AN ANALYTIC SOLUTION TO THE PROPAGATION OF CYLINDRICAL BLAST WAVES IN A RADIATIVE GAS

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In this paper, we have obtained a set of non-similarity solutions in closed forms for the propagation of a cylindrical blast wave in a radiative gas. An explosion in a gas of constant density and pressure has been considered by assuming the existence of an initial uniform magnetic field in the axial direction. The disturbance is supposed to be headed by a shock surface of variable strength and the total energy of the wave varies with time.

Propagation of cylindrical blast waves in a plasma, under a constant axial current, has been studied by Greenspan[1], Greifinger and Cole[2] and Christer and Helliwell[3]. These authors have sought similarity solutions of the problem for a very strong instantaneous line explosion. Korobeinikov[4] has considered the problem of an explosion in a gas of constant density and pressure, by assuming the existence of an initial uniform magnetic field in the axial direction. He reduced the equations of motion in terms of two independent variables in suitable forms to effect numerical computations.

In the present paper, we have considered a problem similar to that of Korobeinikov[4]. We find such solutions which maintain their similarity form except at the shock surface heading the disturbed region. After Sedov[5], we name such motions as non-self-similar. The strength of the shock propagated does not remain constant. The variation of both the Mach number of the shock as well as the energy of the wave with time has been considered. We have also included radiation effects which enter in three forms, (*i*) radiation flux (*ii*) radiation pressure, and (*iii*) radiation energy density. While radiation flux becomes important even in laboratory experiments the effects of radiation pressure and radiation energy density simply get added to the gas pressure and gas energy density. Radiation effects are important for supersonic aerodynamics, nuclear explosions and nuclear energy devices, the defence applications of which cannot be over emphasized.

EQUATIONS OF MOTION AND BOUNDARY CONDITIONS

The equations, governing the motion of the fluid behind a magnetogasdynamic cylindrical shock wave in a radiative gas, can be expressed, in the usual notation as,

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial r} + \rho \frac{\partial u}{\partial r} + \frac{\rho u}{r} = 0 \tag{1}$$

$$\frac{\partial H}{\partial t} + u \frac{\partial H}{\partial r} + H \frac{\partial u}{\partial r} + \frac{uH}{r} = 0 \tag{2}$$

$$\frac{\partial}{\partial t}(E + E_H) + u \frac{\partial}{\partial r}(E + E_H) + (p + p_H) \left\{ \frac{\partial}{\partial t} \left(\frac{1}{\rho}\right) + u \frac{\partial}{\partial r} \left(\frac{1}{\rho}\right) \right\} + \frac{1}{\rho r} \frac{\partial}{\partial r}(Fr) = 0 \tag{3}$$

where,

$$E = E_M + E_R , \quad p = p_M + p_R , \quad E_H = \frac{H^2}{2\rho}$$

and

$$p_H = \frac{H^2}{2}$$

The suffixes M, R and H attached to a symbol denote expressions for material, radiation and magnetic terms respectively. Also we have,

$$E_M = \frac{p_M}{\rho\,(\gamma - 1)}\ ; \quad E_R = \frac{3p_R}{\gamma}$$

where γ, as usual, is the ratio of the specific heats. The radiation flux F is given by

$$F = -\frac{C}{\epsilon\rho}\ \frac{dp_R}{dr} \tag{4}$$

where C is the velocity of light and ϵ is the coefficient of opacity. Also, $p_M = Z_p$; $p_R = (1-Z)_p$; $(0<Z<1)$, so that,

$$E = \frac{p}{\rho\ (\Gamma - 1)} \tag{5}$$

where p is called the Klimshin's coefficient, given by,

$$\Gamma = \frac{4\ (\gamma - 1) + Z\ (4 - 3\gamma)}{3\ (\gamma - 1) + Z\ (4 - 3\gamma)} \tag{6}$$

Also by assuming the adiabacy for each element of the fluid, we have,

$$\frac{\partial p}{\partial t} + u4\ \frac{\partial p}{\partial r} = \frac{\Gamma p}{\rho}\ \left(\frac{\partial \rho}{\partial t} + u4\ \frac{\partial \rho}{\partial r} \right) \tag{7}$$

where u, $P\rho$, and H are the velocity, pressure, density and axially directed magnetic field at a distance r from the line of explosion at any time t. The motion is supposed to be bounded on the outside by a cylindrical shock surface $r = R(t)$, moving outward with a velocity,

$$V = \frac{dR}{dt}\ .$$

With the help of (2) and (5) the equation (3) can be written as,

$$\frac{\partial p}{\partial t} + u\ 4\ \frac{\partial p}{\partial r} + \Gamma p\ \left(\frac{\partial 4u}{\partial r} + \frac{u}{r} \right) + \frac{\Gamma - 1}{r}\ \ \frac{\partial}{\partial r}\,(Fr) = 0 \tag{8}$$

