A MATHEMATICAL GAME OF REACHING AN ISLAND IN A DEEP LAKE WITH THE AID OF A ROPE

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The problem of reaching an island in a deep lake by a man who cannot swim, with the help of just a rope, has been generalised. The rope is such that it can be used to tie the two trees only, one on the edge of the lake and other in the island. The least length of the rope that the man would require has been obtained. The particular case of an elliptical lake with one tree at an end of a latus rectum and the other at the corresponding focus in an island there has been discussed.

The following problem appeared as a Mathematical Game[1]. 'A deep circular lake, 300 yards in diameter has a small island at the centre. There are only two trees, one just on the edge and the other at the centre. A man who cannot swim has a rope, a few yards longer than 300 yards. How does he use it as a means of getting to the island ?'

In this paper, the above problem has been modified and generalised to one of finding the least length of a rope the man would require in order to reach the island in a deep lake having trees in the manner described in the Mathematical game[2]. In solving a trignometric equation that arises, a new graphical method of circular nomogram[3] has been used. A modified form of this example is recommended as a 'Shrewdness Test' in the interview for selection at graduate level in the Defence Services.

Present Problem

A deep lake has a continuous edge and has an island wherein there is a tree; the only other tree in the neighbourhood is situated somewhere on the edge. What is the least length of the rope (say, $L+$a small length 'h') for the purpose of tying a man would require to help him reach the island in the manner already described[2].

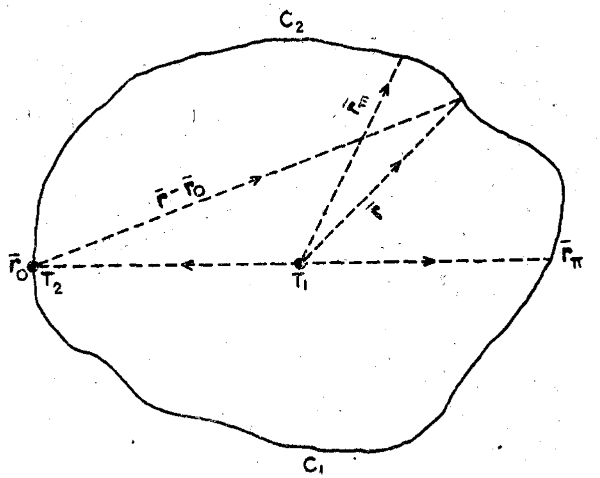

FIG. 1—A deep lake with an island.

Solution

Let the edge C of the lake be the set of points given by $\bar{r}=\bar{f}(\theta)$, $0\leqslant\theta\leqslant 2\pi$ referred to the point on the island where a tree (T_1) is situated as pole and the line joining the two trees as the initial line, so that the other tree (T_2) is located at $\bar{r}_0=\bar{f}(0)$. Let $\bar{r}_1\in C_1$ and $\bar{r}_2\in C_2$ where $C_1 \cup C_2 = C$ and $C_1 \cap C_2 = \{\bar{r}_0, \bar{r}_\pi\}$. We assume that $\bar{r}_m = \max. |\bar{r}|$ and also that $\bar{r}_m \in C_2$. The least length of the rope the man would require to make a double rope stretched firmly between the two trees in the manner described[2] is the greater of max. $\rho(\bar{r}_2, \bar{r}_0)$ (*ie.* max. $|\bar{r}_2-\bar{r}_0|$ and max. $|\bar{r}_1|+|\bar{r}_0|$ with a little surplus length h for the purpose of) tying (Fig. 1.)

Proof

We first consider the route in the increasing order of 'θ' (to C_2 through C_1), along the edge in which the man moves after tying one end of the rope to the tree at $\bar{r}_0$, holding the other end in hand. He can reach the point $\bar{r}_\pi$ opposite to $\bar{r}_0$ only if the rope length is not less than $h +$ max. $\rho\,(\bar{r}_1, \bar{r}_0)$. As and when he crosses $\bar{r}_\pi$, the tree in the island will be caught in the rope and he can return to $\bar{r}_0$, after walking around the lake completely once, only if the rope is of length at least $h + |\bar{r}_m| + |\bar{r}_0|$. As $\rho\,(\bar{r}_1, \bar{r}_0) \leqslant |\bar{r}_m| + |\bar{r}_0|$ the least length required in this case is $|\bar{r}_m| + |\bar{r}_0| + h$.

