PROPAGATION OF SPHERICAL MAGNETO GAS DYNAMIC SHOCK IN A RADIATIVE GAS

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In this paper the propagation of explosion waves, in a conducting gas, produced on account of a point explosion into inhomogeneous self-gravitating gas spheres is considered. Radiation effects have been taken into account and the density of the gas is assumed to vary as $r^{-\alpha}$, r being the distance from the point of explosion. In order that the mass and pressure be positive in the equilibrium state, the choice of α is restricted between 1 and 3. The variation of Mach number of the shock and energy of the wave with time have been taken into consideration.

Kynch[1] and Taylor[2] have studied the propagation of spherical shock waves by assuming the undisturbed density to vary according to some inverse power of the distance from the centre of explosion. They neglected the counter pressure and used similarity concepts to obtain the solution. Taylor's well known problem[3] solved by himself, by numerical methods comes out to be a special case of the above problem. But if counter pressure is also taken into account the problem no longer remains self-similar and numerical methods have to be employed to obtain the solution. Therefore when Sedov[4] took into account counter pressure, he assumed uniform density in the undisturbed state and thus avoided the use of numerical methods. Verma[5] studied in conducting gases the propagation of a cylindrical shock produced on account of instantaneous energy release along a straight line by assuming the density of the undisturbed state to vary as $r^{-\alpha}$, r being the distance from the axis of explosion. On account of their considerable importance and applicability in defence sciences and high speed flow, we have attempted to consider the propagation of explosion waves, in a conducting gas, produced on account of a point explosion into inhomogeneous self-gravitating gas spheres. Radiation effects have been taken into account and the density of the gas is assumed to vary as $r^{-\alpha}$, r being the distance from the point of explosion. In order that the mass and pressure be positive in the equilibrium state, the choice of α is restricted between 1 and 3. The variation of Mach number of the shock and energy of the wave with time have been taken into consideration.

EQUATIONS OF MOTION AND BOUNDARY CONDITIONS

The equations governing the magnetogasdynamic flow behind a spherical shock in a self-gravitating system including radiation effects are

$$\frac{Du}{Dt} + \frac{1}{\rho}\left\{\frac{\partial p}{\partial r} + h\frac{\partial h}{\partial r}\right\} + \frac{Gm}{r^2} = 0 \tag{1}$$

$$\frac{D\rho}{Dt} + \rho\left\{\frac{\partial u}{\partial r} + \frac{2u}{r}\right\} = 0 \tag{2}$$

$$\frac{Dh}{Dt} + h\left\{\frac{\partial u}{\partial r} + \frac{2u}{r}\right\} = 0 \tag{3}$$

$$\frac{D}{Dt}(E + E_h) + (p + p_h)\left\{\frac{\partial}{\partial t}\left(\frac{1}{\rho}\right) + u\frac{\partial}{\partial r}\left(\frac{1}{\rho}\right)\right\} + \frac{1}{\rho r^2}\frac{\partial}{\partial r}(F r^2) = 0 \tag{4}$$

where

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial r}, \quad \frac{\partial m}{\partial r} = 4\pi\rho r^2 \tag{5}$$

$$E = E_M + E_R \ , \ p = p_M + p_R$$

$$E_M = \frac{p_M}{\rho(\gamma - 1)} \ , \ E_R = \frac{3p_R}{\rho} \ , \ E_h = \frac{p_h}{\rho} \tag{6}$$

and the variables u, h, p, ρ are velocity, component of magnetic field, pressure and density of the gas respectively at radial distance r; m denotes the mass of the gas within a sphere of radius r at time t, G the gravitational constant and γ is the ratio of the specific heats.

The suffixes M, R, h attached to a symbol denote expressions for material, radiation and magnetic terms respectively.