If p_∞, ρ_∞ and H_∞ are the pressure, density and magnetic field in the undisturbed state, and p_n, ρ_n, u_n and H_n denote corresponding expressions just behind the shock, we have the following form of usual shock conditions[4].

$$\frac{u_n}{V} = 1 - \frac{\rho_\infty}{\rho_n} \tag{9}$$

$$\frac{H_n}{H_\infty} = \frac{\rho_n}{\rho_\infty} \tag{10}$$

$$\frac{p_n}{p_\infty}\ \frac{1}{\Gamma M^2} + \frac{H^2{}_n}{H^2{}_\infty}\ \frac{1}{2M^2{}_0} - \left(1 - \frac{\rho_\infty}{\rho_n}\right) = \frac{1}{2M^2{}_0} + \frac{1}{\Gamma M^2} \tag{11}$$

$$\left(\frac{p_n}{p_\infty}\ \frac{1}{\Gamma M^2} + \frac{H^2{}_n}{H^2{}_\infty}\ \frac{1}{2M^2{}_0} \right) \left(1 - \frac{\rho_\infty}{\rho_n}\right) + \frac{1}{\Gamma\ (\Gamma - 1)\ M^2} + \frac{1}{2M^2{}_0} -$$
$$- \frac{1}{2}\left(1 - \frac{\rho_\infty}{\rho_n}\right)^2 - \frac{\rho_\infty}{\rho_n}\ \left\{ \frac{1}{\Gamma\ (\Gamma - 1)}\ \frac{p_n}{p_\infty}\ \frac{1}{M^2} + \frac{H^2{}_n}{H^2{}_\infty\ 2M^2{}_0} \right\} = 0 \tag{12}$$

where,

$$M^2 = \frac{\rho_\infty \; V^2}{\Gamma \; p_\infty}$$

and

$$M^2_0 = \frac{4\pi \; V^2 \; \rho_\infty}{H^2_\infty} \; .$$

The parameters M^2 and M^2_0 should be such that $(\rho_\infty \,/\, \rho_n) < 1$, a condition which is easily seen to be satisfied for M^2 and M^2_0 sufficiently large. We now seek a solution of equations (1), (2), (7) and (8) in the form :

$$\left.\begin{aligned} u &= \frac{r}{t} \, U(\eta) \\ p &= r^{K+2} \, t^{\lambda-2} \, P(\eta) \\ \rho &= r^{K} \, t^{\lambda} \, \Omega(\eta) \\ H &= r^{(K+2)/2} \, t^{(\lambda-2)/2} \, L(\eta) \\ F &= r^{K+3} \, t^{\lambda-3} \, \bar{F}(\eta) \end{aligned}\right\} \qquad (13)$$

where,

$$\eta = r^a \, t^b$$

and the constants K, λ, a and b are, determined from the conditions of the problem.

Let the shock surface be given by,

$$\eta_0 = A \, t^\mu$$

where A and μ are constants, $(0 < \mu < 1)$, so that the velocity of the shock surface is given by

$$V = \left(\frac{\mu - b}{a} \right) \frac{R}{t}$$

From the equations (1), (2), (7) and (8), it can be shown that

$$\frac{\partial E}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} \, r\,(uI + F) = 0 \qquad (14)$$

where

$$E = \frac{1}{2} \rho u^2 + \frac{p}{\Gamma - 1} + \frac{H^2}{8\pi} \qquad (15)$$

and

$$I = \frac{1}{2} \rho u^2 + \frac{\Gamma p}{\Gamma - 1} + \frac{H^2}{4\pi} \qquad (16)$$

Now,

$$\frac{p}{\Gamma - 1} = t^{\lambda - 2 - \frac{b}{a}(K+2)} \, \phi_1(\eta)$$

$$\tfrac{1}{2} \rho u^2 = t^{\lambda - 2 - \frac{b}{a}(K+2)} \, \phi_2(\eta)$$

$$\frac{H^2}{8\pi} = t^{\lambda - 2 - \frac{b}{a}(K+2)} \, \phi_3(\eta)$$

so that,

$$E = t^{\lambda - 2 - \frac{b}{a}(K+2)} \phi(\eta) \tag{17}$$

Then from (17) it follows that,

$$\frac{\partial E}{\partial t} = \left\{ \lambda - 2 - \frac{b}{a}(K \times 2) \right\} \frac{E}{t} + \frac{b}{a} \frac{r}{t} \frac{\partial E}{\partial r} \tag{18}$$