If the man goes around the edge in the opposite direction (to C_1 through C_2) he will reach r_π only when the length is $h +$ max. $\rho\,(\bar{r}_2, \bar{r}_0)$ and then go further and return to $\bar{r}_0$ only when the length is $h +$ the greater of max. $|\bar{r}_1| + |\bar{r}_0|$ and max. $\rho\,(\bar{r}_2, \bar{r}_0)$. But $\rho\,(\bar{r}_2, \bar{r}_0)$ and max. $|\bar{r}_1| + |\bar{r}_0|$ are each, $\leqslant |\bar{r}_m| + |\bar{r}_0|$. Therefore, this route is preferable in any case and the least length required is thus $h +$ the greater of max. $\rho\,(\bar{r}_2, \bar{r}_0)$ and max. $|\bar{r}_1| + |\bar{r}_0|$.

ILLUSTRATIVE EXAMPLE

If the edge of the lake is elliptical with eccentricity $\frac{1}{2}$ and latus rectum 200 metres with one tree at a focus in an island and the other tree at an end of the latus rectum through the focus. Referring to the focus as pole and the line joining the trees as the initial line, the edge of the lake is given by $\bar{r} = 200\, e^{i\theta} / (2 - \sin\theta)$. It is easily seen that max. $|\bar{r}| = |\bar{r}_{\pi/2}| = 200$ metres. It follows that the man has to walk along the edge in the increasing order of 'θ' and the least length required for making a double rope stretched between the two trees is the greater of $\max_{\theta \in (0,\pi)} \rho\,(\bar{r}, \bar{r}_0)$ and max $|\bar{r}| + |\bar{r}_0| + h$.

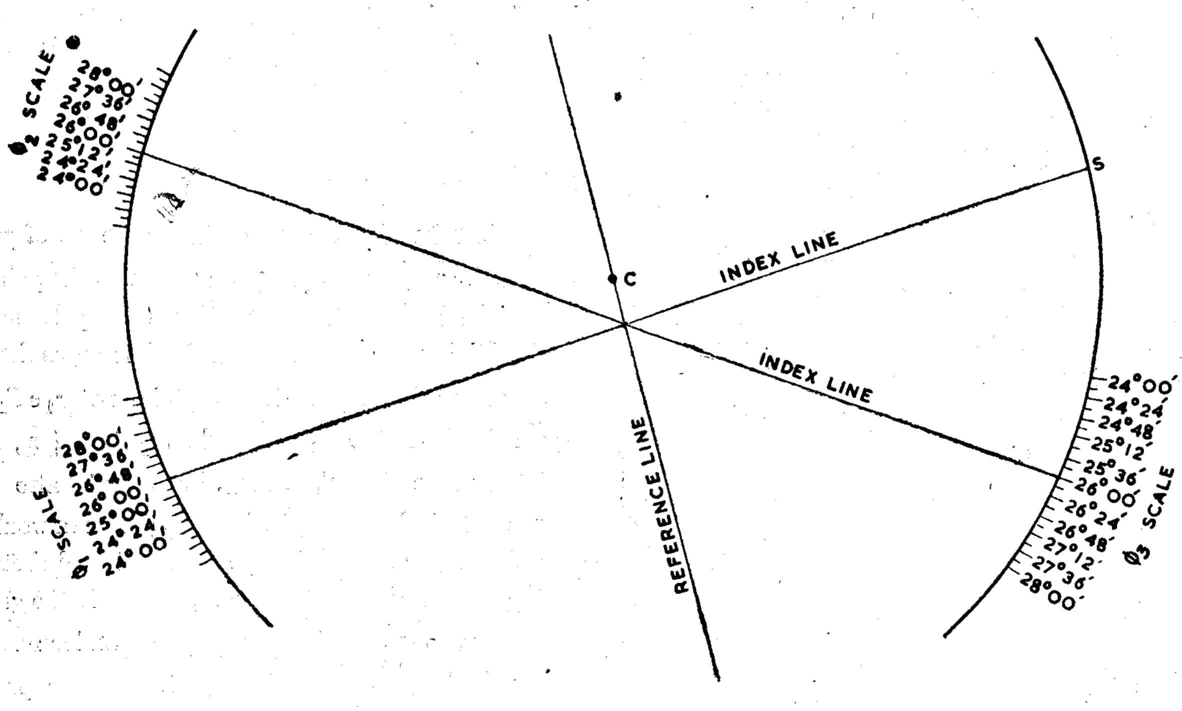

FIG. 2—Circular Nomogram.

