Defence Science Journal, Vol. 66, No. 3, May 2016, pp. 278-286, DOI : 10.14429/dsj.66.9226

RESEARCH PAPER

Profile-based Maximum Penalised Likelihood Trajectory Estimation from Space-borne LOS Measurements

Taihe Yi[#], Zhen Shen[#*], Zhengming Wang[#], Bing Liu[#], and Dongyun Yi[#]

[#]*College of Science, National University of Defense Technology, Changsha, 410 073, China*

[*]*Correspondence e-mail: shenzhen@nudt.edu.cn*

ABSTRACT

Estimating the boost-phase trajectory of a ballistic missile using line of sight measurements from space-borne passive sensors is an important issue in missile defense. A well-known difficulty of this issue is the poor-observability of the target motion. A profile-based maximum penalised likelihood estimator is presented, which is expected to work in poor-observability scenarios. Firstly, a more adaptable boost-phase profile is proposed by introducing unknown parameters. Then, the estimator is given based on the Bayesian paradigm. After that, a special penalty for box constraint is constructed based on a mixed distribution. Numerical results for some typical scenarios and sensitivity with respect to a priori information are reported to show that the proposed estimator is promising.

Keywords: Trajectory estimation, poor-observability, pseudo-measurements, Likelihood estimator

1. INTRODUCTION

Space-borne infrared missile early-warning system is an important component of missile defense systems. By detecting the azimuth and elevation angles of a missile, the early-warning system can provide a real-time report of each occurrence of a missile launch, and estimate launch parameters and trajectory[1]. The estimation of boost-phase trajectory from space-borne line-of-sight (LOS) measurements. Due to the lack of distance information between a missile and observational satellites, the first problem that needs to be settled is the poor-observability[2,3]. There are two solutions to this problem: one is to increase the number of satellites; the other is to incorporate a priori information into estimation. We focus on the later approach in this paper.

A priori information in trajectory estimation can be divided into two levels: *a priori* information about trajectory model and parameters. Generally, a priori information about model is expressed by a parameterless model with high accuracy. A typical parameterless model is the profile-based model[4,5,11], which assumes that one can build a profile database before launch[6]. Since a profile-based model only contains four launch parameters, it has conquered the problem of poor-observability effectively. However, an accurate profile of an incoming missile is hard to obtain. Hence, the 'full knowledge' profile has to be improved to increase its adaptability. The pseudo-measurement approach, which takes constraints on trajectory as measurements, can also be treated as a priori information about trajectory model. For example, equality-constraints are considered as pseudo-measurements in state estimation in[7]. In[8], the constraints of altitude and speed at the first measurement epoch of a ballistic target are considered as pseudo-measurements.

A priori information about parameters refers to prior distribution of model parameters. To include this kind of information, one can employ the so-called Bayesian paradigm. It has been pointed out by[9] that the Bayesian paradigm has no difficulty in treating non-identifiable parameters. But one might be misled by a wrong prior distribution, especially in poor-observability scenarios like the current situation.

Recently Tharmarasa[10], *et al.* proposes a profile-free launch point estimator using smoothing followed by backward prediction. They focus on using less a priori information as far as possible, which makes their method work well only in a multi-sensor scenario. Kim[11] *et al.* proposes a launch point estimator based-on k-NN search. Their method is a profile-based method too, but with a different kind of profile.

An estimator for trajectory estimation from space-borne LOS measurements following the Bayesian paradigm were proposed in the paper.

(i) A new kind of boost-phase trajectory profile, which is more adaptable than other existing profiles;
(ii) A profile-based maximum penalised likelihood estimator (PMPLE), which can incorporate different kinds of information into trajectory estimation;
(iii) A L^1 and log-barrier penalty, which can handle box constraints efficiently.

2. TRAJECTORY ESTIMATION USING LOS MEASUREMENTS

The motion equation of a ballistic target in earth-centered earth-fixed (ECEF) can be described by[5]

Received : 14 September 2015, Revised : 22 March 2016
Accepted : 28 March 2016, Online : 25 April 2016

$$\frac{dv}{dt} = a_N - \mu \frac{r}{\|r\|^3} - 2\omega \times v - \omega \times (\omega \times r) \tag{1}$$

where r and v are position and velocity of the target, ω and μ are the rotational angular velocity and the gravitational constant of the earth, a_N is the net acceleration[5], combined by thrust acceleration, drag acceleration and control acceleration.

Let $a_N(\alpha, t)$ be a model of the net acceleration, where α stands for parameters. And $l_0 = [t_0, \lambda_0, B_0, \alpha_0]^T$ be launch parameters, where t_0 is the launch time, λ_0 the launch longitude, B_0 the launch latitude and α_0 the launch heading. The state of the target $x(t) = [x(t), y(t), z(t), v_x(t), v_y(t), v_z(t)]^T$ can be computed by numerical integration, denoted by

$$x(t) = f(\alpha, l_0, t), \quad \forall t \in [t_0, t_0 + T] \tag{2}$$

where T is the target's boost-phase duration. In the rest of this paper, we denote $[\alpha^T, l_0^T]^T$ by β for simplicity.

The measurement equation at epoch $t_i (i = 1, 2, \cdots, n)$ is

$$\begin{bmatrix} A(t_i) \\ E(t_i) \end{bmatrix} = \begin{bmatrix} \arctan \dfrac{y(t_i) - y_s(t_i)}{x(t_i) - x_s(t_i)} \\ \arctan \dfrac{z(t_i) - z_s(t_i)}{\sqrt{\left[x(t_i) - x_s(t_i)\right]^2 + \left[y(t_i) - y_s(t_i)\right]^2}} \end{bmatrix} + \begin{bmatrix} \varepsilon_1(t_i) \\ \varepsilon_2(t_i) \end{bmatrix} \tag{3}$$

where $[x(t_i), y(t_i), z(t_i)]^T$ and $[x_s(t_i), y_s(t_i), z_s(t_i)]^T$ are locations of the target and sensor respectively. $\varepsilon_1(t_i)$ and $\varepsilon_2(t_i)$ are measurement noises, modelled as zero-mean, white, Gaussian sequences with known covariance matrix $R(t_i)$.

