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Construction of Dual Cyclic Codes over $\mathbb{F}_2[u,v]/\langle u^2, v^2-v, uv-vu\rangle$ for DNA Computation

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ABSTRACT

Here, we assume the construction of cyclic codes over $\Re = \mathbb{F}_2[u,v]/\langle u^2, v^2-v, uv-vu\rangle$. In particular, dual cyclic codes over $\Re_1 = \mathbb{F}_2[u]/\langle u^2\rangle$ with respect to Euclidean inner product are discussed. The cyclic dual codes over $\Re$ are studied with respect to DNA codes (reverse and reverse complement). Many interesting results are obtained. Some examples are also provided, which explain the main results. The GC-Content and DNA codes over $\Re$ are discussed. We summarise the article by giving a special DNA table.

Keywords: Dual cyclic codes; DNA cyclic codes; Reverse constraint codes; Reverse constraint complement codes; The GC content

1. INTRODUCTION

Since last 30 years, cyclic codes have been well studied due to their rich algebraic structures. Cyclic codes also have practical implementations in DNA computing. The cyclic codes constructions over of 16 element have great interest and these are an extension of cyclic codes over the rings of 4 element. Due to which, many authors attracted to study the rings of 16 element in a series of papers[1-3]. In particular, Yildiz[2], *et al.* assumed the ring $\Re$ with $u^2 = u$ *and* $v^2 = v$ of 16 elements, where they studied the cyclic codes over such ring. Recently, Gao[4], *et al.* discussed various categories of linear codes over the ring $\mathbb{Z}_4 + v\mathbb{Z}_4$ of 16 elements.

In 1987, Tom Haed introduced the computing by DNA and Adleman first time used the DNA computation to discuss the Hamiltonian path problem[5]. DNA computing is better than silicon-based computing because of their storage capacity, which attracted several authors to study the cyclic DNA codes over some rings. In particular, Guenda[6], *et al.* considered the structure of $\Re_1$ of 4 element. They discussed the construction of cyclic codes over such ring for DNA computing. Bennenni[7], *et al.* discussed the new DNA cyclic codes over ring. Later, Zhu[8], *et al.* discussed the cyclic DNA codes over $\Re$ and explored the application in DNA computing. Recently, ring $\mathbb{Z}_4[u]/\langle u^2 -1\rangle$ of 16 element is considered[9], where they stabilised the theory for DNA construction. They also explained the GC-content of these codes on the basis of deletion distance. The cyclic codes over the ring $\mathbb{Z}_4 + u\mathbb{Z}_4$ and its DNA computing were also studied. Further, Dinh[10], *et al.* considered the ring of 64 element and discussed the DNA codes by applying the Chinese Remainder Theorem. These works encourage us to study the DNA dual cyclic codes for our purpose.

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2. PRELIMINARIES

Here, we remind some basic facts and results, which will be used throughout the paper. Let η be the gray map, which maps the elements of $\Re$ to the elements of $\Re_1^n$ and is given as,

$$\eta(a+bv) = (a, a+b),$$

where $a, b \in \Re_1$.

Let two n-tuples h and $k \in \Re^n$. The Euclidean inner product is given as $h.k = h_0k_0 + h_1k_1 + ... + h_{n-1}k_{n-1}$. For the code C over $\Re$ of length n, the dual code of C is as given by $C^{\perp} = \{h \in \Re^n \mid h.k = 0, \ \forall\ k \in C\}$.

The set $S_{D_4} = \{A, T, G, C\}$ represents the DNA alphabet. The elements of the ring $\Re_1$ are given as $0, 1, u, 1+u$. Here, we can easily see that the elements of S_{D_4} and $\Re_1$ are related by $\underline{0}$ to A, $\underline{1}$ to G, $\underline{u}$ to $\underline{T}$, and $\underline{1+u}$ to $\underline{C}$. By WCC rule, $\overline{A} = T$, $\overline{T} = A$, $\overline{G} = C$ and $\overline{C} = G$. The set $S_{D_{16}}$ is given below, which is taken from[1].

$$S_{D_{16}} = \begin{Bmatrix} AA, & AT, & AC, & AG, & TT, & TA, & TC, & TG, \\ CC, & CA, & CT, & CG, & GG, & GA, & GT, & GC \end{Bmatrix}$$

