ON THE DYNAMIC STABILITY OF A MISSILE

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The P-method given by Parks and Pritchard has been used to discuss the stability behaviour of a missile in free flight. General stability criteria for aerodynamic stabilisation have been obtained for slowly varying coefficients. The effect of pressure gradient on the stability of a coasting rocket has been explicitly examined. It is observed that the positive Magnus moment parameter ensures stability whereas a negative moment parameter would enhance the requirements of a larger stability margin.

The qualitative methods have been extensively followed in analysing the stability behaviour of bodies in motion. The second method of Lypunov provides, in particular, relevant practical information. Pritchard[1,2] has discussed the P-method of constructing Lypunov functions, in few cases, for the general second order system.

$$L(x) \equiv \ddot{x} + A\dot{x} + Bx = 0 \tag{1}$$

where A and B are (mxm) matrices.

In this paper we construct Lypunov functions for the linear systems with two degrees of freedom with the help of the P-method. The independent variable assumes values on the positive half ray and in stability problems of flight mechanics, this variable is generally the path length 's' of the trajectory. The coefficient matrices A and B are assumed to depend on this variable and vary slowly. An overhead dot denotes differentiation with respect to 's' in this paper. We discuss in detail the differential equation of Murphy[3] for illustrations.

MATHEMATICAL ANALYSIS

The coefficient matrices A and B are split into its symmetric and skew symmetric components as

$$A = A_1 + A_2$$
$$B = B_1 + B_2$$

where subscripts 1 and 2 denote symmetry and skew symmetry respectively. The expanded form of the system is interpreted, following Magnus[4], as

$$-A_1\dot{x} = \text{damping or exciting forces,}$$

$$-A_2\dot{x} = \text{gyroscopic forces,}$$

$$-B_1 x = \text{conservative positional forces,}$$

$$-B_2 x = \text{non-conservative positional forces.}$$

(These are the Magnus type forces in flight mechanics.)

The matrices A_2 and B_2 may be further simplified to

$$\left.\begin{aligned} A_2 &= \mu S \\ B_2 &= \mu\phi S \end{aligned}\right\} \tag{2}$$

where $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and μ and ϕ are scalar functions defined as

$$\left.\begin{aligned} \mu &= \text{gyroscopic parameter} \\ \phi &= \text{Magnus moment parameter} \end{aligned}\right\} \tag{3}$$

The functions μ and ϕ are continuous and twice differentiable along the flight path.

The generating operator $N(x) = 2\dot{x} + Px$ forms the inner product of $L(x)$ as

$$< L(x), N(x) > = L'(x) N(x)$$

where $L'(x)$ indicates the transpose of $L(x)$. Now

$$< L(x), N(x) > + < N(x), L(x) > =$$

$$= 2(\ddot{x}'\dot{x} + \dot{x}'\ddot{x}) + (\ddot{x}' P x + x' P' \ddot{x}) + 4\dot{x}' A_1 \dot{x} +$$

$$+ x'(P' A + 2\dot{B}')x + x'(A'P + 2B)x + x'(B'P + P'B)x$$

and since $L(x) = 0$, the right hand side expression vanishes and further suitably rearranging the terms, we write

$$[2\dot{x}'\dot{x} + \dot{x}' P x + x' P' \dot{x} + x'(2B_1 + 2Q)x]^{\cdot} =$$

$$= \dot{x}'(-4A_1 + P + P')\dot{x} + \dot{x}'(\dot{P} - A'P - 2B_2 + 2Q)x +$$

$$+ x'(\dot{P}' - P'A + 2B_2 + 2\dot{Q})x + x'(-B'P - P'B + 2\dot{B}_1 + 2\dot{Q})x$$

where P and Q are hitherto arbitrary matrices. The matrix Q is symmetric. On substituting for A_2 and B_2 from (2), we get

$$[2\dot{x}'\dot{x} + \dot{x}' P x + x' P' \dot{x} + x'(2B_1 + 2Q)x]^{\cdot} =$$

$$= -[\dot{x}'(4A_1 - P - P')\dot{x} + \dot{x}'(-\dot{P} + A_1 P - \mu S P + 2\mu\phi S - 2Q)x +$$

