RADIAL VELOCITY EFFECTS ON THE CONVECTIVE INSTABILITY INDUCED BY CENTRIFUGAL BUOYANCY

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The effect of radial velocity on the stability of viscous flow between two arbitrarily spaced concentric porous cylinders in the presence of a radial temperature gradient has been examined by numerically solving the resulting differential equation with variable coefficients. The combined influence of suction (or injection) and the temperature gradient has been presented graphically.

Walowit[1], *et. al.*, studied the influence of a radial temperature gradient on the stability of flow between two cylinders with wide gap, by using Galerkin method. Recently Butler and Mckee[2] have attempted the same problem, applying a variational technique. The object of the present paper is to examine the combined effect of radial velocity and temperature gradient with arbitrary gap. The radial velocity means suction or injection at the outer cylinder according as it is directed away from or towards the axis. The problem has been solved numerically by applying the technique used by sparrow, Munro & Jonsson[3] to solve the simpler case when there is no temperature gradient or radial velocity.

It is concluded that in the presence of negative temperature gradient, injection of fluid at the outer cylinder produces a stabilizing effect whereas its effect is reversed when it is coupled with positive temperature gradient. In a similar manner, suction from the outer cylinder produces an opposite effect.

The study is helpful in certain cases where it may be beneficial to inject some coolant into the cylindrical surface of a system to counteract the excess heat generated.

EQUATIONS OF THE PROBLEM

In the presence of suction the steady state solutions of the basic equations governing the axisymmetric flow, give the velocity and temperature distributions in cylindrical coordinates (r, θ, z) as:—

$$\left.\begin{aligned} u_r &= R_1 u_1/r = U(r) \\ u_\theta &= A r^{\lambda+1} + B/r = V(r) \qquad u_z = 0 \end{aligned}\right\} \tag{1}$$

$$T = T_1 - \frac{(r^{\delta'} - R_1^{\delta'})(T_1 - T_2)}{(R_2^{\delta'} - R_1^{\delta'})} \tag{2}$$

where

$$A = -\,\Omega_1 \eta^2 \, \frac{(1 - \mu/\eta^2)}{R_2^{\lambda}\,(1 - \eta^{\lambda+2})}$$

$$B = \Omega_1 R_1^2 \, \frac{(1 - \mu\eta^{\lambda})}{(1 - \eta^{\lambda+2})}$$

$$\lambda = \frac{R_1 u_1}{\nu},\ \delta' = \frac{\lambda\nu}{K},\ \mu = \frac{\Omega}{\Omega_1} \text{ and } \eta = \frac{R_1}{R_2}$$

In the above expressions, u_r, u_θ, v_z are the components of velocity in the increasing r, θ, z directions. R_1 Ω_1, & T_1 are the radius, angular velocity and temperature of the inner cylinder respectively and R_2, Ω_2 & T_2 are the corresponding quantities of the outer one. u_1 is the radial velocity of the fluid at $r = R_1$, ν is the kinematic viscosity and K is the thermal diffusivity.

The equations governing the flow are linearised by applying small perturbation theory and Boussinesq approximation.

The disturbances are analysed into normal modes of the form

$$(u_r, u_\theta, w, \theta) \quad \left[u(r), v(r), \overline{w}(r), \theta(r)\right] e^{pt} \cos kz$$

$$u_z \quad w(r)\, e^{pt} \sin kz$$

where $w = \dfrac{\delta P}{\rho}$; θ and δP are the perturbations in temperature and pressure respectively, k is the wave number of the disturbance and p is a constant.