The elements of the set $S_{D_{16}}$ are known as DNA double pairs. We define a map $\psi : C \rightarrow S_{D_4}^{2n}$, such that

$$(a_0 + b_0v,\ a_1 + b_1v,\ \ldots,\ a_{n-1} + b_{n-1}v)$$
$$\rightarrow (a_0, a_1, \ldots, a_{n-1}, a_0 + b_0, a_1 + b_1, \ldots, a_{n-1} + b_{n-1}),$$

where $a_i, b_i \in \Re_1$. Let $x = (x_0, x_1, ..., x_{n-2}, x_{n-1}) \in \Re^n$. We define the reverse of x to be $x^r = (x_{n-1}, x_{n-2}, ..., x_1, x_0)$, the complement of x to be $x^c = \overline{x}_0, \overline{x}_1, ..., \overline{x}_{n-2}, \overline{x}_{n-1}$ and the reverse-complement of x to be $x^c = \overline{x}_0, \overline{x}_1, ..., \overline{x}_{n-2}, \overline{x}_{n-1}$. A code C is reversible, if each codeword $x \in C$, x^r is also in C and C is called reversible complement if $x^{rc} \in C$, $\forall x \in C$. The reciprocal of a polynomial $c(x) = c_0 + c_1 x + ... + c_{r-2} x^{r-1} + c_r x^r$ with $c_r \neq 0$ is defined as as polynomial $c^*(x) = c_r + c_{r-1} x + ... + c_1 x^{r-1} + c_0 x^r$. If $c^*(x) = c(x)$, then $c(x)$ is known as self-reciprocal polynomial. The number of places in which the DNA codeword has coordinate C or G is known as GC-content.

3. DUAL CODES OVER $\Re_1$

Here, we study the generator polynomials for the dual cyclic codes over $\Re_1$.

Theorem 3.1[11] Let C be a cyclic code in $\Re_{1,n} = \Re_1[x]/(x^n - 1)$. Then

(i). If n be an odd no, then $c = (g, ua) = (g + ua)$, where g and a is polynomials over $\mathbb{F}_2$. Then dual of code C is $C^\perp = (u\tilde{h})$, and $\tilde{h} \mid (x^n - 1)$.

(ii). If n is even, then, let $g = a$, then, $c = (g + ua)$ where $g \mid (x^n - 1)$ mod 2 and $(g + up) \mid (x^n - 1)$ in $\Re$ and $g \mid p\hat{g}$. Then the dual of C is $C^\perp = \left(\left(\frac{x^n - 1}{g}\right)^* + ux^i (m_2)^*\right)$ with $p\left(\frac{x^n - 1}{g}\right) = gm_2$ and

$i = \deg\left(\frac{x^n - 1}{g}\right) - \deg(m_2)$ where $\left(\frac{x^n - 1}{g}\right)^* \mid (x^n - 1)$ mod 2 and

$\left(\left(\frac{x^n - 1}{g}\right)^* + ux^i (m_2)^*\right) \mid (x^n - 1)$ in $\Re$ and $\left(\frac{x^n - 1}{g}\right)^* \mid \left(x^i (m_2)^*\right)\left(\frac{x^n - 1}{g}\right)^*$.

(iii). Let $g \neq a$, then $C = (g + up, ua)$, where g, a, p are polynomials over $\mathbb{F}_2$ with $a \mid g \mid (x^n - 1)$ mod 2, $a \mid p\hat{g}$, where $\hat{g} = \left(\frac{x^n - 1}{g}\right)$ and $\deg p \leq \deg a$. Then the dual of C,

$C^\perp = \left(\left(\frac{x^n - 1}{a}\right)^* + ux^i (m_2)^*, u\left(\frac{x^n - 1}{g}\right)^*\right)$ with $p\left(\frac{x^n - 1}{g}\right) = gm_2$,

$i = \deg\left(\frac{x^n - 1}{g}\right) - \deg(m_2)$, where $\left(\frac{x^n - 1}{g}\right)^*$, $\left(\frac{x^n - 1}{a}\right)^*$, and $x^i (m_2)^*$ are polynomials over $\mathbb{F}_2$ with

$\left(\frac{x^n - 1}{g}\right)^* \mid \left(\frac{x^n - 1}{a}\right)^* \mid (x^n - 1)$ mod 2, $\left(\frac{x^n - 1}{g}\right)^* \mid x^i (m_2)^* \left(\frac{x^n - 1}{a}\right)^*$

and $\deg\left(x^i (m_2)^*\right) \leq \deg\left(\frac{x^n - 1}{g}\right)^*$.

Proof: Proof is similar to paper[13] [Theorem 4].

4. DNA CODES

We mainly study the dual DNA cyclic codes over $\Re$ by using the generators of dual cyclic codes over $\Re_1$ in the present section. First discuss the reverse constraint codes over $\Re$. For this purpose, some useful lemmas are as given, which are easily verified by examples.

Lemma 4.1[10] Let h, k be any two polynomials over with $\deg h \leq \deg k$. Then

(i) $(h.k)^* = h^* k^*$

(ii) $(h + k)^* = h^* + x^{\deg h - \deg k} k^*$.

