WAVES GENERATED AT THE INTERFACE BETWEEN AN ELASTIC AND A THERMO-ELASTIC SOLID

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In the present note, we study the conditions for propagation of interface waves when one of the solids is a thermo-elastic halfspace and the other an elastic halfspace. The propagation of plane waves along the interface is studied and the frequency equation is derived. In view of the complicated form of this equation, two limiting cases corresponding to very high or very low frequencies are studied. It is also shown that when the coupling constant in the thermo-elastic solid is put equal to zero, the frequency equation reduces to the equation derived by Stoneley. As another particular case, the frequency equations corresponding to very high or very low wave lengths are obtained.

The propagation of generalised Rayliegh or Stoneley waves along the surface of separation of two elastic solids, was thoroughly investigated by Stoneley[1] and others. Stoneley derived the corresponding frequency equation, whose solution has been further studied by Koppe, Sholte and others, and conditions under which Stoneley waves can exist have been derived. In the present note, it is proposed to study the conditions for propagation of interface waves when one of the solids is a thermo-elastic halfspace, and the other and elastic halfspace. The basic equations for wave propagation in a thermo-elastic medium adopted here are those given by Chadwick[2]. In a subsequent paper Chadwick[3] has discussed the Rayleigh waves in a thermo-elastic halfspace. As usual, we assume *welded contact* at the interface between the two solids. The elastic halfspace is assumed to be at constant temperature T_o throughout, with no coupling between the two solids *i.e.*, between the elastic and thermal fields. Under these assumptions, the propagation of plane waves along the interface is studied and the frequency equation is derived. In view of the complicated form of this equation, two limiting cases of this equation corresponding to very high or very low frequencies are studied. It is also shown that when the coupling constant in the thermo-elastic solid is put equal to zero, the frequency equation reduces to the equation derived by Stoneley. As another particular case, the frequency equations corresponding to very high or very low wave lengths are obtained.

BASIC EQUATIONS

We set up the coordinate system (x, y, z) with the x-axis along the interface and the z-axis normal to the interface with $z > 0$ in the thermo-elastic medium. The components of the displacement are then given by

$$u = \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial z} \; ; \; v = 0 \; ; \; w = \frac{\partial \phi}{\partial z} + \frac{\partial \psi}{\partial x} \tag{1}$$

Denoting the variation of temperature by θ, the equation satisfied by ϕ, ψ, θ for the thermo-elastic medium[3] are :

$$\left.\begin{aligned} \frac{\partial^2 \phi}{\partial t^2} &= V^2{}_T \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} \right) - \frac{\alpha}{\rho x_T}\, \theta \\ \frac{\partial^2 \psi}{\partial t^2} &= V^2{}_S \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} \right) \\ \rho\, c_\epsilon \frac{\partial \theta}{\partial t} + \frac{\alpha T_o}{x_T} \frac{\partial}{\partial t} \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} \right) &= k \left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial z^2} \right) \end{aligned}\right\} \tag{2}$$

For the purely elastic medium the equations (distinguishing the corresponding quantities by a superscribed prime) are :

$$\left.\begin{aligned} \frac{\partial^2 \phi'}{\partial t^2} &= V'^2{}_T \left(\frac{\partial^2 \phi'}{\partial x^2} + \frac{\partial^2 \phi'}{\partial z^2} \right) \\ \frac{\partial^2 \psi'}{\partial t^2} &= V'^2{}_S \left(\frac{\partial^2 \psi'}{\partial x^2} + \frac{\partial^2 \psi'}{\partial z^2} \right) \end{aligned}\right\} \tag{3}$$

Here (V_T, V_S), (V'_T, V'_S) denote the compressional and shear wave velocities in the two media and ρ, ρ^1 the densities; k, c and T_o are respectively coefficients of the thermal conductivity, the specific heat at constant strain and the initial temperature of the thermo-elastic solid; α is its coefficient of volume expansion and x_T is its isothermal compressibility. The elastic medium is supposed to be of infinite conductivity and at temperature T_o.

