SHEAVES AND SCHEMES: AN INTRODUCTION TO ALGEBRAIC GEOMETRY

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Abstract

The purpose of this report is to serve as an introduction to the language of sheaves and schemes via algebraic geometry. The main objective is to use examples from algebraic geometry to motivate the utility of the perspective from sheaf and scheme theory. Basic facts and definitions will be provided, and a categorical approach will be frequently incorporated when appropriate.

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Chapter 1

1

1.1 Affine Varieties

Definition 1.1.1. *Let k be an algebraically closed field. We define $\boldsymbol{A^n}$ to be the set of all n-tuples of elements of k. This is known as the affine space over k.*

Consider $k[x_1, ..., x_n]$, the polynomial ring in n variables over k. Let $S \subseteq k[x_1, ..., x_n]$ be any subset. We can define the zero set of $S, Z(S)$, as follows:

$$Z(S) = \{a \in \boldsymbol{A^n} | f(a) = 0 \; \forall f \in S\}$$

Definition 1.1.2. *A subset B of $\boldsymbol{A^n}$ is an algebraic set if there exists a subset $S \subseteq k[x_1, ..., x_n]$ with $B = Z(S)$.*

It can be checked that algebraic sets are closed under finite union and arbitrary intersection and that the empty set and whole space are algebraic sets. Thus, we can define a topology on $\boldsymbol{A^n}$ by taking open sets to be complements of algebraic sets. This is known as the Zariski topology.

Example 1.1.1. *Note that if I is an ideal of $k[x_1, ..., x_n]$ generated by a set $S \subseteq k[x_1, ..., x_n]$, Z(S) = Z(I). In light of this, consider $k[x]$. Since $k[x]$ is a principal ideal domain, every* *algebraic set is simply Z(f) for some* $f \in k[x]$. *Since* k *is algebraically closed,* $f(x) = r(x-a_1)\cdots(x-a_m)$ *and so* $Z(f) = \{a_1, ..., a_m\}$.

Definition 1.1.3. *A nonempty subset* Y *of a topological space* X *is irreducible if it cannot be expressed as the union of two proper subsets, each of which is closed in* Y. *An affine variety is an irreducible closed subset of* $\boldsymbol{A^n}$ *with the induced topology. An open subset of an affine variety is called a quasi-affine variety.*

Let $Y \subseteq \boldsymbol{A^n}$. Define the ideal of Y in $k[x_1, ..., x_n]$ by:

$$I(Y) = \{f \in k[x_1, ..., x_n] | f(y) = 0 \ \forall y \in Y\}.$$

A useful way to view I and Z for our purposes is to regard them as inclusion-reversing maps which can be composed with another. In fact, it can be shown that the composition $I \circ Z$ sends ideals in $k[x_1, ..., x_n]$ to their radicals in $k[x_1, ..., x_n]$ (via the Nullstellensatz) and that the composition $Z \circ I$ sends subsets of $\boldsymbol{A^n}$ to their closures in $\boldsymbol{A^n}$ in the Zariski topology. Combining these observations with easily demonstrated properties of I and Z give us our first theorem.

Theorem 1.1.1. *There is a bijective correspondence between algebraic sets in* $\boldsymbol{A^n}$ *and radical ideals in* $k[x_1, ..., x_n]$. *Furthermore, an algebraic set is irreducible if and only if its ideal is a prime ideal.*

Example 1.1.2. *Let f be an irreducible polynomial in* $k[x, y]$. *Since* $k[x, y]$ *is a unique factorization domain, every irreducible element is prime, and therefore* (f)*, the ideal generated by f, is prime. By Theorem 1.1, Z(f) is irreducible. This is known as an affine curve.*

Example 1.1.3. *By inclusion-reversal, a maximal ideal of* $k[x_1, ..., x_n]$ *corresponds to a point of* $\boldsymbol{A^n}$.

To briefly comment on the topology of these affine varieties, we define an important property of a topological space.

Definition 1.1.4. *A topological space X is called Noetherian if it satisfies the descending chain condition for closed subsets. That is, given a sequence $Y_1 \supseteq Y_2 \supseteq \cdots$ of closed subsets Y_i of X, there is an integer r such that $Y_r = Y_{r+1} = \cdots$.*

By the fact that $k[x_1, ..., x_n]$ is a Noetherian ring, it can be deduced that $\boldsymbol{A^n}$ is a Noetherian topological space. In fact, through intrinsic properties of Noetherian topological spaces, it can be shown that every algebraic set in $\boldsymbol{A^n}$ can be expressed as a union of varieties.