Lemma 4.2[12] Let C = (f) be a cyclic code over $\mathbb{F}_2$, where f is a monic polynomial. Then C is a reversible if and only if f is a self-reciprocal polynomial over $\mathbb{F}_2$.

Lemma 4.3 For odd length n, let $C^\perp = (u\tilde{h})$ be a cyclic code over $\Re_1$, where $\tilde{h} = \left(\frac{x^n - 1}{g}\right)^*$. Then necessary and sufficient for reversible of $C^\perp$ is $\tilde{h}$ is self – reciprocal polynomial.

Example 4.1 Let $g = (x - 1)(x^3 + x + 1)$ and hence $\tilde{h} = (x^3 + x^2 + 1)$ be a polynomial in $x^7 - 1$ over $\mathbb{F}_2$, It is easy to see that $\tilde{h}$ is self-reciprocal. Since $C^\perp = (u\tilde{h})$, therefore $C^\perp$ is reversible code.

Lemma 4.4 For even length n, let $C^\perp = \left(\left(\frac{x^n - 1}{g}\right)^* + ux^i (m_2)^*\right)$ be a cyclic code over $\Re_1$. Then necessary and sufficient conditions for reversible of $C^\perp$ are

(1). $\left(\frac{x^n - 1}{g}\right)^*$ is self-reciprocal.

(2). **(i).** $x^j \left(x^i (m_2)^*\right)^* = x^i (m_2)^*$ or

(ii). $\left(\frac{x^n - 1}{g}\right)^* = x^j \left(x^i (m_2)^*\right)^* + x^i (m_2)^*$, where

$j = \deg\left(\frac{x^n - 1}{g}\right)^* - \deg\left(x^i (m_2)^*\right)$.

Example 4.2 Let $x^{10} - 1 = g_1^2 g_2^2 = (x + 1)^2 (x^4 + x^3 + x^2 + x + 1)^2$ over $\mathbb{F}_2$. Let cyclic code $C = (g_1 g_2^2 + ug_2^2 C_0)$ and $C^\perp = \left(\left(\frac{x^n - 1}{g}\right)^* + ux^i (m_2)^*\right)$. Hence, we have $\left(\frac{x^n - 1}{g}\right)^* = g_1,\ m_2 = C_0$ and $x^i (m_2)^* = xC_0$.

$x^j \left(x^i (m_2)^*\right)^* = x^i (m_2)^*$, where $j = \deg\left(\frac{x^n - 1}{g}\right)^* - \deg\left(x^i (m_2)^*\right) = 0.$

Hence, $C^\perp$ is reversible.

Lemma 4.5 For even length n, Let $C^{\perp} = \left(\left(\frac{x^n-1}{a}\right)^* + ux^i(m_2)^*, u\left(\frac{x^n-1}{g}\right)^*\right)$ be a cyclic code over $\Re_1$. Then necessary and sufficient conditions for reversible of $C^{\perp}$ are

(1). $\left(\frac{x^n-1}{g}\right)^*, \left(\frac{x^n-1}{a}\right)^*$ are self-reciprocal.

(2). $\left(\frac{x^n-1}{g}\right)^* \mid \left[x^j\left(x^i(m_2)^*\right)^* + x^i(m_2)^*\right]$,

where $j = \deg\left(\frac{x^n-1}{g}\right)^* - \deg\left(x^i(m_2)^*\right)$.

Example 4.3 Let $x^{10}-1 = g_1^2 g_2^2 = (x+1)^2 (x^4+x^3+x^2+x+1)^2$ over $\mathbb{F}_2$. Let cyclic code and $C^{\perp} = \left(\left(\frac{x^n-1}{a}\right)^* + ux^i(m_2)^*, u\left(\frac{x^n-1}{g}\right)^*\right)$. We have $\left(\frac{x^n-1}{a}\right)^* = g_1g_2$, $\left(\frac{x^n-1}{g}\right)^* = g_1$, $m_2 = 1$ and $x^i(m_2)^* = x$. We can easily check that $\left(\frac{x^n-1}{g}\right)^*$, $\left(\frac{x^n-1}{a}\right)^*$ are self-reciprocal and $\left(\frac{x^n-1}{g}\right)^* \mid \left[x^j\left(x^i(m_2)^*\right)^* + x^i(m_2)^*\right]$, where $j = \deg\left(\frac{x^n-1}{g}\right)^* - \deg\left(x^i(m_2)^*\right) = 4$, hence, follows the result.

