k,j'_l}(\Delta_{p_k})$$

if $\Delta_{p_k} = -\Delta_{q_l}$. In the case $(k,l) \in B$ one of these possibilities happens.

These relations make possible to rewrite Σ_1^γ in a simpler form. It can be rewritten in the form of a Wiener–Itô integral of order $n+m-2|\gamma|$ with integration with respect to the random measure $\prod_{k\in A_1} Z_{G,j_k}(\,dx_k) \prod_{l\in A_2} Z_{G,j'_l}(\,dx_l)$, (where the sets A_1 and A_2 were defined in (6.6) and (6.7)). Then we can rewrite this integral, by reindexing its variables in a right way to an integral very similar to the Wiener–Itô integral (6.15) (with the same parameter γ). The difference between these two expressions is that the kernel function h'_γ of the Wiener–Itô integral expressing Σ_1^γ is slightly different from the kernel function $\bar{\bar{h}}_\gamma$ appearing in the other integral. The main difference between these two kernel functions is that there is a small set in the domain of integration where h'_γ disappears, while $\bar{\bar{h}}_\gamma$ may not disappear. But the two Wiener–Itô integrals are very close to each other. An adaptation of the argument in the proof of Theorem 5.3 in [9] shows that

$$E(\Sigma_1^\gamma - I_\gamma)^2 \le C\varepsilon$$

with an appropriate constant $C > 0$.

We also want to show that Σ_2^γ is a negligibly small error term. To get a good upper bound on $E(\Sigma_2^\gamma)^2$ we write it in the form

$$E(\Sigma_2^\gamma)^2 = \sum\nolimits_2^\gamma h_1(u_{p_1},\dots,u_{p_n})h_2(u_{q_1},\dots,u_{q_m}) h_1(u_{\bar{p}_1},\dots,u_{\bar{p}_n})h_2(u_{\bar{q}_1},\dots,u_{\bar{q}_m}) \Sigma_3^\gamma(p_k,q_l,p_{\bar{k}},q_{\bar{l}},\ k,\bar{k}\in\{1,\dots,n\},\ l,\bar{l}\in\{1,\dots,m\})$$

with

$$\Sigma_3^\gamma(p_k, q_l, p_{\bar k}, q_{\bar l},\ k, \bar k \in \{1, \dots, n\},\ l, \bar l \in \{1, \dots, m\})$$
$$= E\Bigg(\Bigg(\prod_{k \in A_1} Z_{G,j_k}(\Delta_{p_k}) \prod_{l \in A_2} Z_{G,j'_l}(\Delta_{q_l}) \prod_{\bar k \in A_1} Z_{G,j_{\bar k}}(\Delta_{p_{\bar k}}) \prod_{\bar l \in A_2} Z_{G,j'_{\bar l}}(\Delta_{q_{\bar l}}) \Bigg)$$
$$\left[\prod_{(k,l) \in B} Z_{G,j_k}(\Delta_{p_k}) Z_{G,j'_l}(\Delta_{q_l}) - E \prod_{(k,l) \in B} Z_{G,j_k}(\Delta_{p_k}) Z_{G,j'_l}(\Delta_{q_l}) \right]$$
$$\left[\prod_{(\bar k,\bar l) \in B} Z_{G,j_{\bar k}}(\Delta_{p_{\bar k}}) Z_{G,j'_{\bar l}}(\Delta_{q_{\bar l}}) - E \prod_{(\bar k,\bar l) \in B} Z_{G,j_{\bar k}}(\Delta_{p_{\bar k}}) Z_{G,j'_{\bar l}}(\Delta_{q_{\bar l}}) \right] \Bigg),$$

where we sum up in $\sum_2^\gamma$ for such sequences of indices p_k, q_l, $p_{\bar k}$, $q_{\bar l}$, $k, \bar k \in \{1, \dots, n\}$, $l, \bar l \in \{1, \dots, m\}$, $p_k, p_{\bar k}, q_l, q_{\bar l} \in \{\pm 1, \dots, \pm N\}$ which satisfy the following properties. For all indices $k, l, \bar k$ and $\bar l$, $p_k = q_l$ or $p_k = -q_l$ if $(k, l) \in B$, and similarly $p_{\bar k} = q_{\bar l}$ or $p_{\bar k} = -q_{\bar l}$ if $(\bar k, \bar l) \in B$. Otherwise all numbers $\pm p_k$ and $\pm q_l$ are different, and similarly otherwise all $\pm p_{\bar k}$ and $\pm q_{\bar l}$ are different.