The radiation flux F is given by

$$F = -\frac{c}{\epsilon\rho} \frac{dp_R}{dr} \ , \tag{7}$$

where c is the velocity of light and ϵ is the coefficient of opacity.

$$p_M = zp \ , \ p_R = (1 - z) \ p \ , \ (0 < z < 1)$$

so that

$$E = \frac{p}{\rho(\Gamma - 1)} \ , \tag{8}$$

where Γ is called Klimshin's coefficient given by

$$\Gamma = \frac{4(\gamma - 1) + z(4 - 3\gamma)}{3(\gamma - 1) + z(4 - 3\gamma)} \ . \tag{9}$$

The motion is bounded on the outside by a shock surface $r = R\ (t)$ moving outwards with a velocity $V = \frac{dR}{dt}$. As usual, we denote the quantities in the undisturbed region ahead of the shock by suffix 1 and in the region behind the shock by suffix 2, so that the generalised Rankine-Hugoniot shock conditions as given by Whitham[6] become

$$\rho_2 = \rho_1 \phi \tag{10}$$

$$h_2 = h_1 \phi \tag{11}$$

$$u_2 = V \left(\frac{\phi - 1}{\phi} \right) \tag{12}$$

$$p_2 = p_1 + \frac{2\ (\phi - 1)}{(\Gamma + 1) - (\Gamma - 1)\phi} \left\{ \Gamma p_1 + \ (\Gamma - 1)\ h_1^2\ (\phi - 1)^2 \right\} \tag{13}$$

where

$$V^2 = \frac{2\phi}{(\Gamma + 1) - (\Gamma - 1)\ \phi} \left[a_1^2 + b_1^2 \left\{ \left(1 - \frac{\Gamma}{2} \right) \phi + \frac{\Gamma}{2} \right\} \right] , \tag{14}$$

$a_1^2 = \frac{\Gamma p_1}{\rho_1}$ is the ordinary speed of sound and, $b_1^2 = \frac{h_1^2}{\rho_1}$ is the Alfvén velocity. According to our assumption the density in front of the shock, in the undisturbed state is given by

$$\rho_1(r) = \beta r^{-\alpha} \ , \tag{15}$$

where β is a constant. From (1) the equilibrium pressure is obtained as

$$\frac{1}{\rho_1} \left(\frac{\partial p_1}{\partial r} + h_1 \frac{\partial h_1}{\partial r} \right) + \frac{Gm_1}{r^2} = 0 \ , \tag{16}$$

where

$$m_1(r) = 4\pi \int_0^r \rho_1 r^2 \, dr = \frac{4\pi\beta \, r^{3-\alpha}}{3-\alpha)} \tag{17}$$

The equation (16) then integrates to

$$p_1 = \frac{\beta^2}{2r^{2\alpha}} \left\{ \frac{4\pi G r^2}{(\alpha-1)(3-\alpha)} - \nu^2 \right\},$$

where

$$\nu = \frac{h}{\rho} \, .$$

In order to make $m_1(r)$ positive, the equation (17) imposes the restriction

$$3 - \alpha > 0,$$

whereas to make p_1 positive, the equation (18) gives

$$< \alpha < 3$$

and

$$\frac{4\pi G r^2}{(\alpha-1)(3-\alpha)} > \nu^2 \, .$$

With the help of the equations (2), (3) and (8), the equation (4) can be written as

$$\frac{\partial p}{\partial t} + u \frac{\partial p}{\partial r} + \Gamma p \left(\frac{\partial u}{\partial r} + \frac{2u}{r} \right) + \frac{(\Gamma - 1)}{r^2} \frac{\partial}{\partial r} (F r^2) = 0 \, ,$$