Theorem 4.6 For general length n, let $C^{\perp} = vC_1^{\perp} \oplus (1+v)C_2^{\perp}$, be a cyclic code over $\Re$, where $C_1^{\perp}$ and $C_2^{\perp}$ are cyclic codes over $\Re_1$. Then $C^{\perp}$ is reversible if and only if $C_1^{\perp}$ and $C_2^{\perp}$ are reversible cyclic codes respectively

Proof First, we consider $C_1^{\perp}$ and $C_2^{\perp}$ are reversible, which means $(C_1^{\perp})^r \in C_1^{\perp}$ and $(C_2^{\perp})^r \in C_2^{\perp}$ and $d = vd_1 + (1+v)d_2$. Hence, $d^r = vd_1^r + (1+v)d_2^r \in C^{\perp}$ where $d_1 \in C_1^{\perp}$ and $d_2 \in C_2^{\perp}$ .. It is easy to see that $d_1^r \in C_1^{\perp}$ and $d_2^r \in C_2^{\perp}$, thus $d^r = vd_1^r + (1+v)d_2^r \in C^{\perp}$. Therefore the dual of cyclic code $C^{\perp}$ is reversible.

Conversely, if $C^{\perp}$ is reversible, then for any $b_1 \in C_1^{\perp}$, $b_2 \in C_2^{\perp}$, we have $b = vb_1 + (1+v)b_2 \in C^{\perp}$ Therefore $b^r = vb_1^r + (1+v)b_2^r \in C^{\perp}$.Let, where $e_1 \in C_1^{\perp}$ and $e_2 \in C_2^{\perp}$. Then, . Thus, we get $b_1^r = e_1 \in C_1^{\perp}$ and $b_2^r = e_2 \in C_2^{\perp}$. Hence, both $C_1^{\perp}$ and $C_2^{\perp}$ are reversible.

Example 4.4 Let $\tilde{h}_1 = (x^{13}+x^{12}+x^{10}+x^9+x^7+x^6+x^4+x^3+x+1)$, $\tilde{h}_2 = (x^{11}+x^{10}+x^6+x^5+x+1)$ be self-reciprocal polynomials in $x^{15}-1$.

Since $C^{\perp} = \langle g \rangle = \left(v\tilde{h}_1 + (1+v)\tilde{h}_2\right) = vx^{13}+vx^{12}+(1+v)x^{11}+x^{10}+vx^9+vx^7+x^6+(1+v)x^5+vx^4+vx^3+x+1$, and $g^r = x^{13}+x^{12}+vx^{10}+vx^9+(1+v)x^8+x^7+vx^6+vx^4+x^3+(1+v)x^2+vx+v$.

On the other hand

$(v+(1+v)x^2)g = x^{13}+x^{12}+vx^{10}+vx^9+(1+v)x^8+x^7+vx^6+vx^4+x^3+(1+v)x^2+vx+v = g^r$ in $C^{\perp}$. Which means the cyclic code $C^{\perp}$ is reversible.

Example 4.5 Let $\tilde{h}_1 = (x^6+x^3+1)$ and $\tilde{h}_2 = (x^3+1)$ be two self-reciprocal polynomials in x^9-1. As $C^{\perp} = \langle g \rangle = \left(v\tilde{h}_1 + (1+v)\tilde{h}_2\right) = 1+x^3+vx^6$, and $g^r = x^6+x^3+v$. An another way we can obtain as $(v+(1+v)x^3)g = x^6+x^3+v = g^r \in C^{\perp}$. Therefore, $C^{\perp}$ is reversible cyclic code.

Next, the reverse-complement codes over $\Re$ are discussed. Some lemmas are taken from paper[8], which are given below.

Lemma 4.7 $d + \bar{d} = u$, $d \in \Re$.

Lemma 4.8 Let $d, e \in \Re$, then $\overline{d+e} = \bar{d} + \bar{e} + u$.

Lemma 4.9 If $d \in \mathbb{F}_2$, then we get $u + u\bar{d} = ud$.

Using all these Lemmas, we describe the next result.

Theorem 4.10 Let $C^{\perp} = vC_1^{\perp} + (1+v)C_2^{\perp}$ be a cyclic code over $\Re$. Then, $c^{\perp}$ is reversible-compliment if and only if $c^{\perp}$ is reversible and $(\bar{0},\bar{0},\ldots,\bar{0}) \in C^{\perp}$.