By substituting Eqn. (2) into Eqn. (3), the problem of boost-phase trajectory estimation turns into a nonlinear regression problem.

$$z(t_i) \triangleq \begin{bmatrix} A(t_i) \\ E(t_i) \end{bmatrix} = h(\beta, t_i) + \begin{bmatrix} \varepsilon_1(t_i) \\ \varepsilon_2(t_i) \end{bmatrix}, \quad i = 1, 2, \cdots, n \tag{4}$$

Given a set of measurements $Z_n = \{z(t_i), i = 1, 2, \cdots, n\}$, the likelihood of β can be written as

$$p(\beta; Z_n) = \prod_{i=1}^{n} \left(2\pi |R(t_i)|\right)^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} \sum_{i=1}^{n} \left(z(t_i) - h(\beta, t_i)\right)^T R^{-1}(t_i) \left(z(t_i) - h(\beta, t_i)\right) \right\} \tag{5}$$

Let

$$\upsilon(\beta) = \begin{bmatrix} R^{-\frac{1}{2}}(t_i)\left(z(t_1) - h(\beta, t_1)\right) \\ \vdots \\ R^{-\frac{1}{2}}(t_i)\left(z(t_n) - h(\beta, t_n)\right) \end{bmatrix} \tag{6}$$

be the normalised measurement residuals. The maximum likelihood (ML) estimator of β is

$$\beta_{ML} = \arg\min \left\{ \|\upsilon(\beta)\|^2 \right\} \tag{7}$$

Poor-observability means the condition number of the normal matrix of problem Eqn. (7),

$$\left[\frac{\partial^T \upsilon(\beta)}{\partial(\beta)} \frac{\partial \upsilon(\beta)}{\partial(\beta)} \right]_{\beta_{true}} \tag{8}$$

is very large.

3. A PRIORI INFORMATION FOR TRAJECTORY ESTIMATION

A priori information considered in this paper includes the net acceleration profile (NAP) and constraints of trajectory.

3.1 Net Acceleration Profile

There are two kinds of profiles in literature[1,4,11,13,14]. first is the nominal profile, which consists of the horizontal distance and altitude wrt the launch point as functions of time since launch. The other is the thrust acceleration profile, which consists of the magnitude and orientation of the thrust acceleration. Both of them are 'full knowledge' of a missile's boost-phase motion model. Since a missile's trajectory is related to some uncontrollable environmental factors like atmosphere, weather and so on, the 'full knowledge' is not always reliable and useful. What is more, the orientation of net acceleration depends on specific missions. A 'full knowledge' might cause extra troubles when one wants to recognize an incoming missile's type by its motion feature. To overcome those shortcomings, we only consider the magnitude of net acceleration in the NAP, modelling it under a given precision. For the orientation of the net acceleration, we model it by a parsimonious model in the later section.

Definition : For a given type of missile with boost-phase duration T, let $a(t), t \in [0, T]$ be the true magnitude of the net acceleration, δ and w be positive numbers. $a(\alpha, t)$ is a non-negative function defined on $\Omega \times [0, T]$, where Ω is a subset of $\mathbb{R}^d$. If

$$\min_{\alpha \in \Omega} \int_0^T \left| a(t) - a(\alpha, t) \right| dt = \delta \tag{9}$$

$$\max_{\alpha \in \Omega} \int_0^T a(\alpha, t) dt - \min_{\alpha \in \Omega} \int_0^T a(\alpha, t) dt = w \tag{10}$$

then $\{a(\alpha, t) : (\alpha, t) \in \Omega \times [0, T]\}$ is called a d dimensional NAP of the given type of missile, with bias δ and width w.

Note that δ describes the level of similarity between $a(\alpha, t)$ and $a(t)$, and can't be calculated since $a(t)$ is unknown. w, which is determined by Ω, describes the variation range of the corresponding velocity. We require Ω to be a rectangle $[l_1, u_1] \times \cdots \times [l_d, u_d]$ for simplicity. d and the nonlinearity of $a(\alpha, t)$ represent the complexity of the NAP.

The basic idea of the NAP is to introduce some parameters to improve adaptability, and controlling its freedom by the

Table 1. NAP consisted by discrete data ($0 \le \tau_1 < \cdots < \tau_d \le T$, $l_i < u_i, \forall i = 1, \cdots, d.$)

Time epoch	Lower bound (m/s²)	Upper bound (m/s²)
τ_1	l_1	u_1
$\vdots$	$\vdots$	$\vdots$
τ_d	l_d	u_d

range of values of parameters. Based on this idea, we introduce a NAP constructed from discrete data, as shown in Table 1. Parameters $\alpha=[a(\tau_1),\cdots,a(\tau_d)]^T$ are the true magnitude of net acceleration, with range of values $[l_1,u_1]\times\cdots\times[l_d,u_d]$. Given α, $a(t)$ is calculated by Algorithm 1. Note that the width of a NAP constructed by discrete data is determined by the range of each discrete data and sampling frequency, and is easy to calculate.

Algorithm 1 : Hyperbolic interpolation for $a(t)$, where

$$f(i,i+1,t)=\frac{a(\tau_{i+1})a(\tau_i)}{a(\tau_{i+1})-\left[a(\tau_{i+1})-a(\tau_i)\right](t-\tau_i)/(\tau_{i+1}-\tau_i)} \tag{11}$$

if { $d=1$ } return $a(\tau_1)$.
else { $t<\tau_2$ }
if { $a(\tau_1)\le a(\tau_2)$ } return $f(1,2,t)$;
else { $t<0.5(\tau_1+\tau_2)$ }return $a(\tau_1)$;
else{ $d=2$ or $a(\tau_2)>a(\tau_3)$ } return $a(\tau_2)$;
elsereturn $f(2,3,t)$;
end if.
else{ $t\ge\tau_{d-1}$ }
if { $a(\tau_{d-1})\le a(\tau_d)$ } return $f(d-1,d,t)$;
else { $t\ge 0.5(\tau_{d-1}+\tau_d)$ } return $a(\tau_d)$;
else{ $a(\tau_{d-2})\le a(\tau_{d-1})$ } return $f(d-2,d-1,t)$;
elsereturn $a(\tau_{d-1})$;
end if.
else { $\tau_i\le t<\tau_{i+1}$ }
if { $a(\tau_i)\le a(\tau_{i+1})$ } return $f(i,i+1,t)$;
else { $t\le 0.5(\tau_i+\tau_{i+1})$ and $a(\tau_{i-1})\le a(\tau_i)$ } return $f(i-1,i,t)$;
else{ $t\le 0.5(\tau_i+\tau_{i+1})$ and $a(\tau_{i-1})>a(\tau_i)$ } return $a(\tau_i)$;
else{ $t>0.5(\tau_i+\tau_{i+1})$ and $a(\tau_{i+1})\le a(\tau_{i+2})$ } return $f(i+1,i+2,t)$;
else{ $t>0.5(\tau_i+\tau_{i+1})$ and $a(\tau_{i+1})>a(\tau_{i+2})$ } return $a(\tau_{i+1})$;
end if.
end if.