It is easy to see that in order to get a perfect integral of the combination of equations (14) and (18), we must have

$$\lambda - 2 - \frac{b}{a}(K + 2) = \frac{-2b}{a} \tag{19}$$

so that,

$$\frac{\lambda - 2}{K + 4} = \frac{b}{a}$$

We choose without any loss of generality,

$$b = 1,\ a = 2(\mu - 1),\ \lambda = 2,\ K = -4 \tag{20}$$

and then from

$$V = \left(\frac{\mu - b}{a} \right) \frac{R}{t}$$

we get,

$$V = \frac{1}{2} \frac{R}{t} \tag{21}$$

The characteristic parameters M and M_0 at any time t, are given by,

$$M^2 = \frac{A^{\frac{1}{\mu - 1}} \rho_\infty}{4 \Gamma p_\infty t} \tag{22}$$

and

$$M_0^2 = \frac{\pi A^{\frac{1}{\mu - 1}} \rho_\infty}{H^2_\infty t}$$

the constant A being determined from the value of the explosion energy. From (22) we easily deduce that M and M_0 are functions of time.

SOLUTION OF EQUATIONS

The condition inside the wave is obtained from the solution of the equations (1) to (3) and (8).

From the equations (13), (20) and (21), we get

$$\frac{\partial \rho}{\partial t} = \frac{4 V \rho \mu}{(\mu - 1) R} + \frac{1}{(\mu - 1)} V \frac{r}{R} \frac{\partial \rho}{\partial r}, \tag{23}$$

$$\frac{\partial p}{\partial t} = \frac{2pV}{(\mu-1)R} + \frac{1}{(\mu-1)}\frac{r}{R}V\frac{\partial p}{\partial r}, \tag{24}$$

and

$$\frac{\partial H}{\partial t} = \frac{HV}{(\mu-1)R} + \frac{1}{(\mu-1)}\frac{r}{R}V\frac{\partial H}{\partial r}. \tag{25}$$

From the equations (14) and (18), with values of $\dot{R}$, λ, a and b determined above we have,

$$\frac{\partial}{\partial r'}\Bigg[\frac{1}{(\mu-1)}r'^2\left\{\frac{1}{2}\frac{\rho}{\rho_n}\frac{\rho_n}{\rho_\infty}u'^2 + \frac{p}{p_n}\frac{p_n}{p_\infty(\Gamma-1)\Gamma M^2} + \frac{H^2}{H^2_n}\frac{H^2_n}{H^2_\infty 2M^2_0}\right\} +$$
$$+ r'\left\{u'\left(\frac{1}{2}\frac{\rho}{\rho_n}\frac{\rho_n}{\rho_\infty}u'^2 + \frac{p}{p_n}\frac{p_n}{p_\infty(\Gamma-1)M^2} + \frac{H^2}{H^2_n}\frac{H^2_n}{H^2_\infty M_0^2}\right) -\right.$$
$$\left.- \frac{C}{\epsilon\rho}\frac{(1-Z)}{V^3\rho_\infty}\frac{dp}{dr'}\right\}\Bigg] = 0$$

where

$$r' = \frac{r}{R} \quad \text{and} \quad u' = \frac{u}{V}.$$

Integrating the above equation and applying the boundary conditions

$$r' = 1,\ u' = u'_n,\ \rho = \rho_n,\ p = p_n,\ H = H_n$$
$$\frac{dp}{dr'} = 1$$