We get a good estimate on $E(\Sigma_2^\gamma)^2$ by giving a good bound on all terms

$$\Sigma_3^\gamma(p_k, q_l, p_{\bar k}, q_{\bar l},\ k, \bar k \in \{1, \dots, n\},\ l, \bar l \in \{1, \dots, m\}) \tag{6.21}$$

in the formula expressing it. This can be done by adapting the corresponding argument in Theorem 5.3 of [9]. This argument shows that for most sets of parameters $p_k, q_k, p_{\bar k}, q_{\bar l}$ the term in (6.21) equals zero. More explicitly, it is equal to zero if $\mathcal{A} \neq -\bar{\mathcal{A}}$ with

$$\mathcal{A} = \{p_k \colon\ k \in A_1\} \cup \{q_l \colon\ l \in A_2\}, \quad \text{and} \quad \bar{\mathcal{A}} = \{p_{\bar k} \colon\ \bar k \in A_1\} \cup \{q_{\bar l} \colon\ \bar l \in A_2\},$$

and it also equals zero if $\mathcal{F} \cup (-\mathcal{F})$ and $\bar{\mathcal{F}} \cup (-\bar{\mathcal{F}})$ are disjoint, where

$$\mathcal{F} = \bigcup_{(k,l) \in B} \{p_k, q_l\} \quad \text{and} \quad \bar{\mathcal{F}} = \bigcup_{(\bar k,\bar l) \in B} \{p_{\bar k}, q_{\bar l}\}.$$

These statements can be proved by adapting the corresponding argument in Theorem 5.3 of [9]. More precisely, in the proof of the first statement we still need the following additional observation. If (X, Y, Z) is a three-dimensional Gaussian vector with $EX = EY = EZ = 0$, then $EXYZ = 0$. (In the proof of Theorem 5.3 in [9] we needed this statement only in a special case when it trivially holds.)

To prove this statement let us apply the following orthogonalization for the random variables X, Y and Z. Write $Y = \alpha X + \eta$, $Z = \beta_1 X + \beta_2 \eta + \zeta$, where X, η, ζ are orthogonal, (jointly) Gaussian random variables with expectation zero. Then they are also independent, hence $EXYZ = EX(\alpha X + \eta)(\beta_1 X + \beta_2 \eta + \zeta) = 0$.

In the remaining cases the expression in (6.21) can be estimated (again by adapting the argument of Theorem 5.3 in [9]) in the following way.

$$\Sigma_3^\gamma(p_k, q_l, p_{\bar{k}}, q_{\bar{l}},\ k, \bar{k} \in \{1, \dots, n\},\ l, \bar{l} \in \{1, \dots, m\})$$
$$\le C\varepsilon \prod{}' \mu(\Delta_{p_k})\mu(\Delta_{l_q})\mu(\Delta_{p_{\bar{k}}})\mu(\Delta_{q_{\bar{l}}})$$

with some constant C (not depending on ε) and the measure μ dominating the complex measures $G_{j,k}$ with the properties we demanded at the start of the proof. The sign $'$ in the product $\prod'$ means that first we take the sets Δ_{p_k}, Δ_{q_l}, $\Delta_{p_{\bar{k}}}$, $\Delta_{q_{\bar{l}}}$ for all parameters $k, \bar{k} \in \{1, \dots, n\}$ and $l, \bar{l} \in \{1, \dots, m\}$, then if a set Δ appears twice in the sequence of these sets we omit one of them. Then if both the sets Δ and $-\Delta$ appear for some set Δ, then we omit one of them from this sequence. Then we take in $\prod'$ the product of the terms $\mu(\Delta)$ with the sets Δ in the remaining sequence.

It can be proved with the help of the estimates on the terms in (6.21) (see again Theorem 5.3 in [9]) that

$$E(\Sigma_2^\gamma)^2 \le C\varepsilon.$$

It is not difficult to prove part B of Theorem 6.1 with the help of the estimates on $E(\Sigma_1^\gamma - I_\gamma)^2 \le C\varepsilon$ and $E(\Sigma_2^\gamma)^2 \le C\varepsilon$ if h_1 and h_2 are simple functions. One only has to make an appropriate limiting procedure with $\varepsilon \to 0$. Then we can complete the proof of Theorem 6.1 similarly to the proof of Theorem 5.3 in [9] by means of an appropriate approximation of Wiener–Itô integrals with Wiener–Itô integrals of simple functions. In this approximation we have to apply Lemma 5.1 and the properties of the Wiener–Itô integrals, in particular the already proved Part A of Theorem 6.1.

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