3.2 Constraints of Trajectory

In many circumstances, LOS measurements are not the only available information. One could get information like range of values of the missile's range, magnitude of the burn-out velocity and burn-out altitude of a missile before launch. According to the performance of sensors, one can also get the range of values of the target's altitude at the first measurement epoch. All such information should be considered in the estimation. We employ pseudo-measurements to describe this type of information. For example, if $\hat{z}\in\Theta$, we treat $\hat{z}$ as a measurement (i.e. a random variable) and describe it by a distribution with support Θ.

4. PROFILE-BASED MAXIMUM PENALISED LIKELIHOOD ESTIMATION

4.1 Formulation of the Estimator

Suppose:

(a) The NAP of the incoming missile is $\{a(\alpha,t):(\alpha,t)\in\Omega\times[0,T]\}$, and the prior distribution of $\beta=[\alpha^T,l_0^T]^T$ is $\pi(\beta)=\prod_i\pi(\beta_i)$;

(b) There are q mutually independent pseudo-measurements $\hat{\mathbf{z}}_q\triangleq[\hat{z}_1,\cdots,\hat{z}_q]^T$ with conditional distribution $p(\hat{z}_q\mid\beta)=\prod_{i=1}^q p(\hat{z}_i\mid\beta)$;

(c) LOS measurements Z_n and pseudo-measurements $\hat{z}_q$ are independent for a given β, and the conditional distribution of Z_n is $p(Z_n\mid\beta)$ defined by Eqn. (5).

Under those assumptions, the posterior of β according to the Bayesian theorem[9] is

$$p(\beta\mid Z_n,\hat{z}_q)\cong p(Z_n\mid\beta)p(\hat{z}_q\mid\beta)\pi(\beta) \tag{12}$$

where the notation $\cong$ means 'equal up to a constant'. Hence, the maximum a posterior estimate of β is

$$\begin{aligned}\beta_{MPL}&=\arg\max\left\{p(Z_n\mid\beta)p(\hat{z}_q\mid\beta)\pi(\beta)\right\}\\&=\arg\min\left\{\frac{1}{2}\left\|\upsilon(\beta)\right\|^2-\log[p(\hat{z}_q\mid\beta)]-\log[\pi(\beta)]\right\},\end{aligned} \tag{13}$$

Which is a penalised least square estimator with penalty term $-\log[p(\hat{z}_q\mid\beta)]-\log[\pi(\beta)]$. Hence β_{MPL} is named by PMPLE.

To solve Eqn. (13), rewrite it as,

$$\beta_{MPL}=\arg\min\left\{\frac{1}{2}\left\|\upsilon(\beta)\right\|^2+\frac{1}{2}\left\|\hat{\upsilon}(\beta)\right\|^2\right\} \tag{14}$$

where $\hat{\upsilon}(\beta)=\left[\sqrt{-2\log[p(\hat{z}_q\mid\beta)]}\quad\sqrt{-2\log[\pi(\beta)]}\right]^T$. Thus any nonlinear least square algorithm is applicable. We recommend the LM algorithm provided by[15]. Note that $2\log[p(\hat{z}_q\mid\beta)]$ and $2\log[\pi(\beta)]$ are not required to be negative, since in the LM algorithm we do not need to calculate the square roots of $-2\log[p(\hat{z}_q\mid\beta)]$ and $-2\log[\pi(\beta)]$.

The normal matrix of Eqn. (14) is

$$G=\frac{\partial^T\upsilon(\beta)}{\partial\beta}\frac{\partial\upsilon(\beta)}{\partial\beta}+\frac{\partial^T\hat{\upsilon}(\beta)}{\partial\beta}\frac{\partial\hat{\upsilon}(\beta)}{\partial\beta} \tag{15}$$

Comparing Eqn. (15) with Eqn. (8), one can find out that a priori information improves observability by adding a non-negative definite matrix to the normal matrix.

4.2 L^1 and Log-Barrier Mixed Penalty

Without loss of generality, we only consider penalty terms derived from one dimensional distribution. Since it's easy to get prior intervals for all parameters and pseudo-measurements, the beta distribution with support $[l,u]$

$$Beta(x\mid l,u,\delta_1,\delta_2)=\begin{cases}\dfrac{(x-l)^{a-1}(u-x)^{b-1}}{B(a,b)(u-l)^{a+b-1}}, & x\in[l,u]\\0, & x\notin[l,u]\end{cases} \tag{16}$$

is a good choice, where $B(a,b)$ is the beta function, $\delta_1\in(0,1)$, $\delta_2\in(0,1]$ and

$$\begin{cases} a = \delta_1\left[\dfrac{1+\min\{1-\delta_1,\delta_1\}}{\delta_2\min\{1-\delta_1,\delta_1\}}-1\right], \\ b = (1-\delta_1)\left[\dfrac{1+\min\{1-\delta_1,\delta_1\}}{\delta_2\min\{1-\delta_1,\delta_1\}}-1\right]. \end{cases} \tag{17}$$

when $\delta_1 = 0.5$ and $\delta_2 = 1$, Eqn. (16) degenerates to the uniform distribution on $[l,u]$.

The corresponding penalty function is

$$-\log\left[Beta(x\,|\,l,u,\delta_1,\delta_2)\right] \cong \begin{cases} -(a-1)\log(x-l)-(b-1)\log(u-x), & x\in(l,u) \\ +\infty, & x\notin(l,u) \end{cases} \tag{18}$$

Equation (18) coincides with the well-known log-barrier penalty in box constrained optimisation problem. Unlike the barrier method, the penalty factors $a-1$ and $b-1$ do not need to be updated in each iterator, since they have a specific meaning of a priori assumption.

