RELIABILITY OF A MODULAR STANDBY REDUNDANT SYSTEM WITH UNRECOVERABLE FAILURES

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This paper considers a standby redundant system consisting of two identical modules. Each module is composed of N distinct components connected in series. The failure of a module may be attributed due to the failure of any of its N components. The ith component of a failed module has an arbitrary repair time CDF, $G_i(t)$. The standby module has been assumed to have a nonzero hazard rate even when unpowered. The failure of an on-line module is identified through a sensing device which has a probability 'c' of successfully recovering a fault in the on-line module. Expressions for the distribution of the Time to First System Failure (TFSF), the expected TFSF, and the reliability of the system have been derived by using the theory of Markov renewal processes.

Many large systems e.g., computers, radars etc. consist of an arbitrary number N of distinct components connected in series in the sense of reliability; i.e. the system is good if and only if all its N components are good. Each component exhibits its own peculiarities as regards the probability of failure, the times to repair, the cost involved and its effect on the reliability of the entire system. A failure of a component results in breakdown of the entire system, and the time spent in repairing the system varies according to whether to relates to the failure of a major component or a minor component. The reliability characteristics of such systems have been analysed earlier by Garg[1] and Srinivasan[2].

Sometimes, these systems are required to operate for long mission times without interruption because system failure during this time is costly and dangerous. Redundancy is often needed when the approach to increasing system reliability through the advances of component technology is insufficient. Thus, in the event of the failure of the system due to any of its N components, the entire system is replaced with an identical system. Such a procedure is essential in the modern complex electronic equipment such as a computer where a faulty module is replaced by a new module. A recent survey[3] indicates that due to advances in the solid-state technology and integrated circuits, modular redundancy involving standby modules have earned an edge over other types of redundancies.

In this paper, the reliability of certain systems composed of two identical modules connected in standby redundancy have been analysed. Each module is, in turn, composed of a finite number N of dissimilar components connected in series. Both the modules are individually capable of performing the same system function. In the event of failure of the on-line module, the system continues to function with the help of the standby module which becomes on-line by replacing instantaneously the failed module. The ith component ($i=1, 2, \ldots\ldots\ldots, N$) responsible for the failure of the on-line module is repaired with an arbitrary repair time distribution $G_i(t)$, and after repair is completed, the repaired module is put on standby. The process continues in this way until both the modules are down together.

In order to enable immediate identification of a fault in the on-line module, a sensing device is associated with the system which constantly probes the state of all the N components of the on-line module. The sensing device has a probability 'c' of correctly recovering a fault. Consider the case of a warm standby, i.e., a standby module has a non-zero hazard rate even when unpowered. This is a realistic assumption for most of the present day electronic equipments.

The theory of Markov renewal processes[4,5] has been used to derive expressions for (i) the distribution of the TFSF, (ii) the expected TFSF, and (iii) the reliability of the system.

NOTATIONS

CDF = Cumulative distribution function.

$F_i(t) = 1 - \exp(-\lambda_i t)$ = Failure time CDF of the ith component of the on-line module ($i=1, \ldots, N$).

$H_i(t) = 1 - \exp(-\lambda_i' t)$ = Failure time CDF of the ith component of the standby module ($i=1, \ldots, N$).

$\gamma = \sum_{i=1}^{N} \lambda_i$ = Cumulative hazard rate of the on-line module.

$\gamma' = \sum_{i=1}^{N} \lambda_i^1$ = Cumulative hazard rate of the standby module.

$F(t) = 1 - \exp(-\gamma t)$ = Failure time CDF of the on-line module.

$H(t) = 1 - \exp(-\gamma' t)$ = Failure time CDF of the standby module.

$G_i(t)$ = Repair time CDF for the ith component of a failed module.

$g_i(s) = \int_0^\infty e^{-st} \, d\, G_i(t)$ = Laplace-Stieltjes transform (L-S-T) of $G_i(t)$ ($i=1, \ldots, N$).

S_j = State j of the system; j goes from 0 to $N+1$. $S_0, \ldots, S_N$ are transient states, S_{N+1} is the absorbing state.

$Q_{ij}(t)$ = CDF of the sojourn time in S_i ($i=0, \ldots, N$), given that the system next visits S_j ($j=0, \ldots, N+1$).

$\Phi_i(t)$ = First-passage time CDF from S_i ($i=0, \ldots, N$) at an initial moment to the absorbing state S_{N+1}.

$q_{ij}(s)$ = L-S-T of $Q_{ij}(t)$.

$\phi_i(s)$ = L-S-T of $\Phi_i(t)$.

$*$ = Denotes the convolution operation.

— = Implies the complement, e.g., $\bar{G}_i(.) = 1 - G_i(.)$, $\bar{g}_i(.) = 1 - g_i(.)$.

MATHEMATICAL MODEL AND ANALYSIS

Define the following states of a Markov renewal process characterizing the system.

S_0 : One module is operative, the other is in standby.

S_i : One module is operative, the other enters repair due to the failure of its ith component ($i = 1, \ldots, N$).