1.2 Projective Varieties

We take a brief detour into the realm of projective geometry to elucidate the notion of a projective variety.

Definition 1.2.1. *Let V be a vector space of dimension $n+1$ over a field k. The space $\boldsymbol{P^n}$, known as the projective space of dimension n, is the set of 1-dimensional vector subspaces of V.*

To make this notion more explicit, note that the 1-dimensional vector subspaces of V are the sets of scalar multiples of a fixed vector $v \in V$ where $v \neq 0$. We may equivalently define $\boldsymbol{P^n}$ as the set of equivalence classes of $(n+1)$-tuples $(a_0, ..., a_n) \neq (0, ..., 0)$ under the equivalence relation $(a_0, ..., a_n) \sim (\lambda a_0, ..., \lambda a_n)$ for all $\lambda \in k, \lambda \neq 0$.

Now, consider the polynomial ring $A = k[x_0, ..., x_n]$. We can define a grading on $k[x_0, ..., x_n]$ by taking A_d to the be the set of all linear combinations of monomials of total degree d in the variables $x_0, ..., x_n$. If f is a homogeneous polynomial in $k[x_0, ..., x_n]$ we can define a function $f : \boldsymbol{P^n} \to \{0, 1\}$ by:

$$\begin{cases} f(p) = 0 & f(a_0, ..., a_n) = 0 \\ f(p) = 1 & f(a_0, ..., a_n) \neq 0 \end{cases}$$

Now, let S be a set of homogeneous elements of degree d, according to our grading. We can make a definition analogous to our zero set in the affine case:

$$Z(S) = \{p \in \boldsymbol{P^n} | f(p) = 0 \; \forall f \in S\}.$$

If Y $\subseteq \boldsymbol{P^n}$, we say Y is an algebraic set if there exists a set of homogeneous elements $S \subseteq k[x_0, ..., x_n]$ with $Y = Z(S)$. We can define a topology on $\boldsymbol{P^n}$, similarly to the affine case, by letting open sets be complements of algebraic sets. We will now see that our discussion of affine varieties has close analogs in the projective case.

Definition 1.2.2. *A projective algebraic variety is an irreducible algebraic set in $\boldsymbol{P^n}$. An open subset of a projective variety is a quasi-projective variety.*

If $Y \subseteq \boldsymbol{P^n}$, we define the homogeneous ideal of Y in $k[x_0, ..., x_n]$, $I(Y)$, to be the ideal generated by $\{f \in k[x_0, ..., x_n] | f \textit{ is homogeneous and } f(p) = 0 \; \forall p \in Y\}$. It turns out it is not difficult to use the Nullstellensatz to prove a homogeneous version of the Nullstellensatz. To this end, define the ideal $A_+ = \bigoplus_{d>0} A_d$. It can be shown, using the homogeneous Nullstellensatz and properties of the maps Z and I defined above, that we have an inclusion-reversing correspondence between algebraic sets in $\boldsymbol{P^n}$ and homogeneous radical ideals of $k[x_0, ..., x_n]$ not equal to A_+.

We can say more, however, about the relationship between projective and affine varieties. In fact, it turns out that every projective variety has an open covering by affine varieties. We make this notion more precise.

Consider $x_i \in k[x_0, ..., x_n]$. Let H_i denote the zero set of x_i. Note that $\boldsymbol{P^n}$ can be covered by the open sets $\boldsymbol{P^n - H_i}$.

Theorem 1.2.1. *Define a mapping $\phi_i : \boldsymbol{P^n - H_i} \to \boldsymbol{A^n}$ by $\phi_i(a_0, ..., a_n) = (\frac{a_0}{a_i}, ..., \frac{a_n}{a_i})$. This defines a homeomorphism of $\boldsymbol{P^n - H_i}$ to $\boldsymbol{A^n}$.*

In summary, we see that any projective variety can be covered by open sets homeomorphic to affine varieties.

1.3 Morphisms of Varieties

We are now in a position to discuss morphisms between varieties. Later, we will see that some of the theorems in this section can be generalized to a much broader context.