Equation (18) shows that we have to find initial feasible points $x_0 \in (l,u)$ for both parameters and pseudo-measurements, which is difficult in the current situation since pseudo-measurement is generally a nonlinear function of parameters. To overcome this problem, we construct a mixed distribution with support $\mathbb{R}$

$$Mix(x;l,u,\delta_1,\delta_2,\lambda) = \begin{cases} C\lambda\exp\{\lambda(x-\alpha_1)\}, & x < x_1 \\ C\dfrac{(x-l)^{a-1}(u-x)^{b-1}}{B(a,b)(u-l)^{a+b-1}}, & x_1 \le x \le x_2 \\ C\lambda\exp\{\lambda(\alpha_2-x)\}, & x > x_2 \end{cases} \tag{19}$$

where a and b are determined by Eqn. (17). λ is a fairly large number. x_1, x_2, α_1 and α_2 are determined by the continuity of the probability density as well as its derivative. C is a normalisation factor, making $Mix(x;l,u,\delta_1,\delta_2,\lambda)$ a proper probability density function.

The corresponding penalty term is

$$-\log\left[Mix(x;l,u,\delta_1,\delta_2,\lambda)\right] \cong \begin{cases} -\lambda(x-\alpha_1), & x < x_1 \\ -(a-1)(x-l)-(b-1)(u-x), & x_1 \le x \le x_2 \\ -\log\lambda(\alpha_2-x), & x > x_2 \end{cases} \tag{20}$$

which is a L_1 and log-barrier mixed penalty function.

Figure 1 is a schematic plot of the L^2 penalty, log-barrier penalty and mixed penalty. It shows that the mixed penalty is the most appropriate penalty for a nonlinear box constraint. Note that both the log-barrier penalty and the mixed penalty are convex for all $\delta_1 \in (0,1)$, $\delta_2 \in (0,1]$ and $\lambda > 0$.

4.3 Covariance Estimation

Let $\eta(\beta)$ be the parameters we are interested in. The Cramer-Rao Lower Bound (CRLB)[16] of $\eta(\beta)$ is

$$C(\eta) = \left[\frac{\partial^T\eta(\beta)}{\partial\beta}M^{-1}(\beta)\frac{\partial\eta(\beta)}{\partial\beta}\right]_{\beta_{true}} \tag{21}$$

where $M(\beta)$ is the Fisher information matrix of β. Under the assumption of independence, $M(\beta)$ can be decomposed as[16]

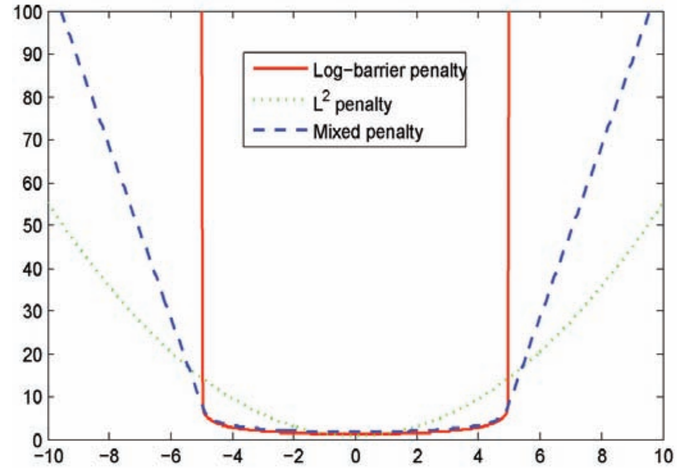

Figure 1. A schematic plot of the three kinds of penalties.

$$M(\beta) = M_{Z_n}(\beta) + \sum_{i=1}^{q} M_{\hat{z}_i}(\beta) + M_\beta(\beta) \tag{22}$$

where $M_{Z_n}(\beta)$, $M_{\hat{z}_i}(\beta)$ and $M_\beta(\beta)$ are the information provided by LOS measurements, pseudo-measurement and a priori information of parameters, respectively. By substituting β_{true} with β_{MPL}, the covariance of η_{MPL} can be estimated by[4,8].

$$V(\eta_{MPL}) = \left[\frac{\partial^T\eta(\beta)}{\partial\beta}M^{-1}(\beta)\frac{\partial\eta(\beta)}{\partial\beta}\right]_{\beta_{MPL}} \tag{23}$$

To evaluate a pseudo-measurement, we can use Eqns. (15) and (22). Equation (15) is used for evaluating the improvement of observability, while Eqn. (22) is used for evaluating the improvement of accuracy.

5. SIMULATION RESULTS

In this section we consider some typical scenarios to investigate the performance of the PMPLE. All the results are based on 50 Monte-Carlo runs.

In a single-sensor scenario, the passive sensor is carried by a GEO satellite located at latitude 140° E. In a double-sensor scenario, passive sensors are carried by two GEO satellites located at 100° E and 180° E, respectively. The LOS error and sampling frequency are set to be 30 μrad and 0.4 Hz. We suppose that the target can be observed when the altitude is larger than 1 km.

At the reference time $t_0 = 0$ s, the target is launched at 20° N latitude, 140° E longitude, with launch heading (wrt true north) 300°. The true magnitude and orientation of the thrust acceleration of the target is shown in Fig. 2. The NAP is given by discrete data in each simulation as

$$\begin{cases} \left(\tau_i, a(\tau_i)-\varepsilon(\tau_i), a(\tau_i)-\varepsilon(\tau_i)+\delta\right) \\ \tau_i = 10.3i, \qquad i = 1,2,\cdots,10 \end{cases} \tag{24}$$

where $a(\tau_i)$ is the magnitude of the true net acceleration, $\varepsilon(\tau_i)$ is a uniformly distributed random variable defined on $[0,\delta]$. δ determines the width of the profile and is set to be 5.0 m/s^2. For the orientation of the net acceleration, we set that the trajectory lies in a plane, and model the angle $\varphi(t)$ between the net thrust and the outward normal vector from the center