For vertically polarised plane waves propagated in the x-direction, we shall seek solutions in the form:

$$(\theta, \phi, \psi) = (\overline{\theta}, \overline{\phi}, \overline{\psi}) \exp\ [i\,(\eta x - \omega t)\,]$$

and in order that their amplitudes become vanishingly small, as $|z| \to \infty$ these solutions must be of the form[2]:

For $z > 0$

$$\theta = \frac{\rho x_T\, V^2_T}{\alpha} \left[A \left(\frac{\omega^2}{V^2_T} - \zeta_1^2 \right) . \exp \left\{ -z\sqrt{\eta^2 - \zeta_1^2} + i\eta x \right\} + B \left(\frac{\omega^2}{V^2_T} - \zeta_2^2 \right) . \exp \left\{ -z\sqrt{\eta^2 - \zeta_2^2} + i\,\eta x \right\} \right]$$

$$\phi = A \,.\, \exp \{ -z\sqrt{\eta^2 - \zeta_1^2} + i\,\eta x \} + B \,.\, \exp \{ -z\sqrt{\eta^2 - \zeta_2^2} + i\eta x \}$$

$$\psi = C.\ \exp \{ -z\sqrt{\eta^2 - \zeta^2_3} + i\eta x \}$$

where ζ_1^2, ζ_2^2 are the roots of the equation:

$$\zeta^4 - \zeta^2 \left\{ \frac{\omega^2}{V^2_T} + \frac{i\,\omega\rho\, C_\epsilon}{k} (1 + \epsilon) \right\} \frac{i\,\omega^3\, \rho C_\epsilon}{k\, V^2_T} = 0$$

and $\zeta_3 = \omega / V_S$, $\epsilon = \alpha^2\ T_o / \rho^2 .\ c x^2_T\ V^2_T$.

For $z < 0$

$$\phi' = A' \,.\, \exp \{ z\sqrt{\eta^2 - \zeta_1'^2} + i\eta x \} \qquad \psi' = C'.\ \exp \{ z\sqrt{\eta^2 - \zeta_3'^2} + i\eta x \}$$

where $\zeta'_1 = \omega / V'_T$ and $\zeta'_3 = \omega / V'_S$.

BOUNDARY CONDITIONS

(*a*) As $|z| \to \infty$, the amplitudes tend to zero. This condition is satisfied by the forms assumed for th above solution.

(*b*) The stresses σ_{zz} and σ_{zx} and the displacements (u, w) are continuous across $z = 0$.

(*c*) $\theta = 0$ for $z = 0$.

Using the expressions:

$$\sigma_{zz} = \rho V^2_T \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{{}^2\phi}{\partial z^2} \right) + 2\,\rho\, V^2_S \left(\frac{\partial^2 \psi}{\partial x\, \partial z} - \frac{\partial^2 \phi}{\partial x^2} \right) - \frac{\alpha\theta}{x_T}$$

$$\sigma_{zx} = \rho\ V^2_S \left(2\ \frac{\partial^2 \phi}{\partial x\, \partial z} + \frac{{}^2\psi}{\partial x^2} - \frac{\partial^2 \phi}{\partial z^2} \right)$$

the boundary conditions yield the following five equations for the constants A, B, C, A', & C':

$$\left. \begin{array}{l} 2\rho V^2_S\, i\eta\sqrt{\eta^2 - \zeta_1^2} \,.\, A - 2\rho\, V^2_S\, i\eta\sqrt{\eta^2 - \zeta_2^2} \,.\, B + \rho\, V^2_S\, (2\eta^2 - \zeta_3^2) \,.\, C \\ + 2\rho'\, V'_S{}^2\, i\eta\sqrt{\eta^2 - \zeta'^2_1} \,.\, A' - \rho'\, V'_S{}^2\, (2\eta^2 - \zeta'^2_3)\ C' \,. \end{array} \right\} = 0$$