Definition 1.3.1. *Let $Y \subseteq \boldsymbol{A^n}$ be quasi-affine. A function $f : Y \to k$ is regular at point $y \in Y$ if there is an open subset U of Y with $y \in U$ and polynomials $g, h \in k[x_1, ..., x_n]$ such that $f = \frac{g}{h}$ on U and $h \neq 0 \; \forall u \in U$.*

An analogous definition can be made for quasi-projective varieties, and it can be shown that regular functions in both the affine and projective cases are continuous. Since affine and projective varieties are quasi-projective (technically we should think of $\boldsymbol{A^n}$ as embedded in $\boldsymbol{P^n}$, so that a closed irreducible subset of $\boldsymbol{A^n}$ becomes an open set of a closed irreducible subset of $\boldsymbol{P^n}$) we will simply refer to quasi-projective varieties as varieties.

Definition 1.3.2. *Let X, Y be two varieties. A morphism $\phi : X \to Y$ is a map satisfying the following conditions:*

i. ϕ is continuous in the Zariski topology.

ii. For every open subset $V \subseteq Y$ and every regular function $f : V \to k$, $f \circ \phi : \phi^{-1}(V) \to k$ is regular.

It can be checked that taking objects to be varieties and morphisms between them to be maps satisfying the conditions of Definition 1.3.2 give us a category.

To any variety, we can associate a ring of functions that we will see plays an important role in understanding morphisms between varieties.

Definition 1.3.3. *Let X be a variety. We denote by $\mathcal{O}(X)$ the ring of all regular functions on X. Let p be a point of X. We define the local ring of X at p to be:*

$$\mathcal{O}_{X,p} = \{f : V \to k | V \subseteq X \text{ is an open neighborhood of } p, \text{ and } f \text{ is regular on } V\} \text{ under the equivalence relation } \langle U_1, f\rangle \sim \langle U_2, g\rangle \text{ if } f = g \text{ on } U_1 \cap U_2.$$

$\mathcal{O}_{X,p}$ is actually a local ring. Its maximal ideal is the set of equivalence classes of regular functions vanishing at p.

Definition 1.3.4. *Let Y be a variety. Define $K(Y)$ to be:*

$$K(Y) = \{f : U \to k | U \subseteq Y \text{ is open, } f \text{ is regular on } U\} \text{ under the equivalence relation}$$
$$\langle U_1, f\rangle \sim \langle U_2, g\rangle \text{ if } f = g \text{ on } U_1 \cap U_2.$$

It can be shown that these regular functions under this equivalence relation define a field. The next theorems sow together the relationship between these rings we have defined.

Theorem 1.3.1. *Let $Y \subseteq \boldsymbol{A^n}$ be an affine algebraic set. Denote by $A(Y)$ the quotient $k[x_1, ..., x_n]/I(Y)$. There is an isomorphism $\phi : \mathcal{O}(Y) \to A(Y)$.*

$A(Y)$ is known as the affine coordinate ring and Theorem 1.3.1 essentially identifies this ring with the set of all regular functions on Y. In turns out this "global" identification can be made local in the sense of the following theorem.

Theorem 1.3.2. *Let $\mathfrak{m}_P \subseteq A(Y)$ be the ideal of functions vanishing at $p \in Y$. There is an isomorphism $\mathcal{O}_{X,p} \to A(Y)_{\mathfrak{m}_P}$ induced by ϕ, where $A(Y)_{\mathfrak{m}_P}$ is the localization of the coordinate ring at $\mathfrak{m}_P$.*

In the projective case, an analogous local statement can be made. However, if $Y \subseteq \boldsymbol{P^n}$ is a projective variety and $\boldsymbol{P^n}$ is the projective n-space over an algebraically closed field k, it can be seen that the ring of regular functions on Y can be identified with k.

Theorems 1.3.1 and 1.3.2 can be used to prove an important categorical statement relating an affine variety to regular functions defined on that variety.

Theorem 1.3.3. *The assignment $\mathcal{F} : X \to A(X)$ defines a contravariant, fully faithful, essentially surjective functor from the category of affine varieties over an algebraically closed field k and the category of finitely generated k-algebras.*

Thus, we can say there is an equivalence between the category of affine varieties over k and the category opposite to the category of finitely generated algebras over k.