Now the elimination of $u(r)$, $w(r)$ and $\theta(r)$ from the linearised flow equations and some simplifications result in the following eighth order differential equation in v.

$$D^8 v + \frac{f_1}{\xi} D^7 v \quad \left(\frac{f_2}{\xi^2} - 3a^2\right) D^6 v + \left(\frac{f_3}{\xi^3} + \frac{a^2 f_4}{\xi}\right) D^5 v +$$

$$+ \left(\frac{f_5}{\xi^4} + \frac{a^2 f_6}{\xi^2} + a^4 f_7\right) D^4 v + \left(\frac{f_8}{\xi^5} + \frac{a^2 f_9}{\xi^3} + \frac{a^4 f_{10}}{\xi}\right) D^3 v +$$

$$+ \left\{\frac{f_{11}}{\xi^6} + \frac{a^2 f_{12}}{\xi^4} + \frac{a^4 f_{13}}{\xi^2} \quad a^6 \quad 2(\lambda + 2)\, a^2\, T_a\, \phi_4 + a^2 N T_a \frac{\phi_0^2}{\xi^{3+\delta}}\right\} D^2 v +$$

$$+ \left\{\frac{f_{14}}{\xi^7} + \frac{a^2 f_{15}}{\xi^5} + \frac{a^4 f_{16}}{\xi^3} \quad \frac{a^6 \phi_1}{\xi} \quad 2(\lambda + 2)\, a^2\, T_a\, \phi_5\right.$$

$$\left. + N T_a \frac{a^2 (1 - \lambda)}{\xi^{4+\delta}} \phi_0^2\right\} Dv + \left\{\frac{f_{17}}{\xi^8} + \frac{a^2 f_{18}}{\xi^6} + \frac{a^4 f_{19}}{\xi^4} - \frac{a^6 \phi_1}{\xi^2}\right.$$

$$\left. 2(\lambda + 2)\, a^2\, T_a\, \phi_6 - N T_a\, a^2 \left(\frac{1 + \lambda}{\xi^5} + \frac{a^2}{\xi^3}\right) \frac{\phi_0^2}{\xi^\delta}\right\} \tag{3}$$

where

$$D = \frac{d}{d\xi}, \quad \frac{r}{r} \quad \xi,\ a^2 = k^2 R_2^2,\ \delta \quad 1 \quad \delta'$$

$$T_a = \frac{\Omega_1^2 R_2^4}{\nu^2}$$

$$R \quad \frac{\alpha \beta \Omega_1^2 R_2^4}{C_1 K \nu}$$

$$N \quad \frac{R}{T_a} = \alpha \beta \nu / C_1 K \text{ and } \beta \quad \frac{(1 - \delta)(T_1 - T_2)}{(\eta^{\delta - 1} - 1)}$$

Here T_a & R are Taylor and Rayleigh number respectively, α is the coefficient of volume expansion, K is the thermal diffusivity, ν the kinematic viscosity and C_1 is a constant to be defined later.

Also

$$f_1 = G_1 + \phi_2$$
$$f_2 = (G_2 - 2G_1) + G_1\phi_2 + \phi_3$$
$$f_3 = (G_3 - 4G_2 + 2G_1) + (G_2 - G_1)\phi_2 + G_1\phi_3$$
$$f_4 = G_4 - 3\phi_2$$
$$f_5 = G_5 - 6G_3 + 6G_2 + (G_3 - 2G_2)\phi_2 + G_2\phi_3$$
$$f_6 = G_6 - 2G_4 - 3\phi_3 + G_4\phi_2$$
$$f_7 = G_7 - 8G_5 + 12G_3 + (G_5 - 3G_3)\phi_2 + G_3\phi_3$$
$$f_8 = G_8 - 4G_6 + 2G_4 + (G_6 - G_4)\phi_2 + G_4\phi_3$$
$$f_9 = G_1 + 3\phi_2$$
$$f_{10} = G_9 - 10G_7 + 20G_5 + (G_7 - 4G_5)\phi_2 + G_5\phi_3$$
$$f_{11} = G_{10} - 6G_8 + 6G_6 + (G_8 - 2G_6)\phi_2 + G_6\phi_3$$
$$f_{12} = G_{11} - 2G_1 + G_1\phi_2 + 3\phi_3$$
$$f_{13} = -12G_9 + 30G_7 + (G_9 - 5G_7)\phi_2 + G_7\phi_3$$
$$f_{14} = -8G_{10} + 12G_8 + (G_{10} - 3G_8)\phi_2 + G_8\phi_3$$
$$f_{15} = -4G_{11} + 2G_1 + (G_{11} - G_1)\phi_2 + G_1\phi_3$$
$$f_{16} = G_9(42 - 6\phi_2 + \phi_3)$$
$$f_{17} = G_{10}(20 - 4\phi_2 + \phi_3)$$
$$f_{18} = G_{11}(6 - 2\phi_2 + \phi_3)$$
$$G_1 = 3 - 6\lambda$$
$$G_2 = 14\lambda^2 - 4\lambda - 6$$
$$G_3 = -8\lambda^3 + 14\lambda^2 + 46\lambda + 12$$
$$G_4 = -2G_1$$
$$G_5 = -(\lambda^4 + 26\lambda^2 + 54\lambda + 27)$$
$$G_6 = 3 - G_2 - 3\lambda^2$$
$$G_7 = -5\lambda^4 + 12\lambda^3 + 92\lambda^2 + 120\lambda + 45$$
$$G_8 = -(15\lambda^2 + 40\lambda + 9)$$
$$G_9 = -3(\lambda^4 + 12\lambda^3 + 36\lambda^2 + 40\lambda + 15)$$
$$G_{10} = -2\lambda^3 + 5\lambda^2 + 16\lambda + 9$$
$$G_{11} = 3(\lambda^2 - 1)$$
$$\phi_0 = (C_1\xi^{\lambda+2} + C_2)$$
$$\phi_1 = -2C_1(\lambda + 2)\xi^{\lambda}/\phi_0$$