$$\left. \begin{array}{l} \rho(2\, V^2_S \eta^2 - \omega^2) \,.\, A + \rho\, (2\, V^2_S\, \eta^2 - \omega^2) \,.\, B - 2\,\rho\, V^2_S\, i\eta\, \sqrt{\eta^2 - \zeta^2_3} \,.\, C \\ + \rho'\, (V'^2_T\ \zeta'^2_1 - 2\ V'^2_S\ \eta^2) \,.\, A' - 2\,\rho'\, V'_S\ i\eta\, \sqrt{\eta^2 - \zeta'^2_3} \,.\, C' \end{array} \right\} = 0$$

$$i\eta \,.\, A + i\eta \,.\, B + \sqrt{\eta^2 - \zeta_3^2} \,.\, C - i\eta \,.\, A' + \sqrt{\eta^2 - \zeta'^2_3} \,.\, C' = 0$$

$$\sqrt{\eta^2 - \zeta_1^2} \,.\, A + \sqrt{\eta^2 - \zeta_2^2} \,.\, B - \eta \,.\, C + \sqrt{\eta^2 - \zeta'^2_1} \,.\, A' - i\eta \,.\, C' = 0$$

$$\left(\frac{\omega^2}{V^2_T} - \zeta_1^2\right) \cdot A + \left(\frac{\omega^2}{V^2_T} - \zeta^2_2\right) \cdot B = 0$$

If now, $\delta_1 = \sqrt{\eta^2 - \omega^2/V^2_T}$; $\delta_2 = \sqrt{\eta^2 - \omega^2/V'_S{}^2}$; $\delta'_1 = \sqrt{\eta^2 - \omega^2/V'_T{}^2}$;

$$\delta'_2 = \sqrt{\eta^2 - \omega^2/V'_S{}^2} \text{ and } \triangle = \sqrt{\eta^2 - \zeta_2^2} \cdot \left(\frac{\omega^2}{V^2_T} - \zeta_1^2\right) - \sqrt{\eta^2 - \zeta_1^2} \cdot \left(\frac{\omega^2}{V^2_T} - \zeta^2_2\right),$$

then the above five equations yield the determinantal equation:

$$\begin{vmatrix} 2\,\delta'_1\,(\rho\,V^2_S - \rho'\,V'^2_S) & -\rho\,\omega^2 & 2\,\eta^2\,(\rho V^2_S - \rho'\,V'^2_S) + \rho'\omega^2 \\ (\rho - \rho')\,\omega^2 - 2\,\eta^2\,(\rho\,V^2_S - \rho'\,V'_S{}^2) & \rho\,\omega^2\,\delta_2\,\delta'_2 & \{\rho\,\omega^2 - 2\,\eta^2\,(\rho V^2_S - \rho'\,V'_S{}^2)\} \\ \triangle + \delta'_1\,(\zeta^2_2 - \zeta^2_1) & \triangle\,\delta_2 - (\zeta^2_2 - \zeta^2_1)\,\eta^2 & \triangle\,\delta'_2 + \eta^2\,(\zeta^2_2 - \zeta^2_1) \end{vmatrix} = 0$$

If $D = \rho\,V'^2_S - \rho'\,V'^2_S$ = constant and $E = \zeta^2_2 - \zeta^2_1$, then the above determinantal equation can be written as:

$$4\,D^2\,\eta^2\,(\eta^2 - \delta'_1\,\delta'_2)\,(\triangle\,\delta_2 - \eta^2\,E) + 4\,D\,\omega^2\,\eta^2\,[\,\rho\,E\,(\eta^2 - \delta'_1\,\delta'_2) + \rho'\,(\triangle\,\delta_2 - \eta^2\,E)\,]$$
$$+\,\omega^4\,[\,(\rho'\,\triangle + E\,\rho\,\delta'_1)\,(\rho\delta'_2 + \rho'\delta_2) - E\eta^2\,(\rho - \rho')^2\,] = 0 \qquad (4)$$

This is the desired frequency equation. Since this is of a complicated form, we discuss some limiting cases.

PARTICULAR CASES

Waves of given period (ω fixed).