we get,

$$r'^2\left\{\frac{1}{2}u'^2\frac{\rho}{\rho_n}\frac{\rho_n}{\rho_\infty} + \frac{1}{\Gamma(\Gamma-1)M^2}\frac{p}{p_n}\frac{p_n}{p_\infty} + \frac{H^2}{2H_n^2}\frac{H_n^2}{H^2_\infty M_o^2}\right\} -$$
$$- (1-\mu)r'\left\{u'\left(\frac{1}{2}u'\frac{\rho}{\rho_n}\frac{\rho_n}{\rho_\infty} + \frac{1}{(\Gamma-1)M^2}\frac{p}{p_n}\frac{p_n}{p_\infty} + \frac{H^2}{H^2_n}\frac{H_n^2}{H^2_\infty M_o^2}\right) -\right.$$
$$\left.- \frac{C(1-Z)}{\epsilon\rho\rho_\infty V^3}\frac{dp}{dr'}\right\} = \left\{\frac{1}{2}u'_n\frac{\rho_n}{\rho_\infty} + \frac{1}{\Gamma(\Gamma-1)M^2}\frac{p_n}{p_\infty} + \frac{H_n^2}{H^2_\infty 2M^2_o}\right\} -$$
$$-(1-\mu)\left\{u'_n\left(\frac{1}{2}u'^2_n\frac{\rho_n}{\rho_\infty} + \frac{1}{(\Gamma-1)M^2}\frac{p_n}{p_\infty} +\right.\right.$$
$$\left.\left.+ \frac{H_n^2}{H^2_\infty}\frac{1}{M_o^2}\right) - \frac{C(1-Z)}{\epsilon\rho_n\rho_\infty V^3}\right\}$$

The equation (1), by using (23) may be written as,

$$\frac{1}{\rho}\frac{\partial\rho}{\partial r'} = -\frac{1}{r'} - \frac{1-(1-\mu)\frac{\partial u'}{\partial r'}}{r'-(1-\mu)u'} - \frac{4\mu}{r'-(1-\mu)u'} + \frac{2}{r'-(1-\mu)u'}$$

which on integration gives,

$$\frac{\rho}{\rho_n} = r'^{-1}(r'-(1-\mu)u')^{-1}.$$

$$\cdot\left\{1-(1-\mu)\mu'_n\right\}\exp\int_1^{r'}\frac{2-4\mu}{r'-(1-\mu)u'}dr' \tag{27}$$

The equation (2), by using (25) may be written as,

$$\frac{1}{H}\frac{\partial H}{\partial r'} = -\frac{1}{r'} - \frac{1-(1-\mu)\dfrac{\partial u'}{\partial r'}}{r'(1-\mu)u'} + \frac{1}{r'-(1-\mu)u'}$$

which on integration gives,

$$\frac{H}{H_n} = r'^{-1}\,(r'-(1-\mu)u')^{-1}\,\{1-(1-\mu)u'_n\}.\ \exp\int_1^{r'}\frac{dr'}{r'-(1-\mu)u'} \tag{28}$$

Similarly, the equation (8), by using (24) may be written as

$$\frac{1}{p}\frac{\partial p}{\partial r'} - \frac{\Gamma}{\rho}\frac{\partial \rho}{\partial r'} = \frac{4\mu\Gamma-2}{r'-(1-\mu)u'} + \frac{(\Gamma-1)(1-\mu)}{r'Vp\{r'-(1-\mu)u'\}}\frac{\partial}{\partial r'}(Fr')$$

Integrating the above and applying the conditions,

$$r' = 1,\ u = u'_n,\ \rho = \rho_n \text{ and } p = p_n = C\rho_n^{\ \Gamma}$$

we obtain,

$$\frac{p}{p_n} = \left(\frac{\rho}{\rho_n}\right)^{\Gamma}\exp\int_1^{r'}\left\{\frac{4\mu\Gamma-2}{r'-(1-\mu)u'} + \frac{(\Gamma-1)(1-\mu)}{r'Vp(r'-(1-\mu)u')}\frac{\partial}{\partial r'}(Fr')\right\}dr' \tag{29}$$

Equations (26) to (29) give the solution of our problem. They constitute a set of non-similarity solutions in closed forms.

ENERGY OF THE WAVE

When

$$r' = (1-\mu)\ u' = \frac{u}{V}(1-\mu),\ tr^{2(\mu-1)} = \text{constant (say } \eta_1)$$

The total energy $\psi\ (t)$ of the configuration after explosion is given by,

$$\psi = 2\pi\int_0^R\left[\tfrac{1}{2}\rho u^2 + \frac{p}{\Gamma-1} + \frac{H^2}{8\pi}\right] r\ dr$$

which, as a consequence of (13) and $\eta = r^a t^b$, takes the form,

$$\psi\ (t) = -\frac{\pi}{(1-\mu)}\int_0^{\eta_0}\left[\tfrac{1}{2}U^2(\eta)\,\Omega(\eta)\eta^{-1} + \frac{P(\eta)\,\eta^{-1}}{(\Gamma-1)} + \frac{L^2(\eta)\,\eta^{-1}}{8\pi}\right]d\eta \tag{30}$$

e see from (30) that $\psi\ (t)$ is a function of η_0 only and by assumption $\eta = At^{\mu}$, $\mu \neq 0$ but $\mu < 1$. Hence ψ is a veriable which changes with time.

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