SIMILARITY TRANSFORMATIONS

We now seek similarity solutions of equations (1) to (4) by making the following transformations

$$u = r t^{-1} U(\eta) \tag{23}$$

$$\rho = r^k t^n \Omega(\eta) \tag{24}$$

$$h = r^{(K+2/2)} t^{(n-2/2)} H(\eta) \tag{25}$$

$$p = r^{k+2} t^{n-2} P(\eta) \tag{26}$$

$$m = r^{k+3} t^n Z(\eta) \tag{27}$$

$$F = r^{k+3} t^{n-3} \overline{F}(\eta) \, ,$$

where

$$\eta = r^a t^b \, . \tag{29}$$

The parameters K, η, a and b are at present free but shall later be fixed according to the requirements of the problem. Let the shock surface be given by

$$\eta_1 = A t^\omega$$

where A and ω are constants. The velocity V of the shock surface is then given by

$$V = \frac{\omega - b}{a} \, \frac{R}{t}$$

From the equations (1) to (5), it can be seen that

$$\frac{\partial E}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \{uI + F\}) \quad 0,$$

where

$$E \quad \tfrac{1}{2} \rho u^2 + \frac{p}{\Gamma - 1} + \frac{h^2}{2} - \frac{G m \rho}{r}$$

and

$$I \quad \tfrac{1}{2} \rho u^2 + \frac{\Gamma p}{\Gamma - 1} + h^2 - \frac{G m \rho}{r} \qquad (34)$$

We have

$$\left.\begin{aligned} \tfrac{1}{2} \rho u^2 &= t^{n-2-b/a(K+2)} f_1(\eta) \\ \frac{p}{\Gamma - 1} &= t^{n-2-b/a(K+2)} f_2(\eta) \\ \frac{h^2}{2} &= t^{n-2-b/a(K+2)} f_3(\eta) \\ \frac{G m \rho}{r} &= t^{2n-2b/a(K+1)} f_4(\eta) , \end{aligned}\right\}$$

where $f_1(\eta)$, $f_2(\eta)$, $f_3(\eta)$ and $f_4(\eta)$ are expressible in terms of η, $P(\eta)$, $\Omega(\eta)$, $U(\eta)$, $H(\eta)$, $Z(\eta)$ and $\bar{F}(\eta)$. Supposing now

$$n - 2 - \frac{b}{a} (K + 2) = 2n - \frac{2b}{a} (K + 1),$$

we have

$$\frac{n + 2}{K} = \frac{b}{a}$$

and then, as a consequence of the equations (33) and (35), we write

$$E = t^{n-2-b/a(K-2)} f(\eta), \qquad (37)$$

so that

$$\frac{\partial E}{\partial t} = \left\{ (n - 2) - \frac{b}{a} (k + 2) \right\} \frac{E}{t} + \frac{b}{a} \frac{r}{t} \frac{\partial E}{\partial r}$$

Supposing

$$n - 2 - \frac{b}{a} (k + 2) = \frac{3b}{a}$$

we have

$$\frac{n - 2}{\;} \qquad (39)$$

From (36) and (39), we may take, without any loss of generality $K=0$, $n=-2$, $a=-5$, $b=4$.

Substituting these values of a and b in the equation (31), we get

$$V = \frac{4-\omega}{5}\ \frac{R}{t}$$

In order that the disturbance may be characterized by an outgoing shock wave, we should have

$$\omega < 4$$

Defining the Mach number M at the shock wave by $M = \frac{V}{a_1}$ and another number $M_A\left(M_A = \frac{b_1}{a_1}\right)$ and using the equations (15), (18), (29), (30) and (41), we obtain

$$M^2 = \frac{M_A^2\,(4-\omega)^2}{25\ \beta\,\nu^2}\ A^{\frac{(\alpha+2)}{5}}\ t^{\frac{\alpha(4-\omega)-2(1+\omega)}{5}}, \tag{43}$$

which shows that M is a function of time and in particular, a decreasing function if

$$\alpha < \frac{2\,(1+\omega)}{4-\omega}$$

From the equations (32) and (38), we have, on using (40)

$$\frac{\partial}{\partial r}\ r^2\,(uI+F) = \frac{\partial}{\partial r}\left(\frac{4}{5}\ \frac{r^3}{t}\ E\right),$$