We now introduce the dimensionless quantity:

$$\chi = \omega/\omega^* \;;\; \omega^* = \rho\,c_\epsilon\,V^2_T/k$$

and consider the cases when: $\chi << 1$ & $\chi >> 1$

(a) Suppose χ is small. ($\chi << 1$). The frequency equation (4) when expanded in powers of χ upto the power χ^7 takes the form:

$$\chi^5\,(\,iP + \chi Q + \chi^2 R + \ldots\,) = 0,$$

where P, Q, R are real valued expressions in ϵ and η. If χ^3 and higher powers are neglected, then the frequency equation takes the form: $iP + \chi\,Q + \chi^2\,R = 0$. Therefore, the wave-velocity depends on the frequency.

(b) Suppose χ is large. ($\chi >> 1$). The frequency equation (4) in this case takes the form:

$$\left.\begin{array}{l} \omega^4\,\{\,\rho' V'_T + \rho\,V_T)\,(\,\rho V_S + \rho' V'_S) - 4\,\rho D\eta^2\,\} \\ +\,\omega^2\eta^2\,\{\,(\rho - \rho')^2\,V_T\,V_S\,V'_T\,V'_S + 4\,D\,\rho'\,V'_T\,V'_S - 4\,D^2\,\} \\ -\,4\,D\,\eta^4\,\{\,D\,V_T\,V_S + (\rho - \rho')\,V_T\,V_S\,V'_T\,V'_S\,\} \end{array}\right\} = 0$$

If we divide throughout by η^4, then we get the following equation for $U = \omega/\eta$, the phase velocity of the interface wave:

$$\left.\begin{array}{l} U^4\,\{\,(\rho\,V_T + \rho'\,V'_T)\,(\rho\,V_S + \rho'\,V'_S) - 4\,\rho\,D\,\eta^2\,\} \\ +\,U^2\,\{\,(\rho - \rho')^2\,V_T\,V_S\,V'_T\,V'_S + 4\,D\,\rho'\,V'_T\,V'_S - 4\,D^2\,\} \\ -\,4\,D\,\{\,D\,V_T\,V_S + (\rho - \rho')\,V_T\,V_S\,V'_T\,V'_S\,\} \end{array}\right\} = 0$$

Therefore the velocity of the wave depends on the wave length. Hence there is dispersion in this case also.

(c) Suppose ϵ is small ($\epsilon << 1$), then ϵ^2, ϵ^3 and higher powers of ϵ may be neglected. Let

$$K^* = \sqrt{\eta^2 - i\,\omega^2/V^2_T\,\chi} \quad ; \quad V^2_Q = \omega^2/V^2_T\,(\chi - i) \quad ;$$
$$M = 4\,D^2\,\eta^2\,\delta_2\,(\eta^2 - \delta'_1\,\delta'_2) + 4\,D\,\omega^2\,\eta^2\,\rho'\,\delta_2 + \omega^4\,\rho'\,(\rho\,\delta'_2 + \rho'\,\delta_2) \;\&$$
$$N = -\,4\,D^2\,\eta^4\,(\eta^2 - \delta'_1\,\delta'_2) + 4\,D\,\omega^2\,\eta^4\,(\rho - \rho')$$
$$-\,\omega^4\,\{\,\rho\,\delta'_1\,(\rho\,\delta'_2 + \rho'\,\delta_2) - \eta^2\,(\rho - \rho')^2\,\}$$

Then the frequency equation (4) takes the form:

$$\left.\begin{array}{l} 2\,(M\,\delta_1 + N)\,\delta_1\,(\chi - i)^2 \\ -\left\{\begin{array}{l} M\,i\,V^2_Q\,(\chi - i)^2 - 2\,M\,i\,\delta_1\,K^*\chi \\ +\,2\,M\,\delta^2_1 - 2\,\delta_1\,N\,i\,(\chi + i) \end{array}\right\} \end{array}\right\} = 0$$