1.4 Sheaves

Definition 1.4.1. *A presheaf $\mathcal{F}$ of abelian groups on a topological space X is an assignment $U \mapsto \mathcal{F}(U)$ of an abelian group $\mathcal{F}(U)$ to every open subset $U \subseteq X$, and a group homomorphism $\phi_{UU'} : \mathcal{F}(U) \to \mathcal{F}(U')$ for every inclusion $U' \subseteq U$ of open subsets, where the homomorphism is subject to the following conditions:*

i. $\mathcal{F}(\varnothing) = 0$.

ii. ϕ_{UU} *is the identity homomorphism.*

iii. The restriction mappings are compatible in the sense that if $U'' \subseteq U' \subseteq U$ are three open subsets of X, $\phi_{UU''} = \phi_{UU'} \circ \phi_{U'U''}$.

To be a sheaf, a presheaf has to additionally satisfy the following properties:

iv. For every open cover $\{U_i\}$ of an open set U and $s \in \mathcal{F}(U)$, $\phi_{UU_i}(s) = 0\ \forall i$ implies $s = 0$.

v. If $\{U_i\}$ is an open covering of an open set U, and if we have elements $s_i \in \mathcal{F}(U_i)$ for each i such that for each pair U_i, U_j, $\phi_{U_i,U_i \cap U_j}(s_i) = \phi_{U_j,U_i \cap U_j}(s_j)$, then there is an $s \in \mathcal{F}(U)$ such that $\phi_{UU_i}(s) = s_i\ \forall i$.

If $\mathcal{F}$ is a presheaf on a topological space X, and x is a point of X, the stalk of $\mathcal{F}$ at x, denoted $\mathcal{F}_x$ is:

$$\mathcal{F}_x := \varinjlim_{x \in U} \mathcal{F}(U)$$

where the order relation on the direct system is $U_1 < U_2$ if $U_1 \supseteq U_2$ for open sets U_1, U_2. Thus, an element of $\mathcal{F}_x$ is represented by a pair $\langle U, s \rangle$ where U is an open neighborhood of x and $s \in \mathcal{F}(U)$. Two pairs $\langle U, s \rangle$ and $\langle V, t \rangle$ define the same element of $\mathcal{F}_x$ if and only if there is an open neighborhood W of x with $W \subseteq U \cap V$ and $\phi_{UW}(s) = \phi_{VW}(t)$.

We can make sense of morphisms of presheaves in the following manner. A presheaf is just a contravariant functor from the category having objects as open sets of a topological space and morphisms as inclusions, to the category of abelian groups. A morphism of presheaves is just a natural transformation between two such functors. Similarly, sheaves can be viewed as such functors satisfying the supplementary conditions *vi* and *vii* of Definition 4, with morphisms as natural transformations. Presheaves on a topological space form a category, and sheaves on this topological space form a full subcategory.

A morphism of any two presheaves induces morphisms on stalks, but in the case the sheaf axioms are satisfied, we can say more.

Theorem 1.4.1. *A morphism of sheaves is an isomorphism if and only if the induced maps on the stalks are isomorphisms. An isomorphism of sheaves can just be interpreted as a natural isomorphism between two functors, as described above.*

This gives us a taste of the local nature of a sheaf. To gain a better understanding of sheaves, however, it is necessary to introduce the sheafification of a presheaf.

Theorem 1.4.2. *Let $\mathcal{F}$ be a presheaf on a topological space. Define*

$$\mathcal{F}^+(U) = \{s : U \to \bigcup_{p \in U} \mathcal{F}_p\}$$

subject to the following conditions

i. $s(p) \in \mathcal{F}_p \; \forall p \in U$.

ii. For every $u \in U$, there exists an open set $V \subseteq U$ with $u \in V$ and a $\sigma \in \mathcal{F}(V)$ such that $\forall v \in V \;\; s(v) = \langle V, \sigma \rangle$ in $\mathcal{F}_v$.

With the obvious restriction maps, $\mathcal{F}^+$ is a presheaf, and in fact satisfies the sheaf axioms. There is a canonical morphism of presheaves $i : \mathcal{F} \to \mathcal{F}^+$ defined as follows: given $s \in \mathcal{F}(U)$ let $i_U(s) \in \mathcal{F}^+(U)$ be defined as $i_U(s)(p) = \langle U, s \rangle \in \mathcal{F}_p$. Furthermore, $\mathcal{F}^+$ satisfies the following universal property: for any sheaf $\mathcal{G}$ and any morphism of presheaves $\phi : \mathcal{F} \to \mathcal{G}$ there is a unique morphism of sheaves $\psi : \mathcal{F}^+ \to G$ such that $\phi = \psi \circ i$.