which integrates to

$$r^2\,(uI+F) - \frac{4}{5}\ \frac{r^3}{t}\,E = Y\,(t),$$

where $Y(t)$ is calculated from the conditions on the inner side of the shock surface. Assuming equilibrium conditions at $r=R$, and

$$\left(\frac{dp}{dr}\right)_{r=R} = \frac{4\pi\,G\,\beta^2}{R^{2\alpha-1}\,(\alpha-1)\,(3-\alpha)},$$

we have

$$Y\,(t) = \beta\nu^2\,R^{1-\alpha}\left\{\frac{2\beta\,(2\alpha-3)\,R^{2-\alpha}}{5\,t} - c\,(1-z)\right\},$$

where we have assumed that

$$\nu^2 = \frac{4\pi\,G\,R^2}{(\alpha-1)\,(3-\alpha)}$$

The equation (46) shows that $Y\,(t)=0$ when $\alpha=\frac{3}{2}$, $Z=1$ and then the equations would admit analytic solutions. But, the condition $Z=1$ corresponds to the non-radiative case which is different from the problem treated here. However, when $Y\,(t)$ is small, its smallness depending not on α but on the factors β and A occurring in the relation

$$R = A^{-1/5}\,t^{(4-\omega/5)},$$

the values of p, ρ and u can be obtained in the present case also.

SOLUTIONS OF EQUATIONS

The condition inside the wave is obtained from the solution of equations (1) to (4). From the equations (24) to (26), (39) and (40), we get

$$\frac{\partial \rho}{\partial t} = -\left(\frac{10}{4-\omega}\right)\rho\,\frac{V}{R} \quad \left(\frac{4}{4-\omega}\right) V\,\frac{r}{R}\,\frac{\partial \rho}{\partial r} \tag{47}$$

$$\frac{\partial p}{\partial t} = -\left(\frac{12}{4-\omega}\right) p\,\frac{V}{R} - \left(\frac{4}{4-\omega}\right) V\,\frac{r}{R}\,\frac{\partial p}{\partial r} \tag{48}$$

and

$$\frac{\partial h}{\partial t} = -\left(\frac{6}{4-\omega}\right) h\,\frac{V}{R} - \left(\frac{4}{4-\omega}\right) V\,\frac{r}{R}\,\frac{\partial h}{\partial r} \tag{49}$$

respectively.

When $Y(t) \approx 0$, the equation (45) gives

$$\lambda = \frac{4}{4-\omega}\,\xi\,\frac{E}{I} - \frac{F}{\ }$$

where $\lambda = \frac{u}{V}$, $\xi = \frac{r}{R}$ and use has been made of the relation (41). Solving for $\frac{p}{\rho}$ with the help of (50), we get

$$\frac{p}{\rho} = \left[\frac{1}{2}\,\frac{V^2}{M^2}\,\frac{\Gamma-1}{\Gamma}\left\{4\xi - \lambda(4-\omega)\right\}\left\{\Gamma\xi\lambda^2 M^2 - 4(\alpha-1)\mu - 2\Gamma(\alpha-1)\mu M_A^2\right\} - 5Ft\,\xi/4R\rho\right] \Big/ \left[\xi\left\{\Gamma\lambda(4-\omega) - 4\xi - \Gamma(\Gamma-1)M_A^2\left(2\xi - \lambda(4-\omega)\right)\right\}\right]$$

where $\mu = \frac{m}{m_1}$. With the help of (2), (4), (22), (47) and (48), we obtain an equation which integrates to

$$\frac{p}{\rho^{\Gamma-1}} = C\,\xi^{-2}\left(\xi - \frac{4-\omega}{4}\lambda\right)^{-1}\left[f(\xi)\right]^{-1}$$

$$- \exp\int \frac{(\Gamma-1)\frac{4-\omega}{4}\lambda}{\left(\frac{4-\omega}{4}\lambda - \xi\right)\xi^2}\;\frac{\partial}{\partial \xi}(F\xi^2)$$

where C is a function of time and

$$\log f(\xi) = -\frac{5}{2}(\Gamma-1)\int_0^{\xi}\frac{d\xi}{\xi - \frac{4-\omega}{4}\lambda} \tag{53}$$