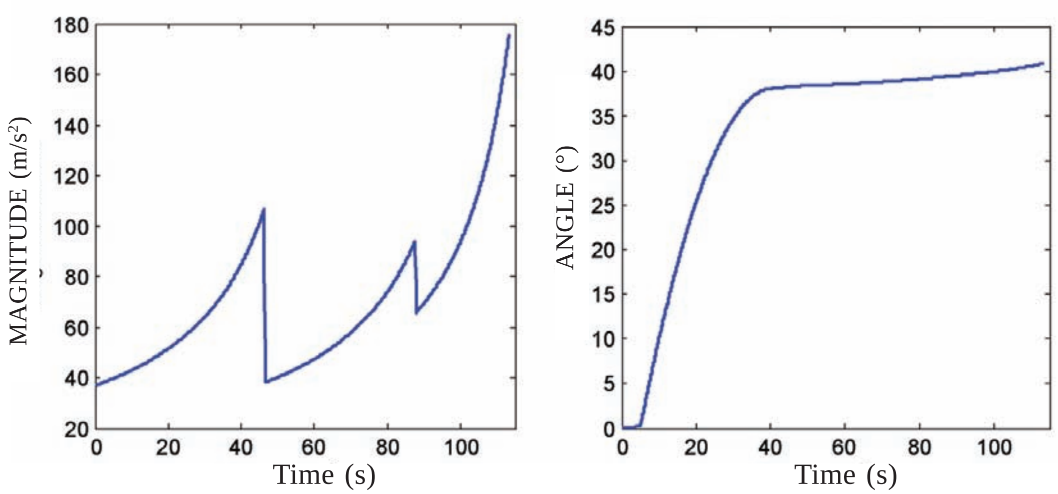

Figure 2. Magnitude and orientation of thrust acceleration of the target during boost-phase.

of the ECEF by,

$$\varphi(t)=\begin{cases}0, t\le\varsigma_1;\\ \varphi_0\left[1-\dfrac{(\varsigma_2-t)^2}{(\varsigma_2-\tau_1)^2}\right], \varsigma_1<t\le\varsigma_2;\\ \varphi_0, t>\varsigma_2.\end{cases} \quad (25)$$

ς_1, ς_2 and φ_0 are parameters, with prior intervals (0,10), (30,100) and $(\pi/6, 5\pi/12)$, respectively. Prior distributions of all parameters are $Mix(x, l, u, 0.5, 0.9, 20)$.

Pseudo-measurements considered in the simulation includes the range $R(\beta)$ of the target, the altitude of the targets at the first measurement epoch $h(t_1)$ and last measurement epoch $h(t_n)$, and the magnitude of the velocity of the target at the last measurement epoch $v(t_n)$.

Interested parameters include the launch position (x_0, y_0, z_0), launch heading α_0, target position $r(t_n)$ and the velocity $v(t_n)$ of the last measurement epoch. The square root of the trace of the CRLB and covariance matrix, and the normalised estimation error squared (NEES)[16].

$$eff(\hat{\eta}) \triangleq (\hat{\eta}-\eta_{true})^T M(\eta_{true})(\hat{\eta}-\eta_{true}) \quad (26)$$

are used to evaluate the performance of the PMPLE. Note that if $\hat{\eta}$ is an unbiased and asymptotically normal distributed estimator, $eff(\hat{\eta})$ is a chi-squared distributed random variable with degrees of freedom $\dim(\hat{\eta})$, where $\dim(\hat{\eta})$ is the dimension of $\hat{\eta}$.

5.1 Results of some Typical Scenarios

In this section, we consider the following four typical scenarios to demonstrate the efficiency of the PMPLE

SS0: single-sensor without pseudo-measurements;
SS1: single-sensor with four pseudo-measurements;
DS0: double-sensor without pseudo-measurements;
DS1: double-sensor with four pseudo-measurements.

Distributions of all pseudo-measurements are set to be $Mix(x\,|\,0.9x_0, 1.1x_0, 0.5, 0.9, 20)$, where x_0 is the true value of the corresponding pseudo-measurement.

Table 2 shows the RMSE of interested parameters. The improvement caused by pseudo-measurements in single-sensor scenarios is more significant than that in double-sensor scenarios. Table 3 shows that the PMPLE is efficient in all the four scenarios. We only present the NEES of $r(t_n)$ and $v(t_n)$, since $r(t_n)$ and $v(t_n)$ are functions of β, $r(t_n)$ and $v(t_n)$ are efficient only if estimate of β is efficient. Note that $\chi^2_{150}(0.025)/50 \approx 2.36$ and $\chi^2_{150}(0.975)/50 \approx 3.72$.

Table 2. RMSE of interested parameters

	Launch point(m)	α_0 (°)	$r(t_n)$(m)	$v(t_n)$(m/s)
SS0	234.83	0.2411	3081.33	70.41
SS1	223.55	0.0745	947.26	28.19
DS0	222.26	0.0632	358.01	19.26
DS1	221.24	0.0575	349.62	17.56

Table 3. NEES of interested parameters

	eff ($r(t_n)$)	*eff* ($c(t_n)$)
SS0	2.41	2.06
SS1	2.42	2.00
DS0	3.70	3.38
DS1	2.73	3.24

5.2 The Performance of Pseudo-Measurements

Table 4 and Table 5 show the CRLB of interested parameters in the double-sensor and single-sensor scenarios respectively, where prior distribution of pseudo-measurement is set to $Mix(x\,|\,0.95x_0, 1.05x_0, 0.5, 0.5, 20)$. The results indicate that the pseudo-measurements play a more important role in the single-sensor scenario, and the effect of using multiple pseudo-measurements simultaneously does not equal the sum of the effect of a single pseudo-measurement.

5.3 Sensitivity with Respect to a Priori Information

In order to study the sensitivity wrt profile width, we consider the four scenarios SS0, SS1, DS0 and DS1 defined in section 5.1. The profile width is controlled by changing δ from 0.2 to 10. Distributions of pseudo-measurements are $Mix(x\,|\,0.9x_0, 1.1x_0, 0.5, 0.9, 20)$.

Table 4. CRLB of the double-sensor scenario

Pseudo-measurements considered	Launch point (m)	α_0 (°)	$r(t_n)$(m)	$v(t_n)$(m/s)
Non	221.16	0.1989	379.0419	19.5992
$R(\beta)$	221.16	0.1988	377.5592	19.5761
$h(t_1)$	221.16	0.1989	378.3838	19.5768
$h(t_n)$	221.16	0.1989	375.5988	19.4680
$v(t_n)$	221.16	0.1989	373.2790	18.7329
$R(\beta)$, $h(t_1)$, $h(t_n)$ and $v(t_n)$	221.16	0.1987	368.2142	18.5837

Table 5. CRLB of the single-sensor scenario

Pseudo-measurements considered	Launch point (m)	α_0 (°)	$r(t_n)$(m)	$v(t_n)$(m/s)
Non	233.39	0.3065	1784.1102	42.0809
$R(\beta)$	222.28	0.2727	1397.6862	35.8447
$h(t_1)$	221.16	0.2873	1396.8637	37.6205
$h(t_n)$	222.28	0.2397	956.3819	28.4291
$v(t_n)$	222.28	0.2928	1597.2533	37.6601
$R(\beta)$, $h(t_1)$, $h(t_n)$and $v(t_n)$	221.16	0.2355	831.6401	26.4374

Figure 3 shows that a NAP of larger width will end up with a larger CRLB and covariance. It once again reveals the fact that pseudo-measurements play more important roles in a single-sensor scenario than in a double-sensor scenario.

