# EXISTENCE AND STABILITY OF MULTIPLE SPOT SOLUTIONS FOR THE GRAY-SCOTT MODEL IN $R^2$

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ABSTRACT. We study the Gray-Scott model in a bounded two dimensional domain and establish the existence and stability of **symmetric** and **asymmetric** multiple spotty patterns. The Green's function and its derivatives together with two nonlocal eigenvalue problems both play a major role in the analysis. For symmetric spots, we establish a threshold behavior for stability: If a certain inequality for the parameters holds then we get stability, otherwise we get instability of multiple spot solutions. For asymmetric spots, we show that they can be stable within a narrow parameter range.

## 1. INTRODUCTION: SELF-REPLICATING SPOTS

We study the existence and stability of multiple spotty patterns in the two-dimensional Gray-Scott model. The Gray-Scott system, introduced in [9], [10], models an irreversible reaction involving two reactants in a gel reactor, where the reactor is maintained in contact with a reservoir of one of the two chemicals in the recation. In nondimensional variables, it can be written as

$$(GS) \qquad \begin{cases} V_t = D_V \Delta V - (F+k)V + UV^2, \ x \in \Omega, \, t > 0 \\ U_t = D_U \Delta U - UV^2 + F(1-U), \ x \in \Omega, \, t > 0 \\ \frac{\partial V}{\partial t} = \frac{\partial U}{\partial t} = 0 \text{ on } \partial\Omega, \end{cases}$$

where $D_U > 0, D_U > 0$ are the two diffusivities, $F$ denotes the feed rate, $k > 0$ is a reaction-time constant, and $\Omega \subset R^N$, $N \leq 3$ is the container. For various ranges of these parameters, (GS) are known to admit a rich solution structure involving pulse, spots,, rings, stripes, traveling waves, pulse-replication pattern, and spatio-temporal chaos. See [21], [22], [23], [14], [15] for numerical simulations and experimental observations.

Some important analytic work is the following, first for the case of 1-D: single and multiple pulse solutions [7], stability [5], [6], stability index [4], slowly modulated two-pulse solutions [2], [3], dynamics of pulses (formal) [22], [23], skeleton structure, spatiotemporal chaos [18], dynamics [8], case of equal diffusivities [11], [12], scattering and separators [19], [20], symmetric and asymmetric patterns, Hopf bifurcation, pulse-splitting in a bounded interval [24].

In higher dimensions there are the following results: formal approach in 2-D and 3-D [16], shadow system in higher dimensions [26], ground state in 2-D [27], bounded domain case in 2-D, symmetric and asymmetric multiple spots, which is the basis for this paper [28], [29].

Let us first rescale the system (GS). Set

$$\epsilon^2 = \frac{D_V}{F+k}, \ D = \frac{D_U}{F}, \ A = \frac{\sqrt{F}}{F+k}, \ \tau = \frac{F+k}{F},$$

$$x = \sqrt{\frac{D_U}{F}}\bar{x}, \ t = \frac{1}{F+k}\bar{t}, \ V(x,t) = \sqrt{F}v(\bar{x},\bar{t}), \ U(x,t) = u(\bar{x},\bar{t}).$$

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Let us drop the bar from now on. Then (GS) is equivalent to

$$(GS1) \qquad \begin{cases} v_t = \epsilon^2 \Delta v - v + Auv^2, \quad x \in \Omega, t > 0 \\ \tau u_t = D\Delta u - uv^2 + (1-u), \ x \in \Omega, t > 0 \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Throughout this paper, we assume that $\epsilon << 1, D = D(\epsilon) \to \infty$ as $\epsilon \to 0$, $A > 0$ may depend on $\epsilon$, $\tau > 1$ does not depend on $\epsilon$, $\Omega \subset R^2$ is a bounded and smooth domain.

We define two important parameters:

$$\eta_\epsilon = \frac{|\Omega|}{2\pi D} \log \frac{\sqrt{|\Omega|}}{\epsilon}, \ L_\epsilon = \frac{\epsilon^2 \int_{R^2} w^2\, dy}{A^2|\Omega|}.$$

Let us assume that

$$\lim_{\epsilon \to 0} L_\epsilon = L_0 \in [0, +\infty], \ \lim_{\epsilon \to 0} \eta_\epsilon = \eta_0 \in [0, +\infty].$$

All our results will be stated in terms of (the real constants) $L_0$, $\eta_0$, and $\tau$.