One of the perks of "sheafifying" is its necessity in formulating notions of kernels and images of sheaves.

Definition 1.4.2. *If $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves, the assignment $U \mapsto ker(\phi(U))$ defines a sheaf, and is known as the kernel of ϕ. The assignment $U \mapsto im(\phi(U))$ defines a presheaf. The sheafification of this presheaf is known as the image of ϕ.*

We now want to convince the reader that the category of sheaves of abelian groups on a topological space X is an abelian category. It is not difficult to show that the category of presheaves of abelian groups on a topological space X is an abelian category. This combined with our notion of sheafification is enough to show that the category of sheaves on X is additive — for instance sheafification can be viewed as a left adjoint functor, thereby preserving finite colimits. What remains to show is the normality of monomorphisms and epimorphisms. For this, we need to define quotient sheaves and we need a concept of cokernel.

Definition 1.4.3. *Let $\mathcal{F}$ be a sheaf. A subsheaf $\mathcal{F}'$ of $\mathcal{F}$ is a sheaf satisfying the following conditions:*

i. For every open set $U \subseteq X$, $\mathcal{F}'(U)$ is a subgroup of $\mathcal{F}(U)$.

ii. If $V \subseteq U$ is open and $\phi : \mathcal{F}(U) \to \mathcal{F}(V)$ is the restriction mapping, then the restriction $\mathcal{F}'(U) \to \mathcal{F}'(V)$ is just ϕ with its domain restricted to $\mathcal{F}'(U)$.

The quotient sheaf $\mathcal{F}/\mathcal{F}'$ is the sheafification of the assignment $U \mapsto \mathcal{F}(U)/\mathcal{F}'(U)$. As for cokernels, if $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, the assignment $U \to coker(\phi(U))$ defines a presheaf, and its sheafification is known as the cokernel of ϕ. It can be shown that $coker(\phi) \cong \mathcal{G}/im(\phi)$ where $im(\phi)$ is the image of ϕ. Thus, giving suitable definitions of injective and surjective morphisms of sheaves (which are easy to define, but will not delve into), the axiom of an abelian category stipulating that monomorphisms be normal is equivalent to showing that if $\mathcal{F}'$ is a subsheaf of $\mathcal{F}$ the sequence:

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}/\mathcal{F}' \to 0$$

is exact. This can be shown by observing that taking kernels and cokernels behaves well when taking stalks, and that showing exactness of sequences of morphisms of sheaves is equivalent to showing exactness of the induced sequences on stalks. The second stipulation for an additive category to be abelian can be shown using a similar method.

1.5 Schemes

We can define a presheaf of rings on a topological space in a manner analogous to presheaves of abelian groups — it is a contravariant functor from the category of open sets of a topological space with morphisms as inclusions to the category of commutative rings. A presheaf of rings is a sheaf of rings if it is a sheaf as a presheaf of abelian groups.

We have many of the tools necessary to discuss the notion of a scheme.

Definition 1.5.1. *Let A be a commutative ring. We define the spectrum of A, or Spec A, to be the set of all prime ideals of A. If I is an ideal of A, we define $V(I)$ to be the set of all prime ideals containing I.*

One can check that a topology can be defined on Spec A by taking the closed sets to be sets of the form $V(I)$. We can define a sheaf of rings on Spec A with this topology. Let $U \subseteq$ Spec A be open. Define:

$$\mathcal{O}(U) := \{s : U \to \coprod_{p \in U} A_p\} \text{ with the functions } s \text{ subject to the following conditions:}$$

i. $s(p) \in A_p$

ii. For each $p \in U$ one can find an open set V with $V \subseteq U, p \in V$ and elements $a, b \in A$ such that $\forall v \in V\ s(v) = \frac{a}{b}$ with $b \notin v$ in A_v.

It can be checked that this assignment forms a sheaf. We shall see that it is in fact similar to the sheaf of regular functions defined on a variety. To make this notion more precise, we need to introduce several concepts.