[illegible]

$$\phi_3 = (\lambda - 3)(\lambda - 2 - \delta) + \phi_1\xi^2(\delta + 7 - \lambda) + \frac{3}{2}\xi^4\phi_1^2 - \xi^2 a^2$$

$$\phi_4 = \frac{\xi^{\lambda-2}}{\phi_0}$$

$$\phi_5 = (2 + \delta)\frac{\xi^{\lambda-3}}{\phi_0} - 2C_1(\lambda + 2)\xi^{2\lambda-1}$$

$$\phi_6 = \xi^{\lambda-4}\frac{(\delta - a^2\xi^2)}{\phi_0} - C_1(\lambda + 2)\xi^{2\lambda-2}\left\{(\lambda + 3) + \delta + \phi_1\xi^2\right\},$$

where

$$C_1 = \left[\frac{1 - \mu\eta^{-2}}{\eta^{\lambda} - \eta^{-2}}\right]^2$$

$$C_2 = \eta^2 \left[\frac{(1 - \mu\eta^{-2})\,(1 - \mu\eta^{\lambda})}{(1 - \eta^{\lambda+2})\,(\eta^{\lambda} - \eta^{-2})} \right]$$

BOUNDARY CONDITIONS

From the requirement that the perturbations in velocity and temperature (i.e. u, v, w & θ) all vanish at the boundaries $\xi = \eta$ and $\xi = 1$, we get the following boundary conditions under which the solution of (3) is sought.

At $\xi = \eta$

$$v = 0,\ D^2v + \frac{1-\lambda}{\eta} Dv = 0,\ D^3v - \left(\frac{H_1}{\eta^2} + a^2\right) Dv = 0$$

and (4)

$$D^6v + \frac{G_1}{\eta} D^5v + \left(\frac{G_2}{\eta^2} - 3a^2\right) D^4v + \left(\frac{H_2}{\eta^5} + \frac{a^2 H_3}{\eta^3} + \frac{a^4 H_4}{\eta}\right) Dv = 0$$

Similarly the boundary conditions at $\xi = 1$ are obtained by substituting $\eta = 1$ in (4)

where

$$H_1 = \lambda^2 - 2\lambda + 3, \quad H_2 = G_3 H_1 + G_7 + G_5 (\lambda - 1)$$
$$H_3 = G_4 H_1 + G_3 + G_8 + (\lambda - 1) G_6, \quad H_4 = G_4 + 3(\lambda - 1) + G_1$$