Now, $\rho(\bar{r}, \bar{r}_0)$ is maximum when the chord $(\bar{r} - \bar{r}_0)$ is normal to the ellipse at $\bar{r}$. When the chord $\bar{r} - \bar{r}_0$ is normal at $\bar{r}$, $2\cos\theta\,(1 + \cos\theta) = 4 - 5\sin\theta$. We introduce a novel method of solving such equations as the one above using Adikesavan's Circular Nomogram[3]. Changing 'θ' to $\pi/2 + \phi$, the equation becomes :

$$2 \sin\phi\,(1 - \sin\phi) = 5\cos\phi - 4 \tag{1}$$

We construct three scales using the scale equations $2\sin\phi_1 = \tan(\alpha/2)$, $1 - \sin\phi_2 = \tan(\beta/2)$ and $5\,\text{Cos}\,\phi_3 - 4 = \tan(\gamma/2)$ with the fourth scale as a degenerate point S where $\delta = \pi/2$ (fig. 2). Small range of ϕ's are plotted on the respective scales around a guessed solution ϕ of the equation (1). By gently moving the index lines we arrive at an approximate solution $\phi_1 = \phi_2 = \phi_3 = \phi = 26°\,2'$. Thus $\theta = \pi/2 + \phi = 116°\,2'$, approximately. It follows that $\max\limits_{\theta \in (0,\pi)} \rho(\bar{r}, \bar{r}_0) = 242{\cdot}7$ metres, approximately.

$$\text{Also, } \max_{\theta \in (\pi, 2\pi)} |\bar{r}| + |\bar{r}_0| = 2\bar{r}_0 = 200 \text{ metres.}$$

Thus, the length of the rope should be at least $242{\cdot}7 + h$ metres. It should be noted that had he moved in the opposite direction ($\theta\downarrow$) he would have required $(300 + h)$ metres of the rope.

Remarks

1. If the max. $|\bar{r}|$ occurs at two points, one in each of $\theta \in (0, \pi)$ and $\theta \in (\pi, 2\pi)$, the least length is max. $|\bar{r}| + |\bar{r}_0| + h$ no matter in what direction the man goes around the lake.

A SHREWDNESS TEST

The above example suitably modified as follows is suggested as a test at graduate level to find suitability of a candidate in the interview, for selection to Defence Services.

An ellipse is marked around a tall tree (cocoanut tree is suitable) such that the tree is at a focus S of the ellipse as shown in Fig. 3. At the end L of a semi-latus rectum through that focus, a pole is firmly planted vertically with some provision for knotting a rope at a height of about 1.5 metres from the ground level. Adjoining the semi-latus rectum a trench LS of small width is dug and the depth is varied in such a manner that a man can pull himself along a rope firmly stretched between the tree and the pole at a considerable height without touching the ground.

Treating the area of the ellipse except a small portion surrounding the tree as 'out of bounds', the candidate shall be asked to reach the base of the tree and return with the aid of a rope of just sufficient length and nothing more. The dimensions of the ellipse and the type of the tree should be so chosen as to make it impossible for him either to make a long jump and reach the focus or catch the tree by throwing the rope from the nearest point viz. the nearer end of the major axis or make a loop on the tree on reaching the opposite end of the latus rectum and return in the opposite sense. For this purpose a tree like cocoanut slightly leaning towards the pole with the choice of semi-axes, 20 m and 17·32 m respectively for the surrounding ellipse may be quite suitable (Fig. 3). A length little more than 36·4 metres of the rope will be just sufficient for the above choice of axes.

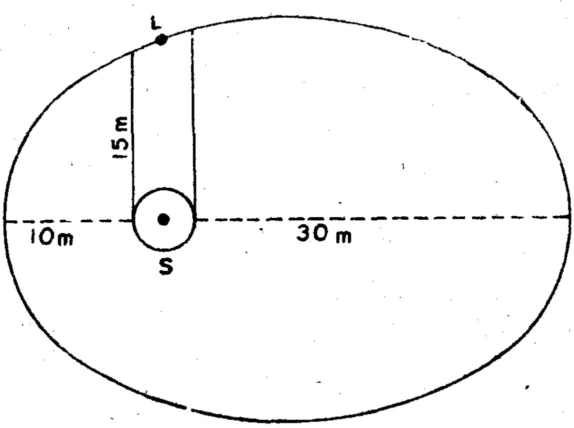

FIG. 3—Design for a "Shrewdness Test".

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1. Gardener Martin, *Scientific American*, 225 (1971), 108-109.

2. Gardener Martin, *Scientific American*, 225, (1971), 105.

3. Adikesavan, A.S., *Def. Sci. J.*, 21 (1971), 149-154.