Definition 1.5.2. *Let A be a commutative ring and $g \in A$. Define $D(g)$ to be the set of ideals of A not containing (g), the ideal in A generated by g.*

These sets from a basis of topology for Spec A.

Definition 1.5.3. *Let X be a topological space, and $\mathcal{F}$ a sheaf of abelian groups or commutative rings. Let $V \subseteq X$ be an open set. An element $s \in \mathcal{F}(V)$ is called a section over V.*

Let $Sh(X, \mathcal{C})$ denote the category of sheaves on X with values in $\mathcal{C}$ where $\mathcal{C}$ is the category of abelian groups or commutative rings. We can define a covariant functor $\Gamma(X, \_)$ from $Sh(X, \mathcal{C})$ to C via the assignment $\Gamma(X, \mathcal{F}) \coloneqq \mathcal{F}(X)$. This is known as the global sections functor.

We can now state two theorems similar to Theorems 1.3.1 and 1.3.2 for varieties.

Theorem 1.5.1. *Let A be a commutative ring and $\mathcal{O}$ the sheaf of rings on Spec A as defined above. Then, $\Gamma(\text{Spec } A, \mathcal{O}_{Spec\ A}) \cong A$.*

Theorem 1.5.2. *For any prime ideal $\mathfrak{q} \in Spec\ A$, $\mathcal{O}_{Spec\ A, \mathfrak{q}} \cong A_{\mathfrak{q}}$ where $A_{\mathfrak{q}}$ is the localization away from $\mathfrak{q}$.*

Definition 1.5.4. *Let X, Y be topological spaces and let $f : X \to Y$ be a continuous map. Let $U \subseteq Y$ be an open set, and $\mathcal{F}$ a sheaf on X. We can define a functor f_* from sheaves on X to sheaves on Y by the assignment*

$$f_*\mathcal{F}(U) \coloneqq \mathcal{F}(f^{-1}(U))$$

This is known as the direct image functor.

Definition 1.5.5. *Let X, Y be topological spaces and G a sheaf on Y. Let $f : X \to Y$ be a continuous map. Consider the assignment $U \mapsto \varinjlim_{V \supseteq f(U)} G(V)$*

where the order relation on the direct system is $V_2 < V_1$ if $V_1 \subseteq V_2$ and V runs over all open sets containing $f(U)$. This assigment defines a presheaf on X. Sheafifying gives us a sheaf on X and thus a functor f^{-1} from the category of sheaves on Y to the category of sheaves on X.

The functors f^{-1} and f_* form an adjoint pair, in the sense that there is a natural adjunction

$$Hom_{Sh(X)}(f^{-1}\mathcal{G}, \mathcal{F}) \cong Hom_{Sh(Y)}(\mathcal{G}, f_*\mathcal{F})$$

where $Sh(X)$ and $Sh(Y)$ denote, respectively, the categories of sheaves of abelian groups on X and Y.

Now, let X, Y be topological spaces and suppose there is a continuous map $f : X \to Y$. Let $\mathcal{F}$ be a sheaf of rings on X and $\mathcal{G}$ a sheaf of rings on Y. Given a point $p \in X$ any morphism of sheaves $\mathcal{G} \to f_*\mathcal{F}$ induces, for every open $U \subseteq Y$ with $f(p) \in U$, a ring homomorphism $\mathcal{G}(U) \to \mathcal{F}(f^{-1}(U))$. Taking direct limits we obtain a map $\mathcal{G}_{Y,f(p)} = \varinjlim_U \mathcal{G}(U) \to \varinjlim_U \mathcal{F}(f^{-1}(U))$. As U ranges over all open neighborhoods of $f(p)$, $f^{-1}(U)$ ranges over a subset of all open neighborhoods of p. Thus, we have a map $\varinjlim_V \mathcal{F}(f^{-1}(U) \to \mathcal{F}_{X,p}$ and an induced map $\mathcal{G}_{Y,f(p)} \to \mathcal{F}_{X,p}$.

We now have the groundwork necessary to introduce the notion of scheme.