Proof Suppose that $C^{\perp}$ is a cyclic code over. For any $c = (c_{n-1}, c_{n-2},\ldots, c_1, c_0) \in C^{\perp}$, $c^{rc} = (\bar{c}_{n-1}, \bar{c}_{n-2},\ldots, \bar{c}_1, \bar{c}_0) \in C^{\perp}$ as $C^{\perp}$ is reverse compliment. It is well known that $0 \in C^{\perp}$, its compliment is also in $c^{\perp}$, then $(\bar{0},\bar{0},\ldots,\bar{0}) \in C^{\perp}$. Whence

$$\begin{aligned} c^r &= (c_0, c_1, \ldots, c_{n-2}, c_{n-1}) \\ &= (\bar{c}_0, \bar{c}_1, \ldots, \bar{c}_{n-2}, \bar{c}_{n-1}) \\ &+ (\bar{0}, \bar{0}, \ldots, \bar{0}, \bar{0}) \in C^{\perp} \end{aligned}.$$

On the other side, the cyclic code $c^{\perp}$ is reversible, which means for any $c \in C^{\perp}$, then c^r is in $C^{\perp}$. Since, $(\bar{0},\bar{0},\ldots,\bar{0}) \in C^{\perp}$, thus $c^{rc} = (\bar{c}_{n-1}, \bar{c}_{n-2},\ldots, \bar{c}_1, \bar{c}_0) = (c_{n-1}, c_{n-2},\ldots, c_1, c_0) + (\bar{0}, \bar{0}, \ldots, \bar{0}, \bar{0}) \in C^{\perp}$. Then, cyclic code $c^{\perp}$ is reversible-complement.

5. THE GC WEIGHT

In present section, we discuss the construction of GC weight over $\Re$. Therefore, some results are given below, which will be used in main result.

Lemma 5.1 Let n be an odd number, and C be a cyclic codes over $\Re_1$, then $C = (g, ua) = (g+ua)$, where g and a are polynomials over $\mathbb{F}_2$. Then dual of code C is

$C^{\perp} = (u\tilde{h})$, with rank $n - \deg(\tilde{h})$ and $\mathbb{F}_2$- basis is given by $\{u\,\tilde{h}, u\,x\,\tilde{h},..., u\,x^{n-\deg\tilde{h}-1}\tilde{h}\}$.

Lemma 5.2 Let C be a cyclic codes of even length over $\Re_1$.

(1). If $g = a$, then $C = (g + up)$, with $p\left(\frac{x^n - 1}{g}\right) = gm_2$, we have, $C^{\perp} = \left(\left(\frac{x^n-1}{g}\right)^* + ux^i(m_2)^*\right)$ with $p\left(\frac{x^n-1}{a}\right) = gm_2$ and $i=\deg\left(\frac{x^n-1}{g}\right) - \deg(m_2)$. Let $\left(\frac{x^n-1}{g}\right)^* = \tilde{h}$, then $C^{\perp} = \left(\tilde{h} + ux^i(m_2)^*\right)$ has the rank $n - \deg(\tilde{h})$ and $\mathbb{F}_2$ - basis is given as $\{(\tilde{h} + ux^i(m_2)^*), x(\tilde{h} + ux^i(m_2)^*), ..., x^{n-\deg\tilde{h}-1}(\tilde{h} + ux^i(m_2)^*), u\tilde{h}, ux\tilde{h},..., ux^{n-\deg\tilde{h}-1}\tilde{h}\}$.

(2) Let $C = (g + up, ua)$, with $p\left(\frac{x^n-1}{g}\right) = gm_2$, $p\left(\frac{x^n-1}{g}\right) = gm_2, C^{\perp} = \left(\left(\frac{x^n-1}{a}\right)^* + ux^i(m_2)^*, u\left(\frac{x^n-1}{g}\right)^*\right)$ where $i = \deg\left(\frac{x^n-1}{g}\right) - \deg(m_2)$. Let $\left(\frac{x^n-1}{a}\right)^* = \tilde{h}$ and $\left(\frac{x^n-1}{g}\right)^* = \tilde{r}$, then $C^{\perp} = \left(\tilde{h} + ux^i(m_2)^*, u\tilde{r}\right)$ is of the rank $n - \deg\tilde{r}$ and $\mathbb{F}_2$ – basis is $\{(\tilde{h} + ux^i(m_2)^*), x(\tilde{h} + ux^i(m_2)^*), ..., x^{n-\deg\tilde{h}-1}(\tilde{h} + ux^i(m_2)^*), u\tilde{h}, ux\tilde{h},..., ux^{n-\deg\tilde{h}-1}\tilde{h}, u\tilde{r}, ux\tilde{r},..., ux^{\deg\tilde{h}-\deg\tilde{r}-1}\tilde{r}\}$.

Next, we discuss the minimally generating set of $C^{\perp}$ with the help of all $\mathbb{F}_2$-basis.

Theorem 5.3 Suppose $C^{\perp} = vC_1^{\perp} + (1+v)C_2^{\perp}$ is a cyclic code of over $\Re$. Then $C^{\perp}$ has a minimally generating set $\Delta = v\pi + (1+v)\theta$, where π and θ are minimally generating set of $C_1^{\perp}$ and $C_2^{\perp}$, respectively.