$$+ x'(-\dot{P}' + P' A_1 + \mu P' S - 2\mu\phi S - 2\dot{Q})\dot{x} + x\{(B_1 P + P' B_1) +$$

$$+ \mu\phi(P'S - SP) - 2\dot{B}_1 - 2\dot{Q}\} x] \tag{4}$$

AERODYNAMIC STABILITY

The linear equation of missile motion for pitch-yaw coupling at small angles of attack and in normal flight assumes the form (1). We intend to discuss the motion independent of roll-velocities such that the missile can be inertia stabilised and an inspection of the equation (4) reveals that a choice $P = 2\phi I$ would eliminate the spin parameter μ. A straight forward simplification of this equation yields

$$[\dot{x}'\dot{x} + 2\phi\dot{x}' x + x'(B_1 + Q)x]^{\cdot} =$$

$$= -[2\dot{x}'(A_1 - \phi I)\dot{x} + \dot{x}'(-\dot{\phi} I + \phi A_1 - Q)x +$$

$$+ x'(-\dot{\phi} I + \phi A_1 - Q)\dot{x} + x'(2\phi B_1 - \dot{B}_1 - \dot{Q})x]$$

It is obvious now that a choice $Q = \phi A_1 - \dot{\phi} I$ would eliminate the skew terms on the right and the asymptotic stability is ensured. The reduced form is

$$[\dot{x}' \dot{x} + 2 \phi \dot{x}' x + x' (B_1 + \phi A_1 - \dot{\phi} I) x]^{\cdot} =$$

$$= - [2 \dot{x}' (A_1 - \phi I) \dot{x} + x' \{2 \phi B_1 - \dot{B}_1 - (\phi A_1)^{\cdot} + \ddot{\phi} I \} x] . \tag{5}$$

A choice of the Lypunov function

$$V = \dot{x}' \dot{x} + 2 \phi \dot{x}' x + x' (B_1 + \phi A_1 - \dot{\phi} I) x \tag{6}$$

which is continuous, possesses partial derivatives and is positive definite, provided $B_1 + \phi A_1 - \dot{\phi} I - \phi^2 I$ is positive, yields a derivative $\dot{V}$ along the motion of the body as

$$\dot{V} = - [2 \dot{x}' (A_1 - \phi I) \dot{x} + x' \{2 \phi B_1 - \dot{B}_1 + \ddot{\phi} I - (\phi A_1)^{\cdot} \} x]. \tag{7}$$

This is again negative definite provided $A_1 - \phi I$ and $2 \phi B_1 - \dot{B}_1 + \ddot{\phi} I - (\phi A_1)^{\cdot}$ are positive. The solution of $\dot{V}=0$ is the null solution $\dot{x} = x = 0$. When the matrices $A - \phi I$ and $2 \phi B_1 - \dot{B}_1 + \ddot{\phi} I - (\phi A_1)^{\cdot}$ are singular, the paths given by $(A_1 - \phi I) \dot{x} = 0$ and $\{ 2 \phi B_1 - \dot{B}_1 + 2 \ddot{\phi} I - (\phi A_1)^{\cdot}\} x = 0$ corresponding to $\dot{V} = 0$ are not the half trajectories of the motion and therefore the integral curves do not coincide with the curves $V =$ constants, at any point of the phase space. V and $\dot{V}$ are bounded on the half real line and a $V_0 (x)$ can be selected such that $V (x, s) \geqslant V_0 (x)$. We can express them as

$$M \| x \|^2 \geqslant V (x, s) \geqslant m \| x \|^2 \text{ and } \dot{V} \leqslant - l \| x \|^2$$

where M, m and l are positive constants. The motion represented by (1) is asymptotically stable in the large with respect to the norm $\| x \|$.

For the slowly varying matrices A and B, we may ignore $\ddot{\phi} I$ and state the conditions for asymptotic stability as follows.