Definition 1.5.6. *A ringed space is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of commutative rings on X. A morphism from a ringed space $(X, \mathcal{O}_X)$ to a ringed space $(Y, \mathcal{O}_Y)$ is a pair (f, ϕ) where $f : X \to Y$ is continuous and $\phi : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a morphism of sheaves. A locally ringed space is a ringed space $(X, \mathcal{O}_X)$ such that $\mathcal{O}_{X,p}$ is a local ring for every $p \in X$. A morphism of locally ringed spaces is a morphism of ringed spaces such that the induced morphism $\Phi : \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$ is a local ring homomorphism; that is, the image of the maximal ideal of $\mathcal{O}_{Y,f(p)}$ is contained in the maximal ideal of $\mathcal{O}_{X,p}$.*

Definition 1.5.7. *An affine scheme is a locally ringed space $(X, \mathcal{O}_X)$ that is isomorphic as a locally ringed space to $(Spec\ A, \mathcal{O}_{Spec\ A})$ for some ring A.*

Theorem 1.5.3. *For any ring A, the assignment $A \mapsto (Spec\ A, \mathcal{O}_{Spec\ A})$ defines an equivalence of the category of commutative rings and the the category opposite to the category of affine schemes.*

In fact, for rings A and B, any morphism $(Spec\ B, \mathcal{O}_{Spec\ B}) \to (Spec\ A, \mathcal{O}_{Spec\ A})$ is determined by a ring homomorphism $A \to B$. Thus, there is a contravariant equivalence of the categories of affine schemes and commutative rings. The following is a simple example of an affine scheme.

Example 1.5.1. *Let R be a principal ideal domain. Consider the polynomial ring $R[y]$ in one indeterminate. Using elementary abstract algebra, it can be shown that the prime ideals of $R[y]$ are precisely (0), the principal ideals $(f(y))$ where $f(y)$ is irreducible, and the* *maximal ideals. If k is algebraically closed, this fact characterizes Spec $k[x, y]$. Since in the Zariski topology maximal ideals correspond to closed points, the Nullstellensatz tells us the closed points of Spec $k[x, y]$ are just ordered pairs of elements of k. Now, suppose $f(x, y)$ is irreducible in $k[x, y]$. Then, $(f(x, y))$ is prime. Since the only primes lying above $(f(x, y))$ are maximal ideals of the form $(x - a, y - b)$, we can see that $(x - a, y - b) \supseteq (f(x, y))$ if and only if $f(x, y) = 0$. Thus, viewing $(f(x, y))$ as a point of Spec $k[x, y]$ we see that its closure consists of $(f(x, y))$ along with all closed points (a, b) for which $f(a, b) = 0$. Finally, since $k[x, y]$ is an integral domain, the zero ideal (0) corresponds to a point of Spec $k[x, y]$ and its closure is the entire space. This is known as the generic point.*

Let A be the polynomial ring $k[x_0, ..., x_n]$ with the grading defined in Section 2. Define Proj A to be the set of all homogeneous prime ideals $\mathfrak{p}$ which do not contain all of A_+ (as defined earlier). Letting I be a homogeneous ideal of A, we can define:

$$V(I) = \{\mathfrak{p} \in Proj\ A | \mathfrak{p} \supseteq I\}.$$

As in the affine case, we can define a topology on A by taking closed sets to be sets of the form $V(I)$. With slightly more care than in the affine case, we can define a sheaf of rings on Proj A. We will not do this explicitly.

Definition 1.5.8. *A scheme is a locally ringed space $(X, \mathcal{O}_X)$ such that for every $p \in X$ there exists an open set $U \subseteq X$ with $p \in U$ so that $(U, \mathcal{O}_X|_U)$ is an affine scheme.*

Definition 1.5.9. *Let S be a fixed scheme. A scheme over S is a scheme X equipped with a morphism $p : X \to S$. If Y is a scheme over S with a morphism $q : Y \to S$, then we say a morphism of schemes $\phi : X \to Y$ is an S-morphism if $p = q \circ \phi$. The category comprised of schemes over S as objects and S-morphisms is known as the category of schemes over S.*

Theorem 1.5.4. *For an algebraically closed field k, the category of varieties over a field k embeds into the category of schemes over k. That is, there is a fully faithful functor from the category of varieties over k to schemes over k.*

For any variety, its topological space is homeomorphic to the set of closed points of the topological space of the image of the variety under the functor. The sheaf of regular functions of the variety is obtained by restricting the structure sheaf of the image to the set of closed points.