We take $R(\beta)$ and the estimated result of the launch heading as an example to study the influence of prior distribution of pseudo-measurements. We consider the following four scenarios:

SRM: single-sensor with $R(\beta) \sim Mix(R \,|\, 0.7R_0, 1.3R_0, \rho, \tau, 20)$;

DRM: double-sensor with $R(\beta) \sim Mix(R \,|\, 0.7R_0, 1.3R_0, \rho, \tau, 20)$;

SRB: single-sensor with $R(\beta) \sim Mix(R \,|\, \rho x_0, (\rho+0.4)x_0, \rho+0.2, \tau, 20)$;

DRB: double-sensor with $R(\beta) \sim Mix(x \,|\, \rho x_0, (\rho+0.4)x_0, \rho+0.2, \tau, 20.$

τ is set to be 0.1, 0.5 and 0.9 in each scenario. SRM and DRM are designed for learning the influence of the prior mean, while SRB and DRB for the influence of the prior bound.

Figures 4 and 5 are results of scenarios SRM and DRM. These figures demonstrate that in a single-sensor scenario, bias in the prior mean of the range will lead to bias in the estimate result, however, when $\tau > 0.5$, the bias becomes smaller. In addition, under a double-sensor scenario, the enhanced observability eliminates the bias effectively.

Figures 6 and 7 are results for scenarios SRB and DRB. They once again demonstrate that a bigger τ is more robust than a smaller τ, and double-sensor scenarios are insensitive

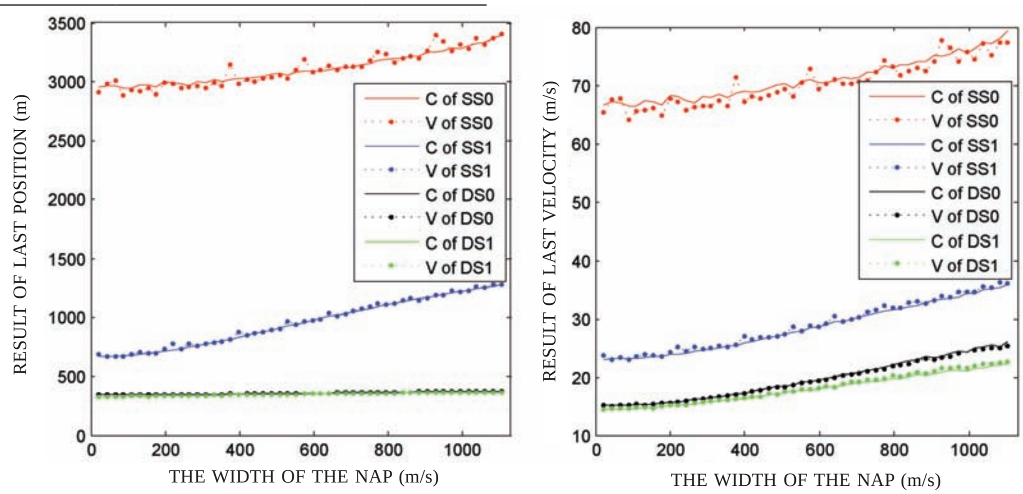

Figure 3. Influence of the width of the NAP (C stands for the CRLB, and V stands for the estimate variance).

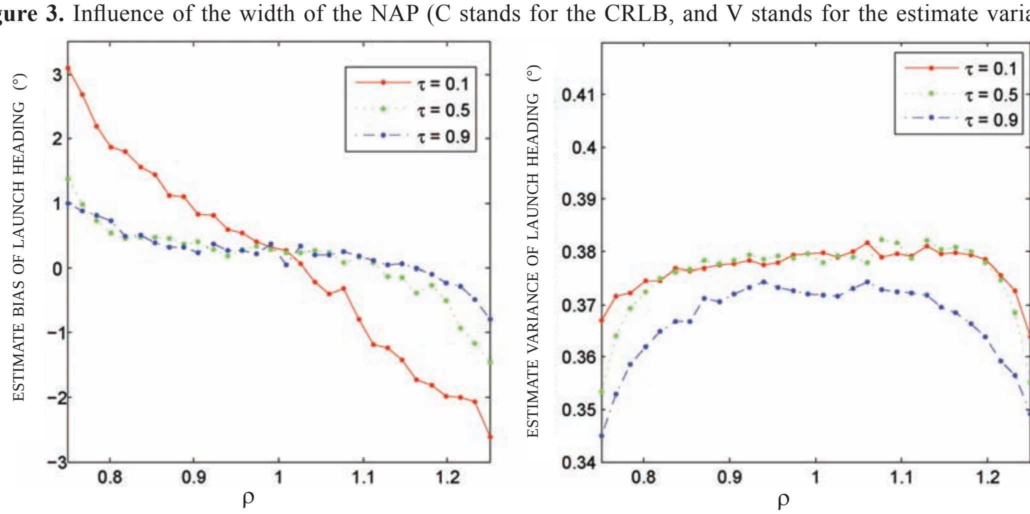

Figure 4. Influence of the mean of the range in the single-sensor scenario.