In particular, if $\epsilon = 0$, the frequency equation reduces to

$$M\delta_1 + N = 0$$

Or

$$\left.\begin{array}{l} \eta^2\left[\eta^2 - \dfrac{\rho'\omega^2}{2D}\right]^2 - \delta_1\,\delta'_1\left(\eta^2 - \dfrac{\rho'\,\omega^2}{2D}\right)^2 \\ -\,\delta_2\,\delta'_2\left(\eta^2 + \dfrac{\rho\omega^2}{2D}\right)^2 - (\delta_1\,\delta'_2 + \delta_2\,\delta'_1)\dfrac{\omega^4\,\rho\rho'}{4D^2} + \delta_1\,\delta_2\,\delta'_1\,\delta'_2\,\eta^2 \end{array}\right\} = 0.$$

This equation is nothing but Stoneley's frequency equation. This is as it should be, since for $\epsilon = O$, there is no coupling between the elastic and the thermal fields, and the case reduces to that of two elastic solids. In view of this, it may be concluded that for small values of ϵ, interface waves exist under the same conditions as in the case of pure Stoneley waves.

Waves of given length/(η fixed).

We now introduce the dimensionless quantity:

$$\xi = \eta/\eta^* \quad ; \quad \eta^* = \omega^*/V_T$$

and consider the cases when $\xi << 1$ and $\xi >> 1$.

(*a*) Suppose ξ is small. ($\xi << 1$). Then expanding in powers of ξ, the frequency equation takes the form

$$\left.\begin{array}{l} 2\,V^2_T\,\chi^2\,\zeta_1\,\zeta_2(\rho V_S + \rho'\,V'_S)\,[\,V'_T\,\rho'\,(\zeta_2 - \zeta_1)\,(\chi^2\,\eta^{*2} + \zeta_1\zeta_2) + E\,\rho\eta^*\,V_T\,\chi\,] \\ +\,\xi^2\left[\begin{array}{l} (\zeta_2 - \zeta_1).\,(\chi^2\eta^{*2} + \zeta_1\,\zeta_2).\left\{\begin{array}{l} 8D\zeta_1\,\zeta_2\,(D + \rho'\,V'_T\,V'_S) - \rho'\,V'_S\,V'_T\,V'_S\,\zeta_1\,\zeta_2\,(\rho\,V'_S + \rho' V_S) \\ +\,\eta^{*2}\,\rho'\,\chi^2\,V^2_T\,V'^2_T\,(\rho\,V_S + \rho'\,V'_S) \end{array}\right\} \\ +\,2\,\eta^*\,\chi\,V_T\,V_S\,E\,\zeta_1\,\zeta_2\,\{\,V'_T\,V'_S\,(\rho - \rho')^2 - 4\,D\rho\,\} \\ +\,\eta^*\,\chi\,V_T\,V'_T\,(\rho\,V_S + \rho'\,V'_S)\,[\,\eta^*\,\chi\,V_T\,\rho'\,(\zeta_2 - \zeta_1)^3 + \rho\,V'_T\,E\,\zeta_1\,\zeta_2\,] \\ -\,E\,\rho\,\eta^*\,\chi\,V_T\,V_S\,V'_S\,\zeta_1\,\zeta_2\,(\rho\,V'_S + \rho'\,V_S) \end{array}\right] \end{array}\right\} = 0.$$

(*b*) Suppose ξ is large. ($\xi >> 1$). Then the frequency equation (4) takes the form:

$$\left.\begin{array}{l} V^2_T\,\chi^2\,E\,(\rho + \rho')\,[\,\rho\,V'^2_T\,V^2_S + \rho' V'^2_T\,V'^2_S\,] \\ -\,2\,\xi^2\left\{\begin{array}{l} 4\,\rho\,\rho'\,E\,V'^2_T\,V'^2_S\,V'^2_S + 2\,D\rho\,E\,V^2_S\,V'^2_S \\ +\,2\,D\,\rho\,E\,V^2_S\,V'^2_T - 2\,D\rho'\,V'^2_T\,V'^2_S \\ -\,D^2\,V'^2_S - D^2\,V'^2_T \end{array}\right\} \end{array}\right\} = 0.$$

It is observed that in case (*a*) the wave-velocity depends on the frequency, but is a constant in case (*b*)

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