1.6 Sheaves of Modules

Definition 1.6.1. *Let $(X, \mathcal{O}_X)$ be a ringed space. A sheaf $\mathcal{F}$ of abelian groups on X is an $\mathcal{O}_X$-module if for every open set U of X, the group $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module and the restriction maps $\mathcal{F}(U) \to \mathcal{F}(V)$ are compatible with the restriction maps $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$: for every $r \in \mathcal{O}_X(U)$ and $s \in \mathcal{F}(U)$, $(r \cdot s)|_V = r|_V \cdot s|_V \in \mathcal{O}_X(V)$. A morphism of $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}$ is a morphism of sheaves $\phi : \mathcal{F} \to \mathcal{G}$ such that for each open set $U \subseteq X$, the morphism $\mathcal{F}(U) \to \mathcal{G}(U)$ is an $\mathcal{O}_X(U)$ module homomorphism.*

Using a similar approach to the case of sheaves of abelian groups on a topological space X, one can show that $\mathcal{O}_X$ modules form an abelian category.

Let $\mathcal{F}$ and $\mathcal{G}$ be two $\mathcal{O}_X$ modules. The assignment $U \to \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$ defines a presheaf. Taking the sheafification gives us an $\mathcal{O}_X$ module $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ known as the tensor product of $\mathcal{F}$ and $\mathcal{G}$.

Let $f : (\mathcal{O}_X, X) \to (\mathcal{O}_Y, Y)$ is a morphism of locally ringed spaces. It can be shown that if $\mathcal{F}$ is an $\mathcal{O}_X$-module, $f_*\mathcal{F}$ is an $f_*\mathcal{O}_X$ module. Since the morphism f gives us a morphism of sheaves on Y $\phi : \mathcal{O}_Y \to f_*\mathcal{O}_X$ we can define, for any open set $V \subseteq Y$ an action of $\mathcal{O}_Y(V)$ on $f_*\mathcal{F}(V)$ that is compatible with the restriction homomorphisms of $f_*\mathcal{F}$. Thus, $f_*\mathcal{F}$ has an $\mathcal{O}_Y$-module structure. If $\mathcal{G}$ is an $\mathcal{O}_Y$ module, then $f^{-1}\mathcal{G}$ is an $f^{-1}\mathcal{O}_Y$-module. By the natural adjunction of Section 1.5, we have a morphism $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ and thus for every open $U \subseteq X$ we can define an action of $f^{-1}\mathcal{O}_Y(U)$ on $\mathcal{O}_X(U)$. Furthermore, this action is compatible with the restriction morphisms of $\mathcal{O}_X$. Thus $\mathcal{O}_X$ is an $f^{-1}\mathcal{O}_Y$-module, and we can make the following definition: $f^*\mathcal{G} := f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$. f^* and f_* form an adjoint pair.

Definition 1.6.2. *Let R be a ring, and M an R-module. For each prime ideal $\mathfrak{p} \subseteq R$, let $M_\mathfrak{p}$*

be the localization of M away from $\mathfrak{p}$. *Let* $U \subseteq$ *Spec R. We make the following definition:*

$$\tilde{M}(U) = \{s : U \to \coprod_{\mathfrak{p} \in U} M_{\mathfrak{p}}\} \text{ subject to the following conditions:}$$

i. For each $\mathfrak{p} \in U$, $s(\mathfrak{p}) \in M_{\mathfrak{p}}$

ii. For each $\mathfrak{p} \in U$ *there exists an open set* $V \subseteq U$ *with* $\mathfrak{p} \in V$ *and elements* $m \in M$ *and* $r \in R$ *such that for each* $\mathfrak{q} \in V$ $s(\mathfrak{q}) = \frac{m}{r}$ *where* $r \notin \mathfrak{q}$.

This assignment defines a sheaf on Spec R. $\tilde{M}$ *is known as the sheaf associated to* M.

Definition 1.6.3. *Let* $(X, \mathcal{O}_X)$ *be a scheme. An* $\mathcal{O}_X$*-module* $\mathcal{F}$ *is quasi-coherent if the restriction of* $\mathcal{F}$ *to each open affine* $U \subseteq X$, *is isomorphic to* $\tilde{M}$ *for some R-module M.*

Many results in algebraic geometry are formulated in terms of quasi-coherent sheaves.

Bibliography

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