S_{N+1} : System failure.

System moves to the absorbing state S_{N+1} in the following two mutually exclusive ways :

(i) The failure of a component of the on-line module is undetected by the sensing device.

(ii) The on-line module fails during the repair duration of a component of the other module.

The first-passage time CDF $\Phi_i(t)$ from S_i to S_{N+1} can be represented as

$$\Phi_i(t) = Q_{i,N+1}(t) + \sum_{j=0}^{N} Q_{ij}(t) * \Phi_j(t) \, ; \; i = 0, 1, \ldots, N. \quad (1)$$

Equation (1) implies that the system moves to S_{N+1} either directly or via S_j.

Taking L-S-T on both sides of (1), we get

$$\phi_i(s) = q_{i,\,N+1}(s) + \sum_{j=0}^{N} q_{ij}(s)\,\phi_j(s)\;;\; i = 0, 1, \ldots\ldots, N. \tag{2}$$

The various $q_{ij}(s)$ appearing in (2) may be obtained by considering transitions from the transient states.

Transitions from S_0

From S_0, $N+1$ mutually exclusive and exhaustive transitions emanate. These can be grouped as under :

(i) *From S_0 to S_i* $(i = 1, \ldots\ldots, N)$: The ith component of either the on-line or the standby module fails and repair on the corresponding module starts.

(ii) *From S_0 to S_{N+1}* : The failure of the on-line module is not detected successfully by the sensing device.

The L-S-T of the sojourn time CDF's corresponding to the above transitions are given by

$$q_{0i}(s) = \int_0^\infty e^{-st}\,\bar{H}_j(t) \prod_{j\neq i} \bar{F}(t)\,.\,c\,dF_i(t) + \int_0^\infty e^{-st}\,\bar{F}(t) \prod_{j\neq i} \bar{H}_j(t)\,.\,dH_i(t)$$

$$= \frac{c\,\lambda_i + \lambda'_i}{s+\gamma+\gamma'}\;,\; i = 1, \ldots\ldots, N; \tag{3}$$

$$= q_{0,\,N+1}(s) = \int_0^\infty e^{-st}\,\bar{H}(t)\,\bar{c}\,dF(t) = \frac{\bar{c}\,\gamma}{s+\gamma+\gamma'}\,. \tag{4}$$

Transitions from S_i $(i = 1, \ldots\ldots, N)$

From S_i two mutually exclusive and exhaustic transitions emanate ;

(i) *From S_i to S_0* : Repair of the ith component of the failed module is completed before the on-line module fails.

(ii) *From S_i to S_{N+1}* : The on-line module fails during the repair duration of the ith component of the failed module.

The L-S-T of the sojourn time CDF's corresponding to these transitions are given by

$$q_{i0}(s) = \int_0^\infty e^{-st}\,\bar{F}(t)\,dG_i(t) = g_i(s+\gamma)\,,\; i = 1, \ldots\ldots, N; \tag{5}$$

$$q_{i,N+1}(s) = \int_0^\infty e^{-st}\,\bar{G}_i(t)\,dF(t) = \left(\frac{\gamma}{s+\gamma}\right)\left[1 - g_i(s+\gamma)\right],\; i = 1, \ldots\ldots, N. \tag{6}$$

It can be easily verified from (3) to (6) that

$$\sum_{i=1}^{N+1} q_{0i}(0) = 1 \; ; \tag{7}$$

$$q_{i0}(0) + q_{i,N+1}(0) = 1, \; i = 1, \ldots\ldots, N. \tag{8}$$

The L-S-T $\phi_0(s)$ of the first-passage time CDF from S_0 at $t=0$ to S_{N+1} can be obtained by solving the system of simultaneous linear equations (2) i.e., by the set of equations.

$$\phi_0(s) = q_{0,N+1}(s) + \sum_{J=1}^{N} q_{0J}(s)\,\phi_j(s) \; ;$$

$$\phi_i(s) = q_{i,N+1}(s) + q_{i0}(s)\,\phi_0(s), \; i = 1, \ldots\ldots, N.$$

Thus, we obtain

$$\phi_0(s) = \frac{q_{0,N+1}(s) + \sum_{i=1}^{N} q_{0i}(s)\,q_{i,N+1}(s)}{1 - \sum_{i=1}^{N} q_{0i}(s)\,q_{i0}(s)} \tag{9}$$

On substituting from (3) to (6) in (9), we obtain

$$\phi_0(s) = \left(\frac{\gamma}{s+\gamma}\right) \cdot \left[\frac{cs+\gamma+\gamma' - \sum_{i=1}^{N} (c\lambda_i + \lambda_i')\, g_i(s+\gamma)}{s+\gamma+\gamma' - \sum_{i=1}^{N} (c\lambda_i + \lambda_i')\, g_i(s+\gamma)} \right] \tag{10}$$

It can be verified from (10) that $\phi_0(0) = 1$, i.e., the distribution $\Phi_0(t)$ of the time to first system failure, is a proper distribution, with $\Phi_0(\infty) = 1$.