Let $\psi(\Delta) = x^n\pi + \theta$, where π and θ are minimally generating set of $C_1^{\perp}$ and $C_2^{\perp}$, respectively. Next, result explains the GC-content.

Theorem 5.4 For general length n, let $C^{\perp} = vC_1^{\perp} + (1+v)C_2^{\perp}$ be a cyclic code of over $\Re$ and $C_1^{\perp} = \left(\tilde{h}_1 + ux^i(m_{12})^*, u\tilde{r}_1\right), C_2^{\perp} = \left(\tilde{h}_2 + ux^i(m_{22})^*, u\tilde{r}_2\right)$, with $\tilde{r}_1 \mid \bar{h}_1 \mid (x^n-1), \tilde{r}_2 \mid \bar{h}_2 \mid (x^n-1)$ and we have $\deg(m_{12}) \le \tilde{r}_1, \deg(m_{22}) \le \tilde{r}_2$. Then hamming weight calculator of $\chi = x^n\{\tilde{h}_1, x\,\tilde{h}_1,..., x^{n-\deg\tilde{h}_1-1}\tilde{h}_1\} + \{\tilde{h}_2, x\,\tilde{h}_2,..., x^{n-\deg\tilde{h}_2-1}\tilde{h}_2\}$ gives the GC weight over $\Re$.

Proof Using the fact that the GC-content of $C^{\perp}$ is the u times of $\psi(\chi)$. Using above Theorem,

$$u\psi(x) = u\,x^n\{\tilde{h}_1, x\,\tilde{h}_1,..., x^{n-\deg\tilde{h}_1-1}\tilde{h}_1\} + u\{\tilde{h}_2, x\,\tilde{h}_2,..., x^{n-\deg\tilde{h}_2-1}\tilde{h}_2\}.$$

Hence GC weight is obtained as the Hamming weight of

$$\chi = x^n\{\tilde{h}_1, x\,\tilde{h}_1,..., x^{n-\deg\tilde{h}_1-1}\tilde{h}_1\} + \{\tilde{h}_2, x\,\tilde{h}_2,..., x^{n-\deg\tilde{h}_2-1}\tilde{h}_2\}.$$

6. DNA CODES OVER $\Re$

Definition Let $\tilde{h}_1(x), \tilde{h}_2(x)$ be two polynomials, where $\tilde{h}_i = ((x^n-1)/f_i)^*$ and $f_i \in \Re_1$ with i = 1,2 and both dividing $x^n - 1$ over $\Re_1$. Let $\deg\tilde{h}_1(x) = t_1, \deg\tilde{h}_2(x) = t_2$, $k = \min\{n - t_1, n - t_2\}$ and we have $g = v\tilde{h}_1(x) + (1+v)\tilde{h}_2(x)$. $L(g)$ is said to be ρ-set and defined as $L(g) = \{\varepsilon_0, \varepsilon_1, ..., \varepsilon_{k-1}, \xi_0, \xi_1,..., \xi_{k-1}\}$, where $\varepsilon(i) = x^i g$, $\xi(i) = x^i\rho(h)$, $0 \le i \le k-1$ and $h = vx^{t_2-t_1}\tilde{h}_1 + (1+v)\tilde{h}_2$, if $t_1 \le t_2$, $h = v\tilde{h}_1 + (1+v)x^{t_1-t_2}\tilde{h}_2$, otherwise. Let $C^{\perp} = \langle g\rangle_{\tilde{n}}$ be the linear code over $\Re$. Note that $\langle L(g)\rangle$ or $\langle g\rangle_{\tilde{n}}$ is the $\Re$-module. Let $g = \alpha_0 + \alpha_1 x + ... + \alpha_t x^t$ over $\Re$, $\rho(h) = \beta_0 + \beta_1 x + ... + \beta_s x^s$ and the set $L(g)$ is obtained by generator matrix

$$L(g) = \begin{bmatrix} \varepsilon_0 \\ \xi_0 \\ \varepsilon_1 \\ \xi_1 \\ . \\ . \\ . \end{bmatrix} = \begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 & ... & \alpha_t & ... & 0 & ... & ... & 0 \\ \beta_0 & \beta_1 & \beta_2 & ... & ... & ... & \beta_s & ... & ... & 0 \\ 0 & \alpha_0 & \alpha_1 & ... & ... & \alpha_t & ... & ... & ... & 0 \\ 0 & \beta_0 & \beta_1 & ... & ... & ... & ... & \beta_s & ... & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & ... & ... & ... & ... \end{bmatrix}$$

Theorem 6.1 Let $\tilde{h}_1$ and $\tilde{h}_2$ be self-reciprocal polynomials dividing x^n-1 over $\Re_1$ with degree t_1 and t_2. If $\tilde{h}_1 \ne \tilde{h}_2$, then $g = v\tilde{h}_1 + (1+v)\tilde{h}_2$ and $|\langle L(g)\rangle| = 16^k$ where, $k = \min\{n - t_1, n - t_2\}$. Also, $C^{\perp} = \langle L(g)\rangle$ is linear code over $\Re$ and $\psi(C^{\perp})$ is reversible DNA code.