If the system (1) is to be asymptotically stable in the sense of Lypunov then matrices

$$B_1 + \phi A_1 - \dot{\phi} I, \; - \phi^2 I, \quad A_1 - \phi I \text{ and } 2 \phi B_1 - \dot{B}_1 - (\phi A_1)^{\cdot} \tag{8}$$

are to be positive definite.

The stability conditions given above are independent of the gyroscopic parameter μ. The conservative positional forces in this case are acting as the restoring agency for the body motion to ensure stability. If one ignores the Magnus type of forces ϕ, the ordinary conditions (B_1 and A_1 are to be positive definite) are obtained from the same Lypunov function. On the other hand if one assumes the coefficient matrices A and B to be constants, the reduced conditions B_1 and $A_1 - \phi I$ (Cf Pritchard[2]) are to be positive, follow from (4).

The contributions occurring from the derivatives need further examination. For this we analyse the equation of Murphy[3] where

$$x = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}, \text{ the yaw and pitch components of a missile.}$$

$$A_1 = \rho D I$$
$$A_2 = G S$$
$$B_1 = -\rho M I$$
$$B_2 = \rho G T S.$$

Since it is intended to examine the effect of density variations $\rho = \rho_0 \exp(-ks)$, the aerodynamic parameters D, M, T and the roll G are assumed to be non-varying.

The stability conditions (8) work out to be

$$M < T \{ \rho (D - T) + k \} \tag{9}$$

$$D - T > 0 \tag{10}$$

$$M (2 \rho T + k) < 2 \rho k T D. \tag{11}$$

The inequalities (9) and (11) provide the aerodynamic arm of the missile. Since D is generally positive and if for the given shape of the missile T is also positive, the requirements on stability are met as the aerodynamic moment parameter M is negative and D is normally sufficiently greater than T. If the Magnus moment parameter T is negative, the damping $D - T > 0$ is satisfied, whereas the inequalities (9) and (11) pecify the moment arm of M which is obviously a definite improvement over the stability requirement of $M > O$.

We consider the vertical flight of a sounding rocket at 80,000 ft. above the ground level with the following aerodynamic coefficients[5].

$$M = -1 \cdot 6 \times 10^{-2}$$
$$D = 4 \cdot 43 \times 10^{-1}$$
$$G = 2 \cdot 5 \times 10^{-5}$$
$$\rho = \rho_0 \exp(-\beta h) = \rho_0 \exp(-ks)$$

where $k = \beta \sin \psi$, (ψ being the average flight path angle) and

$$\rho_0 = 0 \cdot 002377 \text{ slugs/cubic ft.}$$
$$\beta = 1/22{,}000$$
$$k = 4 \cdot 54 \times 10^{-5}$$

The valid limits for the Magnus moment parameter T calculated from equations (9) through (11) are

$$-0 \cdot 360 < T < 0 \cdot 443$$

and the corresponding values, in the absence of pressure gradient are $0 < T < 0 \cdot 443$. The negative range is a useful information and improvement over the stability range in the constant density atmosphere. The Magnus coefficients are sensitive to Reynold numbers and the angle of attack. The limits given by (9) through (11) will serve to determine the range of angle of attack, if the tabulated values of the Magnus coefficient for the rocket are available. Secondly, if one considers even typical values of damping and Magnus moment parameter D, T of the same order, the aerodynamic coefficient for the given damping parameter is $M < 1 \cdot 107 \times 10^{-5}$ and improves upon the requirement of $M < 0$.

CONCLUSION

Introduction of the arbitrary symmetric matrix Q in the Lypunov function (6) has an advantage over the form given by Pritchard[2]. This form is suitable for generalising both the aerodynamic and gyroscopic form of stability behaviour for slowly varying coefficients along the path of the missile. The method does not assume the smallness of any aerodynamic coefficient in the comparisions and also of their products as is necessary to obtain meaningful results from the WBJK method. Secondly, there is no choice to differentiate between the spin and inertia stability in the WBJK technique.

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