The Laplace transform, $\hat{R}(s)$, of the reliability $R(t)$ of the system, given that the system started in S_0, can be obtained from

$$\hat{R}(s) = \bar{\phi}_0(s)/s \tag{11}$$

Substituting from (10) into (11), we obtain

$$\hat{R}(s) = \frac{1}{s+\gamma} + c\left(\frac{\gamma}{s+\gamma}\right) \Big/ \left[s+\gamma+\gamma' - \sum_{i=1}^{N} (c\lambda_i + \lambda_i')\, g_i(s+\gamma) \right] \tag{12}$$

The expected TFSF, T_0, given that the system started in S_0 can be obtained from (12) by taking the limit as $s \to 0$. Thus

$$T_0 = \frac{1}{\gamma} + \frac{c}{(\gamma+\gamma')\left[1 - \sum_{i=1}^{N} \left(\frac{c\lambda_i + \lambda_i'}{\gamma+\gamma'} \right) g_i(\gamma) \right]} \tag{13}$$

An examination of (13) reveals that T_0 depends very strongly on 'c', the probability of successfully recovering a fault in the on-line module. It is therefore, essential that the sensing device should be made extremely reliable.

Let T_0' denote the expected TFSF corresponding to a repair policy in which each failure of a module has the same repair time CDF, $G(t)$, irrespective of the type of component responsible for its failure. Then (13) reduces to

$$T_0' = \frac{1}{\gamma} + \frac{c}{(\gamma+\gamma')\left[1 - \dfrac{c\gamma+\gamma'}{\gamma+\gamma'}\, g(\gamma)\right]} \tag{14}$$

Comparing (13) and (14), we can obtain a condition in which the former repair policy is superior to the latter. Thus, $T_0 > T'_0$ if

$$\left(\frac{1}{\gamma+\gamma'}\right) \sum_{i=1}^{N} (c\lambda_i + \lambda'_i)\, g_i(\gamma) > \left(\frac{c\gamma+\gamma'}{\gamma+\gamma'}\right) g(\gamma) \tag{15}$$

PARTICULAR CASES

Sensing Device Perfect, Warm Standby

Putting $c = 1$ in (10) and (13), we obtain

$$\phi_0(s) = \frac{\left(\dfrac{\gamma}{s+\gamma}\right)\left(\dfrac{\gamma+\gamma'}{s+\gamma+\gamma'}\right)\left[1 - \displaystyle\sum_{i=1}^{N}\left(\dfrac{\lambda_i+\lambda'_i}{\gamma+\gamma'}\right) g_i(s+\gamma)\right]}{1 - \displaystyle\sum_{i=1}^{N}\left(\dfrac{\lambda_i+\lambda_i'}{s+\gamma+\gamma'}\right) g_i(s+\gamma)}; \tag{16}$$

$$T_0 = \frac{1}{\gamma} + \frac{1}{(\gamma+\gamma')\left[1 - \displaystyle\sum_{i=1}^{N}\left(\dfrac{\lambda_i+\lambda_i'}{\gamma+\gamma'}\right) g_i(\gamma)\right]} \tag{17}$$

Results (16) and (17) are the generalizations of the results obtained by Osaki & Nakagawa[6]. Further, condition (15), reduces to

$$\left(\frac{1}{\gamma+\gamma'}\right) \sum_{i=1}^{N} (\lambda_i + \lambda_i')\, g_i(\gamma) > g(\gamma) \tag{18}$$

Sensing Device Perfect, Cold Standby

Putting $\lambda_i' = 0, i = 1, \ldots\ldots, N$; in (16) and (17), we obtain

$$\phi_0(s) = \frac{\left(\dfrac{\gamma}{s+\gamma}\right)^2\left[1 - \displaystyle\sum_{i=1}^{N}\dfrac{\lambda_i}{\gamma}\, g_i(s+\gamma)\right]}{1 - \displaystyle\sum_{i=1}^{N}\left(\dfrac{\lambda_i}{s+\gamma}\right) g_i(s+\gamma)}; \tag{19}$$

$$T_0 = \frac{1}{\gamma} + \frac{1}{\gamma\left[1 - \displaystyle\sum_{i=1}^{N}\dfrac{\lambda_i}{\gamma}\, g_i(\gamma)\right]}. \tag{20}$$

(19) and (20) are generalizations of the results obtained by Gaver[7]. Further condition (18) reduces to

$$\frac{1}{\bar{\gamma}} \sum_{i=1}^{N} \lambda_i \, g_i(\gamma) > g(\gamma) \tag{21}$$

Equation (21) may be interpreted as follows: If the weighted average of the functions $g_i(\gamma)$ with weights λ_i, $i = 1, \ldots\ldots\ldots, N$ is greater than $g(\gamma)$, then $T_0 > T_0'$. In other words, we can improve the expected TFSF if the components that fail more frequently are repaired in shorter intervals of time than those components which fail less frequently.

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