Using (51), the equation (52) yields

$$\rho^{\Gamma-2} = \Big[\frac{1}{2} \frac{V^2}{M^4} \frac{(\Gamma-1)}{\Gamma} \Big\{ 4\xi - \lambda (4-\omega) \Big\} \Big\{ \Gamma \xi \lambda^2 M^2 - 4(\alpha-1) \ \mu^{-2} \ (\alpha-1) \ \mu M_A^2 \Big\}$$

$$-5 Ft\xi/4R\rho \Big] \Big/ \Big[\xi \Big\{ \Gamma\lambda (4-\omega) - 4\xi - \Gamma(\Gamma-1) M_A^2 \Big(2\xi - \lambda(4-\omega)\Big)\Big\} . C\xi^{-2} \Big[f(\xi) \Big]^{-1}$$

$$\cdot \Big(\xi - \frac{4-\omega}{4} \lambda \Big)^{-1} - \exp \int \Big\{ (\Gamma-1) \Big(\frac{4-\omega}{4} \Big) \lambda \Big\} \Big/ \Big\{ \Big(\frac{4-\omega}{4} \lambda - \xi \Big) \xi^2 \frac{\partial}{\partial \xi} (F\xi^2) \Big] \qquad (54)$$

Further from equations (3) and (49), we obtain an equation which integrates to

$$h = D\xi^{-2} \Big(\frac{4-\omega}{4} \lambda - \xi \Big)^{-1} g(\xi)$$

where D is a function of time, and

$$\log g(\xi) = \frac{3}{2} \int_1^{\xi} \frac{d\xi}{\xi - \frac{4-\omega}{4} \lambda}$$

From (53) and (55), we see that the singularity in our solution is given by $\xi = \lambda \frac{4-\omega}{4}$, which can be identified with the inner boundary of the wave.

INNER BOUNDARY AND ENERGY OF THE WAVE

When

$$\xi = \lambda \frac{4-\omega}{4}$$

we have

$$\frac{dr_1}{dt} = \frac{4}{5} \frac{r_1}{t}$$

which on integration gives

$$t^4 r_1^{-5} = \text{constant},$$

where r_1 is the radius of the inner boundary. In terms of η, we have

$$\eta_1 = r_1^{-5} t^4 = \text{constant}$$

In equilibrium, the original gravitational, magnetic and heat energy of the gas enclosed within a radius equal to the radius of the shock is given by

$$4\pi \int_0^R \Big[\frac{p_1}{\Gamma-1} \times \frac{h_1^2}{2} - \frac{Gm_1 P_1}{} \Big] r^2 dr = \frac{8\pi^2 \beta^2 G (3-2\alpha) R^{5-2\alpha}}{(3-\alpha)(\alpha-1)(5-2\alpha)}$$

When $\alpha = 3/2$ but not equal to 5/2, the above expression vanishes showing that the original energy of the gas is zero. Hence, we have to make a continuous set of explosions in order that the energy of the wave may vary as it progresses. After the explosion, the total energy T of the configuration is given by

$$T = - \frac{4\pi}{5} \int_{\eta_1}^{\eta_0} \left[\frac{1}{2} \, \Omega \, U^2 + \frac{P}{\Gamma - 1} + \frac{H^2}{2} \quad Gz \, \Omega \right] \eta^{-2} \, d\eta \tag{57}$$

where η_0 and η_1 are the values of η at the shock front and on the surface of the inner boundary respectively. Since η_1 has been shown to be constant, it is easily seen that T is a function of η_0 only. But by assumption, $\eta_0 = At^{\omega}$ and hence T changes with time. By taking ω sufficiently small, the energy can be made to increase or decrease with time at a reasonable rate. When $\omega = 0$ the total energy is constant.

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