Proof Already, we have discussed algebraic structures, which make proof complete. Here, we notice that the reverse of $C^{\perp} = \langle L(g)\rangle$ is given as $\left(\psi\left(\sum\theta_i\varepsilon_i + \sum\gamma_i\xi_i\right)\right)^r = \psi\left(\sum\rho(\theta_i)\xi_{k-1-i} + \sum\rho(\gamma_i)\xi_{k-1-i}\right)$, where $\theta_i, \gamma_i \in \Re$ and $0 \le i \le k-1$.

Example 6.1 Let $\tilde{h}_1 = x^4 + x^3 + x^2 + x + 1, \tilde{h}_2 = x + 1$, where both divides $x^5 - 1$ over $\mathbb{F}_2$. Hence, $g = v\tilde{h}_1 + (1+v)\tilde{h}_2 = vx^4 + vx^3 + vx^2 + x + 1$, $h = x^4 + x^3 + vx^2 + vx + v$ and we get $\rho(h) = x^4 + x^3 + (1+v)x^2 + (1+v)x + (1+v)$. Thus, $C^{\perp} = \langle L(g)\rangle$ and $\psi(C^{\perp})$ satisfy the reverse constraint. The generator matrix is defined as,

$$L(g) = \begin{bmatrix} \varepsilon_0 \\ \xi_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & v & v & v \\ 1+v & 1+v & 1+v & 1 & 1 \end{bmatrix}$$

Let $\theta_0 = v$ *and* $\gamma_0 = v + u$, and $\theta_0\varepsilon_0 + \gamma_0\xi_0 = (v + u + uv) + (v + u + uv)x + (v + u + uv)x^2 + ux^3 + ux^4$ $C^{\perp} = (v + u + uv, v + u + uv, v + u + uv, u, u)$. Hence, $\psi(C^{\perp}) = (TTTTTGGGTT)$. Again, $\psi(\theta_0)\xi_0 + \psi(\gamma_0)\varepsilon_0 = u + ux + (1 + v + uv)x^2 + (1 + v + uv)x^3 + (1 + v + uv)x^4$ and $C_2^{\perp} = (u, u, 1 + v + uv, 1 + v + uv, 1 + v + uv)$ *and* $\psi(C_2^{\perp}) = (TTGGGTTTTT)$. Hence, we have $(\psi(C_1^{\perp}))^r = \psi(C_2^{\perp})$. Therefore, $\psi(C^{\perp})$ is reversible.

Corollary 6.2 Consider the code $C^{\perp} = vC_1^{\perp} \oplus (1+v)C_2^{\perp}$, where $C_1^{\perp}$ and $C_2^{\perp}$ are reversible code and $C^{\perp} = \langle L(g)\rangle$ be a linear code over $\Re$. If $(\bar{0}, \bar{0}, ..., \bar{0}) \in C^{\perp}$, then $\psi(C^{\perp})$ gives reversible-compliment code.

Example 6.2 Let the polynomial $\tilde{h}_1$ and $\tilde{h}_2$, where both divide $x^7 - 1$ over $\mathbb{F}_2$. Thus we have the polynomial $g = v\tilde{h}_1 + (1+v)\tilde{h}_2 = \langle 1 + x + vx^2 + vx^3 + vx^4 + vx^5 + vx^6 \rangle_{\tilde{n}}$. Here, we can see that $(\bar{0}, \bar{0}, ..., \bar{0}) \in C^{\perp}$, thus follows the result.

Corollary 6.3 Let $C^{\perp} = vC_1^{\perp} \oplus (1+v)C_2^{\perp}$ is a cyclic code over $\Re$, $C_1^{\perp}$ and $C_2^{\perp}$ are reversible code and $C^{\perp} = \langle L(g)\rangle$ be a linear code over $\Re$ and $\psi(C^{\perp})$ is a reversible-DNA code. If we add compliment of 0 vector to $L(g)$ then $\psi(C^{\perp})$ satisfies the reversible-compliment constraint.