SOLUTION

Let v_1, v_2, v_3, and v_4 be the four solutions of (3) which satisfy the conditions (4) and further we assume :

$$Dv = 1,\ D^4v = D^5v = D^7v = 0$$

RESULTS

The critical Taylor numbers $(T_a)_c$ depicting the onset of instability have been computed numerically for $\eta = 0.95$, 0.75 and 0.5 and $\mu = 0.450$, 0.280 and 0.125 respectively. For all the cases, values of radial velocity parameter λ have been taken as ± 1 and 0 and those of Prandtl number as 1 and 7. The cases for $\lambda = 0$, which imply the effect of a radial temperature gradient only are true for all Prandtl numbers. In case of cylinders moving in opposite directions ($\mu < 0$) results have been obtained for gap ratio 0.95 and angular velocity ratio -0.2. Other gap ratios with negative μ have been excluded due to the involvement of large amount of numerical work. These can be computed if so desired.

In the absence of suction (or injection) our results are in agreement with those obtained by Butler and Mckee[2] applying a variational technique. In figure 4, the values of $(T_a)_c$ for the parameters $\eta = 0.5$ $\mu = 0.2$, $N = \pm 30$, ± 60, 0.0 and $a = 6.30$ are suitable for this comparison. The values of N and a shown in figure 4 are equivalent to ± 0.5, ± 1.0, 0.0 and 3.15 respectively of Butler and Mckee. This is due to the difference in the definitions of Rayleigh number and Taylor number. In all the figures the numerical values of $(T_a)_c$ for $N = 0$ and $\lambda = 0$ can also be compared with results of Sparrow *et. al.*

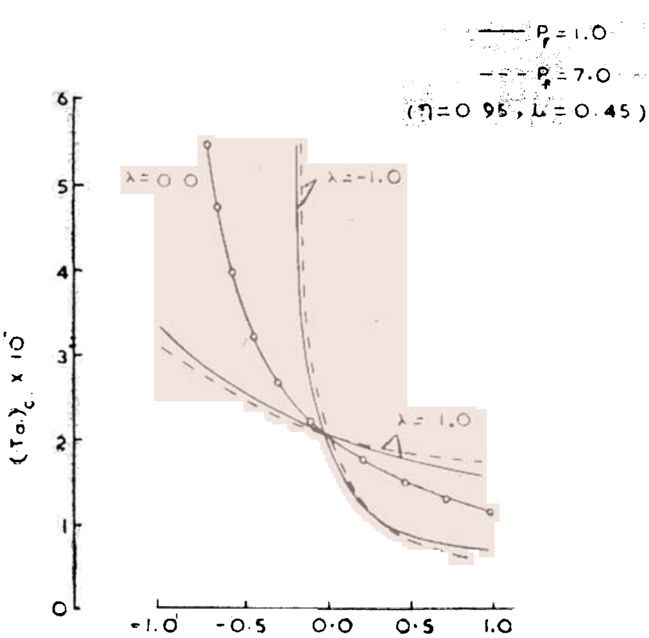

Fig. 1—Variation of critical Taylor Number $(T^*)_c$ with N for $\eta = 0{\cdot}95$, μ 0.45 and various values of λ and Pr.

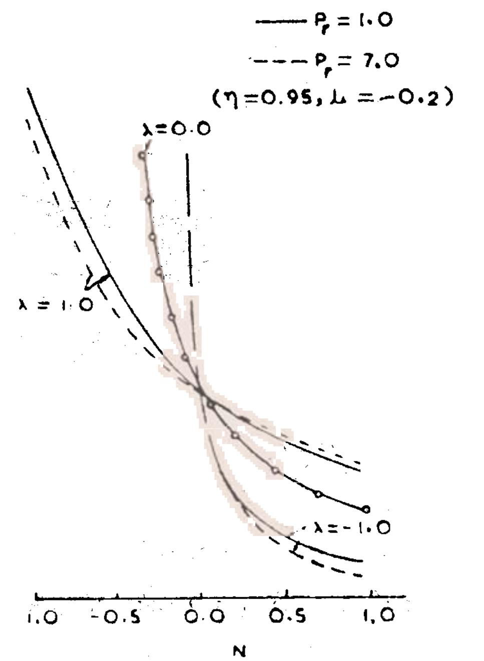

Fig. 2—Variation of critical Taylor Number $(T^*)_c$ with N for $\eta = 0{\cdot}95$, $\mu = -0{\cdot}2$ and in various values of λ and Pr.