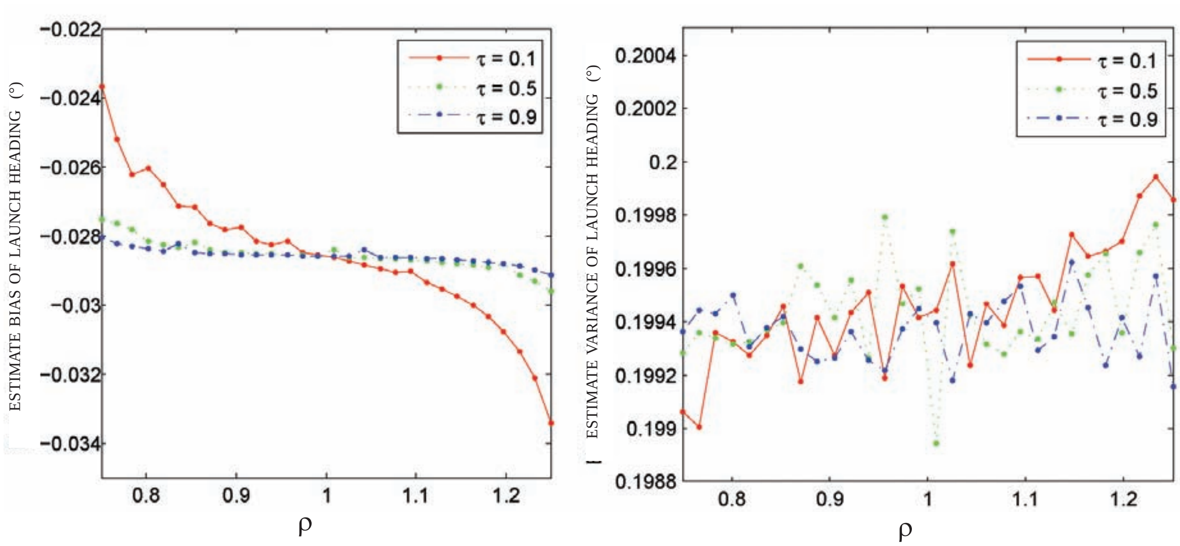

Figure 5. Influence of the mean of the range in the double-sensor scenario.

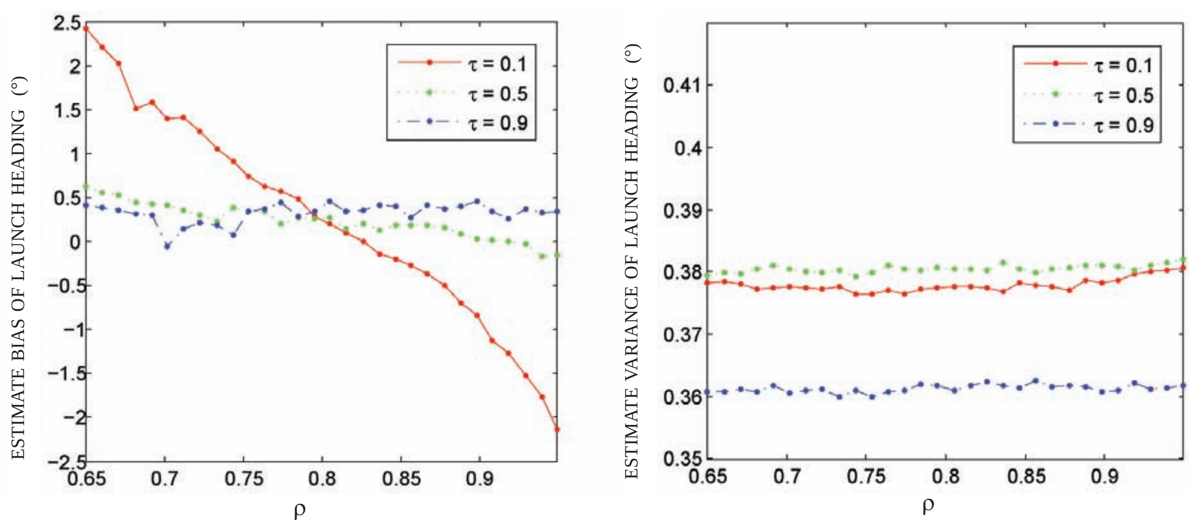

Figure 6. Influence of the bound of the range in the single-sensor scenario.

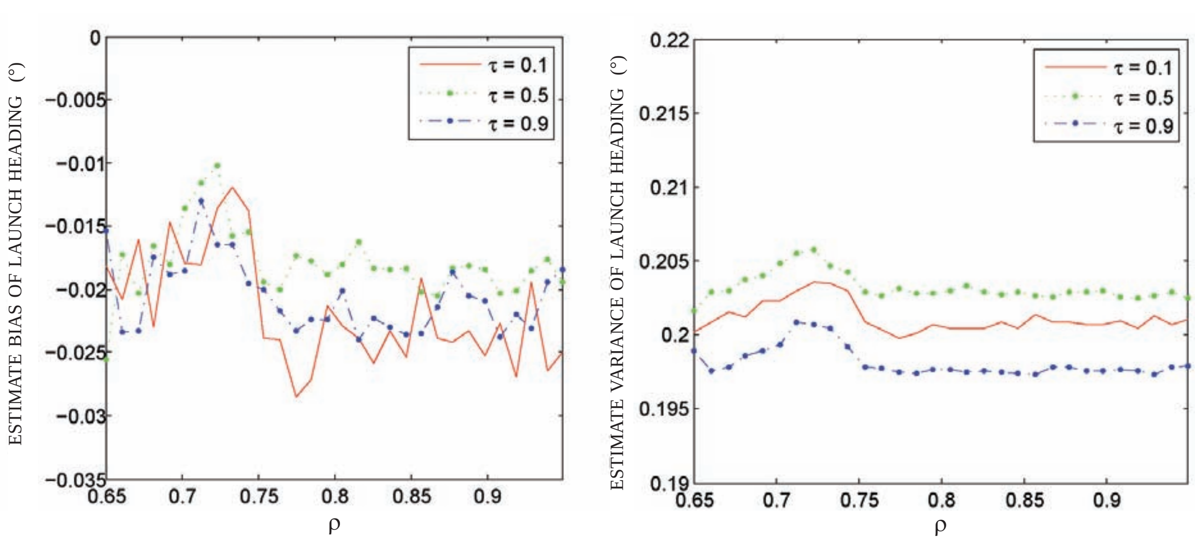

Figure 7. Influence of the bound of the range in the double-sensor scenario.

to prior information. They also suggest that if a pseudo-measurement $z \in [l, u]$, then $Mix(z; l, u, 0.5, \tau, 20)$ is a robust prior probability density function when $0.5 < \tau < 1$.

6. CONCLUSION

We have presented a novel profile-based maximum penalised likelihood estimator for trajectory estimation. The estimator handles the problem of poor-observability by introducing *a priori* information including net acceleration profile, constraints of trajectory and prior intervals of parameters. In order to test the performance of the estimator, we have conducted a series of simulations. The results indicate that:

(i) The estimator is efficient in both single-sensor and double-sensor scenarios;

(ii) By introducing a priori information, the accuracy of interested parameters (especially $r(t_n)$ and $v(t_n)$) has been improved dramatically in the single-sensor scenario. While in a double-sensor scenario, the enhanced observablility makes a priori information less important.