Let $w$ be the unique solution of the following problem (ground state):

$$(G) \quad \Delta w - w + w^2 = 0, w > 0 \ \text{ in } R^2, \ w(0) = \max_{y \in R^2} w(y), \ w(y) \to 0 \text{ as } |y| \to +\infty.$$

The uniqueness of the solution to (G) was proved [13]. We shall prove the existence and stability of steady-state solutions to (GS1) of the following shape:

$$v_\epsilon \sim \sum_{j=1}^K \frac{1}{A\xi_j^\epsilon} w(\frac{x - P_j^\epsilon}{\epsilon}), \ u_\epsilon(P_j^\epsilon) \sim \xi_j^\epsilon,$$

where $K$ is the number of spots, $P_j^\epsilon \in \Omega, j = 1, ..., K$ is the location of the spots, $\frac{1}{A\xi_j^\epsilon}$, $j = 1, ..., K$ is the amplitude of the spots, $w$ is the shape of the spots.

We call the steady state **"$K-$ symmetric spots"** if the amplitudes of the spots are asymptotically the same in the leading order, i.e.,

$$\lim_{\epsilon \to 0} \frac{\xi_i^\epsilon}{\xi_1^\epsilon} = 1, \quad \text{for all } \ i = 2, ..., N.$$

Otherwise, we call it **"$K-$asymmetric spots"**, i.e., if

$$\lim_{\epsilon \to 0} \frac{\xi_i^\epsilon}{\xi_1^\epsilon} \neq 1, \quad \text{for some } \ i = 2, ..., N.$$

## 2. Existence and Stability of Multiple Symmetric Spots

The result on the existence of multiple symmetric spots is the following

**Theorem 2.1.** *Suppose that*

$$(T1) \qquad 4(\eta_0 + K)L_0 < 1$$

*and*

$$(T2) \qquad \frac{(2\eta_0 + K)^2}{\eta_0} L_0 \neq 1.$$

*Then, for $\epsilon$ sufficiently small and $D$ not too large, problem (GS1) has two steady-state solutions $(v_\epsilon^\pm, u_\epsilon^\pm)$ with the following properties:*

*(1) $v_\epsilon^\pm(x) = \sum_{j=1}^K \frac{1}{A\xi_\epsilon^\pm}(w(\frac{x-P_j^\epsilon}{\epsilon}) + o(1))$ uniformly for $x \in \bar{\Omega}$. Here*

$$\xi_\epsilon^\pm = \frac{1 \pm \sqrt{1 - 4(\eta_0 + K)L_0}}{2}.$$

*(2)* $u_\epsilon^\pm(x) = \xi_\epsilon^\pm(1 + o(1)$ *uniformly for* $x \in \bar{\Omega}$.
*(3)* $P_j^\epsilon \to P_j^0$ *as* $\epsilon \to 0$ *for* $j = 1, ..., K$.

*The locations of the spots are determined by using a Green's function and its derivatives as follows: Define* $G_0(x, y)$ *by*

$$\Delta G_0(x,y) - \frac{1}{|\Omega|} + \delta(x-y) = 0, \quad x \in \Omega, \ \frac{\partial G_0(x,y)}{\partial \nu_x} = 0, \quad x \in \partial\Omega,$$

*where* $y \in \Omega$. *Set* $H_0(x, y) = \frac{1}{2\pi} \log \frac{1}{|x-y|} - G_0(x, y)$.
*For* $K \in N$ *and* $\mathbf{P} = (P_1, ..., P_K) \in \Omega^K$ *with* $P_j \neq P_l$ *for* $j \neq l$ *we define*

$$F_0(P_1, ..., P_K) = \sum_{i=1}^{K} H_0(P_i, P_i) - \sum_{j \neq l} G_0(P_j, P_l).$$

*Then, if* $\mathbf{P^0} = (P_1^0, ..., P_K^0)$ *is a nondegenerate critical point of* $F_0$, *the solutions exist.*

We call $(v_\epsilon^-, u_\epsilon^-)$ the **small** solution and $(v_\epsilon^+, u_\epsilon^+)$ the **large** solution.