Corollary 6.4 Let $C_1^{\perp}$ be a reversible and $g = v\tilde{h}_1 + (1+v)\tilde{h}_1$ over $\Re$. Then $C^{\perp} = (g)$ is a reversible cyclic code over $\Re$ and $\psi(C^{\perp})$ is a reversible DNA code. If $x - 1$ does not divide g, then $\psi(C^{\perp})$ follows the reversible-compliment constraint.

Proof Rows of generator matrix of $C^{\perp}$ are given as $g, xg, ..., x^{t-1}g$, where t is dimension of $C^{\perp}$. Using the ρ - set $\langle L(g)\rangle$, we get $\left(\psi\left(\sum_i \theta_i x^i g\right)\right)^r = \psi\left(\sum_i \rho(\theta_i) x^{t-1-i} g\right)$ with $\theta_i \in \Re$ and $0 \le i \le t-1$, thus $C^{\perp}$ is reversible code. Since $\tilde{n}$ does not affect the coefficients, so, we can use ρ - set $\langle L(g)\rangle$ as linear code. Dual code $C^{\perp}$ contains $1 + x + x^2 + ... + x^{n-1}$ as $x - 1$ does not divide g. Thus reversible-complement DNA code of $\psi(C^{\perp})$ is obtained from corollary 6.2.

Example 6.3 Let $\tilde{h}_1 = 1 + x + x^3 + x^4 = \tilde{h}_2$ be polynomial, then $C^{\perp} = (g)$ is reversible code over $\Re$. Using the paper[13] [Theorem, 2.6], we get parameter [6, 2, 4]. In this example, we get 256 DNA code words in the decimal form in Table 1. For example, 985685 represent CCAAGGTTTT.

Table 1. DNA Correspondence of $C^{\perp} = (g)$ explained in Example 6.3

0	325	650	975	1331200	1331525	1331850
1332175	2662400	2663050	2662725	2663375	3993600	3993925
3994250	3994575	1300	1105	1950	1755	1332500
1332305	1333150	1334125	2663700	2663505	2664350	2664155
3994900	3994705	3995550	3995355	2600	2925	2210
2535	1333800	1334125	1333410	1333735	2665000	2665325
2664610	2664935	3996200	3996525	3995810	3996135	3900
3705	3510	3315	1335100	1334905	1334710	1334515
2666300	2666105	2665910	2665715	3997305	3997110	3996915
3997500	5324800	5325125	5325450	5325775	4526080	4526405
4526730	4527055	7987200	7987525	7987850	7988175	7188480
7188805	7189130	7189455	5326100	5325905	5326750	5326555
4527380	4527185	4528030	4527835	7988500	7988305	7989150
7988955	7189780	7189585	7190430	7190235	5327400	5327725
5327010	5327335	4528680	4529005	4528290	4528615	7989800
7990125	7989410	7989735	7191080	7191405	7190690	7191015
5328700	5328505	5328310	5328115	4529980	4529785	4529590
4529395	7991100	7990905	7990710	7990515	7192380	7192185
7191990	7191795	5324800	5325125	5325450	5325775	11980800
11981125	11981450	11981775	9052160	9052485	9052810	9053135
10383360	10383685	10384010	10384335	10650900	10650705	10651550
10651355	11982100	11981905	11982750	11982555	9053460	9053265
9054110	9053915	10384660	10384465	10385310	10385115	10652200
10652525	10651810	10652135	11983400	11983725	11983010	11983335
9054760	9055085	9054370	9054695	10385960	10386285	10385570
10385895	10653500	10653305	10653110	10652915	11984700	11984505
11984310	11984115	9056060	9055865	9055670	9055475	10387260
10387065	10386870	10386675	15974400	15974725	15975050	15975375
15175680	15176005	15176330	15176655	14376960	14377285	14377610
14377935	13578240	13578565	13578890	13579215	15975700	15975505
15976350	15976155	15176980	15176785	15177630	15177435	14378260
14378065	14378910	14378715	13579540	13579345	13580190	13579995
15977000	15977325	15976610	15976935	15178280	15178605	15177890
15178215	14379560	14379885	14379170	14379495	13580840	13581165
13580450	13580775	15978300	15978105	15977910	15977715	15179580
15179385	15179190	15178995	14380860	14380665	14380470	14380275
13582140	13581945	13581750	13581555			

7. CONCLUSIONS

In this article, the algebraic structures of dual cyclic codes over $\mathfrak{R}$ are discussed. The necessary and sufficient condition of DNA codes properties over $\mathfrak{R}$ have been discussed. The GC-content and DNA codes over $\mathfrak{R}$ are also discussed with help of examples and a special DNA Table. The discussion on DNA dual cyclic codes over the generalised ring may be an open problem.

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