(iii) In the double-sensor scenario, the PMPLE is robust wrt l, u, δ_1 and δ_2 of the prior distribution $Mix(z; l, u, \delta_1, \delta_2, \lambda)$. And in the single-sensor scenario, the PMPLE is robust wrt *l*, *u* and δ_1 by setting $\delta_2 \in (0.5, 1)$.

We hold that the PMPLE is highly useful in space-borne missile defense system for its adaptability.

REFERENCES

1. Rudd, J.G.; Marsh, R.A. & Roecher, J. A. Surveillance and tracking of ballistic missile launches. *IBM J. Res. Develop.*, 1994, **38**(2), 195-216.
doi: 10.1147/rd.382.0195
2. Song, T.L. Observability of targettracking with bearings-only measurements. *IEEE Trans. Aero. Electron. Sys.*, 1996, **32**(4), 1468-1472.
doi: 10.1109/7.543868
3. Jauffret, C. & Pillon, D. Observability in passive target motion analysis. *IEEE Trans. Aero. Electron. Sys.*, 1996, **32**(4), 1290-1300.
doi: 10.1.1109/7.543850
4. Li, Y.; Kirubarajan, T.; Kirubarajan, T.; Bar-Shalom, Y. & Yeddanapudi, M. Trajectory and launch point estimation for ballistic missiles from boost phase LOS measurements. *In* the IEEE Proceedings on Aerospace Conference, Snowmass, Aspen, CO, USA, 1999, pp. 425-442.
5. Li, X.R. & Jilkov, V. P. Survey of maneuvering target tracking Part II: Motion models of ballistic and space targets. *IEEE Trans. Aero. Electron. Sys.*, 2010, **46**(1), 96-119.
doi: 10.1109/TAES.2010.5417150
6. Jerardi, T. TBM profile data. Johns Hopkins University Applied Physics Laboratory, 1998.
7. Zhansheng, D. & Li, X.R. The role of pseudo-measurements in equality-constrained state estimation. *IEEE Trans. Aero. Electron. Sys.*, 2013, **49**(3), 1654-1666.
doi: 10.1109/TAES.2013.6558010
8. Yeddanapudi, M.; Bar-Shalom, Y.; Pattipati, K.R. & Deb, S. Ballistic missile track initiation from satellite observations. *IEEE Trans. Aero. Electron. Sys.*, 1995, **31**(3), 1055-1071.
doi: 10.1109/7.395236
9. Francisco J. S. A Comparison of the Bayesian and frequentist approaches to estimation. Springer, 2010.
doi: 10.1007/978-1-4419-5941-6
10. Tharmarasa, R.; Kirubarajan, T.; Nandakumaran, N.; Bar-Shalom, Y. & Thayaparan, T. Profile-free launch point estimation for ballistic targets using passive sensors. *J. Adv. Info.Fusion*, 2012, **7**(1), 46-50.
doi: 10.1109/AERO.2011.5747433
11. Kim, S.; Kim, H. & Cho, S. Fast parameterless ballistic launch point estimation based on k-NN search. *Def. Sci. J.*, 2014, **64**(1), 41-47.
doi: 10.14429/dsj.64.2952
12. Kim, J.K.; Kim, T.H. & Song, T.L. Launch point prediction employing the smoothing IPDA algorithm in 3-D cluttered environments. *Int. J. Control, Automation, Sys.*, 2015, **13**(4), 808-815.
doi: 10.1007/s12555-014-0023-6
13. Ming, J. & Fannie, A.R. Angle-only tracking and prediction of boost vehicle position. *Signal Data Proces. Small Missiles*, 1991, **1481**, 281-291.
doi: 10.1117/12.45662
14. Danis, N.J. Space-based tactical ballistic missile launch parameter estimation. *IEEE Trans. Aero. Electr. Sys.*, 1993, **29(2)**, 412-424.
doi: 10.1109/7.210079
15. Lourakis, M.I.A. levmar: Levenberg-Marquardt nonlinear least squares algorithms in C/C++. http://www.ics.forth.gr/~lourakis/levmar/, 2004 (Accessed on 31 Jan. 2005)
16. Bar-Shalom, Y. & Li, X.R. Estimation and tracking: Principles, techniques and software. Norwood, MA: Artech House, 1993.

CONTRIBUTORS

Mr Taihe Yi received his BS (Applied Mathematics) from National University of Defense Technology, China, in 2011. Currently pursuing his PhD in Applied Mathematics at the National University of Defense Technology. His area of expertise are : Constrained statistical inference, design of experiment, and target tracking.
In the current study, provide the initial idea, performed experiments and prepared the manuscript.

Dr Zhen Shen received his BS (Applied Mathematics) from National University of Defense Technology, China, in 2002, MS and PhD in Systems Analysis and Integration from National University of Defense Technology, in 2005 and 2010, respectively. Currently, working as a Lecturer at the Department of Mathematics and Systems Science, College of Science, National University of Defense Technology. His research interests include: Data analysis and systems engineering.
In the current study, he conceived, designed, and performed the experiment.

Dr Zhengming Wang received his BS and MS in Applied Mathematics and PhD in System Engineering in 1982, 1986,

and 1998, respectively. He has co-published five monographs (all ranked first) and well over 90 papers. Currently, he is a Professor in Applied Mathematics. His research interests include: Experiment evaluation and data fusion, mathematical modelling in tracking data, image processing.
In the current study, he provides initial idea for the study and prepared the manuscript.

Mr Bing Liu received his BS (Applied Mathematics) from National University of Defense Technology, China in 2009, and MS (System Analysis and Integration) from National University of Defense Technology, 2011. Currently pursuing his PhD in System Analysis and Integration at the National University of Defense Technology and currently works on target tracking and system engineering.
In the current study, he performed the experiments.

Mr Dongyun Yi received his BS and MS in Applied Mathematics and PhD in System Engineering in 1985, 1992, and 2003, respectively. He has co-published two monographs and published more than sixty papers. Currently, working as a Professor in systems analysis and integration. His research interests include : Data fusion, parameter estimation of satellite positioning, mathematical modelling, and analysis of financial data.
In the current study, he conceived and designed the experiments.