Next we consider the stability of such solutions

**Theorem 2.2.** *Assume that (T1) and (T2) hold. Let* $\mathbf{P^0} = (P_1^0, ..., P_K^0) \in \Omega^K$ *be a nondegenerate local maximum point of* $F_0$. *Let* $(v_\epsilon^\pm, u_\epsilon^\pm)$ *be the* $K-$*spot solutions constructed in Theorem 2.1.*

*Then, for* $\epsilon$ *sufficiently small and* $D$ *not too large, the solution* $(v_\epsilon^+, u_\epsilon^+)$ *is* **linearly unstable** *for all* $\tau \geq 0$. *For the* **small** *solutions the following holds:*

**Case 1.** $\eta_\epsilon \to 0$.

*If* $K = 1$, *there exists a* **unique** $\tau_1 > 0$ *such that for* $\tau < \tau_1$, $(v_\epsilon^-, u_\epsilon^-)$ *is* **linearly stable**, *while for* $\tau > \tau_1$, $(v_\epsilon^-, u_\epsilon^-)$ *is* **linearly unstable**.

*If* $K > 1$, $(u_\epsilon^-, v_\epsilon^-)$ *is* **linearly unstable** *for any* $\tau \geq 0$.

**Case 2.** $\eta_\epsilon \to +\infty$.

*Then* $(v_\epsilon^-, u_\epsilon^-)$ *is* **linearly stable** *for any* $\tau \geq 0$.

**Case 3.** $\eta_\epsilon \to \eta_0 \in (0, +\infty)$, *(i.e.,* $D \sim \log \frac{1}{\epsilon}$*).*

*If* $L_0 < \frac{\eta_0}{(2\eta_0+K)^2}$, *then* $(u_\epsilon^-, v_\epsilon^-)$ *is* **linearly stable** *for* $\tau$ *small enough or* $\tau$ *large enough.*

*If* $K = 1$, $L_0 > \frac{\eta_0}{(2\eta_0+1)^2}$, *there exist* $\tau_2 > 0$, $\tau_3 > 0$ *such that* $(v_\epsilon^-, v_\epsilon^-)$ *is* **linearly stable** *for* $\tau < \tau_2$ *and* **linearly unstable** *for* $\tau > \tau_3$.

*If* $K > 1$ *and* $L_0 > \frac{\eta_0}{(2\eta_0+K)^2}$, *then* $(v_\epsilon^-, u_\epsilon^-)$ *is* **linearly unstable** *for any* $\tau \geq 0$ .

## 3. Existence and Stability of $K$-Asymmetric Spots

Our first theorem shows that the asymmetric patterns exist only in a **narrow parameter range**.

**Theorem 3.1** *Asymmetric patterns can exist only if* $\lim_{\epsilon \to 0} \eta_\epsilon = \eta_0 \in (0, +\infty)$. *In other words,* $D \sim C \log \frac{1}{\epsilon}$.

Our next theorem shows that the asymmetric patterns are **generated by exactly two types of spots**.

**Theorem 3.2** *The asymmetric solutions is generated by exactly two kinds of spots, called type* **A** *and type* **B**, *respectively, which differ by their amplitudes.*

The following theorem gives the existence of $K-$asymmetric spots.

**Theorem 3.3.** *Fix any two integers $k_1 \geq 1, k_2 \geq 1$ such that $k_1 + k_2 = K \geq 2$. If*

$$L_0 \leq \frac{\eta_0}{4(\eta_0 + k_1)(\eta_0 + k_2)},$$

*there are asymmetric $K-$spotty solutions with $k_1$ type* **A** *spots and $k_2$ type* **B** *spots. The locations of these $K$ spots are determined by a Green's function which depends on the number $k_1, k_2$ of the spots.*

The last theorem classifies the **stability** of asymmetric patterns

**Theorem 3.4.**

*(1) (***stability***) The $K$-asymmetric spots are stable if*

$$\frac{\eta_0}{(2\eta_0 + K)^2} < L_0 \leq \frac{\eta_0}{4(\eta_0 + k_1)(\eta_0 + k_2)}$$

*and $\tau$ small.*

*(2) (***Instability***) Assume that*

$$L_0 < \frac{\eta_0}{(2\eta_0 + K)^2}$$

*or*

$$\tau \text{ is large enough.}$$

*Then the $K$-asymmetric spots are unstable.*

## 4. Main Steps in the Existence Proof

The existence of $K$ symmetric and asymmetric spots is obtained by the **Liapunov-Schmidt** reduction method.