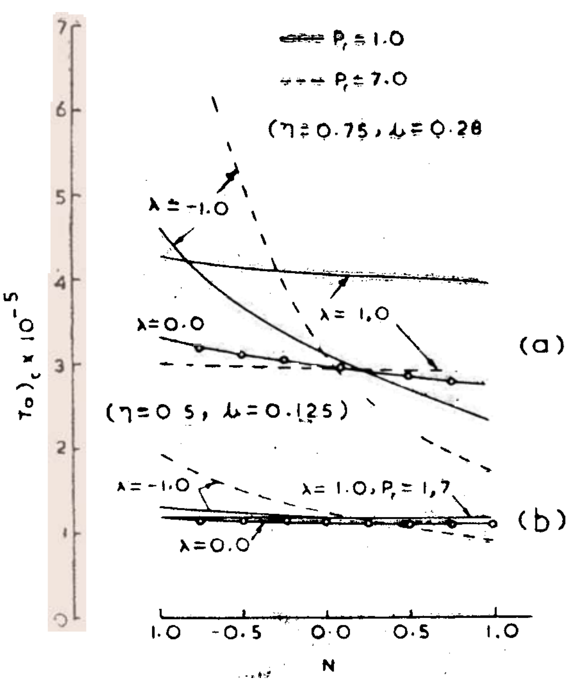

Fig. 3—Variation of critical Taylor Number $(T^*)_c$ with N for; (a) $\eta = 0{\cdot}75$, $\mu = 0{\cdot}28$; (b) —$\eta = 0{\cdot}5$, $\mu = 0{\cdot}125$ and various values of λ and Pr.

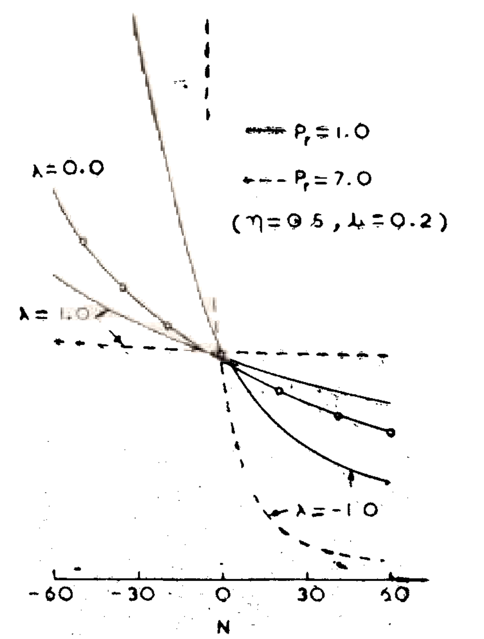

Fig. 4—Variation of critical Taylor Number $(T^*)_c$ with N for $\eta = 0{\cdot}5, \mu = 0{\cdot}2$ and various values of λ and Pr.

In general it can be concluded from figures 1 to 4 that the injection of fluid at the outer cylinder promotes an inhibiting influence on the instability of the flow in the presence of adverse temperature gradient while in case of positive temperature gradient it tends to destabilize the flow. On the other hand suction from the outer cylinder produces the reverse phenomena. Furthermore it is observed that stabilizing or destabilizing effect of the radial velocity increases with the increase in the +ve or —ve values of N. The effect is minimum in the neighbourhood of $N = 0$. Another important observation is that for wider gap ratios of the cylinders ($\eta = 0.75$ and 0.5), as the value of P_r increases from 1 to 7, the radial velocity becomes more important and largely contributes to the stability or instability of the fluid. This is well exhibited in Fig. 3.

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