We calculate the **equations for the amplitudes** by solving the second equation of (GS2) with a suitable Green's function and expanding this Green's function (here $G_0$ is needed). Assuming asymptotically that

$$\lim_{\epsilon \to 0} \xi_{\epsilon,j} = \xi_j, \quad j = 1, ..., K,$$

we obtain the following system of algebraic equations

$$1 - \xi_i - \frac{\eta_0 L_0}{\xi_i} = \sum_{j=1}^{K} \frac{L_0}{\xi_j}, \quad i = 1, ..., K.$$

In the **symmetric case**, i.e., $\xi_1 = ... = \xi_K$, we have

$$\xi_1 = ... = \xi_K = \xi,$$

where $\xi$ satisfies the quadratic equation

$$1 - \xi - \frac{\eta_0 L_0}{\xi} = \frac{K L_0}{\xi}.$$

This gives **two branches** of solutions.

In the **asymmetric case**, it follows by an elementary argument that (assuming w.l.o.g. that $\xi_2 \neq \xi_1$) for $\xi_j$, $j = 3, \ldots, K$, we have either $\xi_j = \xi_1$ or $\xi_j = \xi_2$. This shows that asymmetric patterns are generated by **exactly two types** of spots.

Let $k_1$ be the number of $\xi_1$'s in $\{\xi_1, \ldots, \xi_K\}$ and $k_2$ the number of $\xi_2$'s in $\{\xi_1, \ldots, \xi_K\}$. Then $\xi_1$ must satisfy

$$1 - \xi_1 = \frac{(k_1 + \eta_0) L_0}{\xi_1} + \frac{k_2}{\eta_0} \xi_1$$

and therefore

$$(k_2 + \eta_0)\xi_1^2 - \eta_0 \xi_1 + (k_1 + \eta_0)\eta_0 L_0 = 0,$$

which has a solution if and only if

$$\eta_0 \geq 4(k_1 + \eta_0)(k_2 + \eta_0)L_0.$$

Thus we have determined the amplitudes of the spots. Now we have to glue the spots together in $\Omega$. This is be done by the **Liapunov-Schmidt** reduction process and a **fixed point theorem** argument.

## 5. Stability Proof: Systems of NLEPs

To study the stability of $K$-spots, we consider two cases: **Large eigenvalues** $\lambda_\epsilon \to \lambda_0$ and **small eigenvalues** $\lambda_\epsilon \to 0$. The small eigenvalues are related to the domain geometry via the Green's function $G_0$.

The analysis of the large eigenvalues gives us the **critical threshold**.

By some lengthy asymptotic analysis, we arrive at the following **system of nonlocal eigenvalue problems (NLEP-system)**:

$$\Delta\Phi - \Phi + 2w\Phi - 2\mathcal{B}\frac{\int_{R^2} w\Phi}{\int_{R^2} w^2} w^2 = \lambda_0 \Phi, \qquad \Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_K \end{pmatrix} \in (H^2(R^2))^K,$$

where

$$\mathcal{B} = L_0 \left(\mathcal{F} + \frac{L_0}{1+\tau\lambda_0}\mathcal{E}\right)^{-1} \left(\eta_0 \mathcal{I} + \frac{1}{1+\tau\lambda_0}\mathcal{E}\right),$$

$$\mathcal{F} = \begin{pmatrix} \xi_1^2 + L_0\eta_0 & & \\ & \ddots & \\ & & \xi_K^2 + L_0\eta_0 \end{pmatrix}, \; \mathcal{E} = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \vdots & \vdots \\ 1 & \cdots & 1 \end{pmatrix},$$

and $\mathcal{I}$ is the identity matrix.

Diagonalizing the matrix $\mathcal{B}$, we are lead to the study of the following single NLEPs :

$$\text{(NLEP)} \qquad \Delta\phi - \phi + 2w\phi - 2\mu_i(\tau\lambda_0)\frac{\int_{R^2} w\phi}{\int_{R^2} w^2} w^2 = \lambda_0\phi, \; i = 1, 2, \; \phi \in H^2(R^2),$$

where $\mu(z)$ is an analytic function of $z = \tau\lambda_0$.

We need a key result about (NLEP) from [25]: if $\tau = 0$, then (NLEP) is stable if $2\mu_i(0) > 1$. To prove instability, we first show that (NLEP) admits a positive real eigenvalue under the condition $2\mu_i(0) < 1$ and then apply a compactness argument of Dancer [1] to show the original eigenvalue problem has also a positive eigenvalue provided that $\epsilon$ is sufficiently small.

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