ON CERTAIN DUAL INTEGRAL EQUATIONS

R. S. PATHAK

Banaras Hindu University, Varanasi

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Dual integral equations involving H-Functions have been solved by using the theory of Mellin transforms. The proof is analogous to that of Busbridge on solutions of dual integral equations involving Bessel functions.

Dual integral equations involving H-functions have been solved by Fox[1] and Saxena[2-3] by using the theory of fractional integrals and solutions have been expressed in terms of certain integral operators. In the present paper, we have shown that Titchmarsh's method[4] will also work for the present problem and solutions are expressible in terms of definite integrals.

Let us consider the pair of equations

$$\int_0^\infty H^{(1)}(xy)\, f(y)\, dy = u(x) \qquad 0 < x < 1 \tag{1}$$

$$\int_0^\infty H^{(2)}(xy)\, f(y)\, dy = v(x) \qquad x > 1 \tag{2}$$

where $u(x)$, $v(x)$ are two given functions and $f(x)$ is to be determined and $H^{(1)}(x)$, and $H^{(2)}(x)$ are Fox H-functions defined as

$$H^{(1)}(x) = H^{m\,n}_{p\,q}\left(x \,\middle|\, \begin{matrix} \{(a_p, \alpha_p)\} \\ \{(b_q, \beta_q)\} \end{matrix} \right) = \frac{1}{2\pi i} \int_L \frac{\prod_1^m \Gamma(b_j - \beta_j s) \prod_1^n \Gamma(1 - a_j + \alpha_j s)}{\prod_{m+1}^q \Gamma(1 - b_j + \beta_j s) \prod_{n+1}^p \Gamma(a_j - \alpha_j s)}\, x^s\, ds \tag{3}$$

where $\{(a_p, \alpha_p)\}$ stands for the set of parameters $(a_1, \alpha_1)\,(a_2, \alpha_2)\ldots,(a_p, \alpha_p)$, empty product is always interpreted as unity, $0 \leqslant m \leqslant q$, $0 \leqslant n \leqslant p$, $\alpha_i > 0$ $i = 1, 2, \ldots,$ $\beta_j > 0$ $(j = 1, 2, \ldots, q)$. The contour L is a straight line parallel to the imaginary axis in the $s = \sigma + it$ plane such that all the poles of $\Gamma(b_j - \beta_j s)$, $(j = 1, 2, \ldots, m)$ lie to the right and those of $\Gamma(1 - a_j + \alpha_j s)$, $(j = 1, 2, \ldots, n$ to the left of it. The integral converges absolutely under certain conditions[5].

The asymptotic behaviours of $H^{(1)}(x)$, given by Braaksma[6], are

$$H^{(1)}(x) = O(|x|^{\rho_1}) \qquad \text{for small } x,$$

where

$$\sum_1^q \beta_j - \sum_1^p \alpha_j = \tau > 0 \quad \text{and} \quad \rho_1 = \min R(b_j/\beta_j),\ (j = 1, 2, \ldots, m)$$

$$H^{(1)}(x) = O(|x|^{\rho_2}) \qquad \text{for large } x,$$

where

$$\tau > 0,\ \alpha = \sum_{1}^{n} \alpha_j + \sum_{1}^{m} \beta_j - \sum_{m+1}^{q} \beta_j - \sum_{n+1}^{p} \alpha_j > 0,\ |\arg x| < \tfrac{1}{2}\alpha\pi$$

and

$$\rho_2 = \max R\,[(a_j - 1)/\alpha_j],\ (j = 1, 2, \ldots, n).$$

A similar definition is assumed, for $H^{(2)}(x)$ in which the parameters with dashes *i.e.*, m', n', p', q', a'_i, b'_i, α_i', β'_i, τ', α', ρ'_1, ρ'_2 are taken corresponding to the parameters without dashes involved in $H^{(1)}(x)$.

We assume that equations (1) and (2) have a solution $f(x)$, whose Mellin transform $F(s)$ is regular for $\sigma < Q$ and uniform in any interior strip so that

$$F(s) = O(|t|^{\tau(\sigma - Q) + \epsilon}). \tag{4}$$

It follows from this that by a suitable choice, of ϵ that if $\sigma < R(Q - \frac{1}{2})$, then $F(k + it) \epsilon L^2(-\infty, \infty)$,

and therefore

$$x^{k - \frac{1}{2}} f(x) \epsilon L^2(0, \infty).$$

Now, we see that if

$$\max_{1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant n^1} R\,[(a_i - 1)/\alpha_i + \tfrac{1}{2}, (\alpha'_j - 1)/\alpha'_j + \tfrac{1}{2}] < k < \min_{1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant m'} R\,[Q - \tfrac{1}{2}, b_i/\beta_i + \tfrac{1}{2}, b'_j/\beta'_j + \tfrac{1}{2}] = N - \tfrac{1}{2}$$

(say)

$$\tau > 0,\ \tau' > 0,\ \alpha > 0,\ \alpha'\ 0,$$

then $x^{k - \frac{1}{2}} f(x)$, $x^{-k + \frac{1}{2}} H^{(1)}(xy)$ and $x^{-k + \frac{1}{2}} H^{(2)}(xy)$ belong to $L^2(0, \infty)$

We therefore apply Parseval's theorem[4] for functions of L^2 to the left-hand side of (1), and get

$$\frac{1}{2\pi i} \int_{k - i\infty}^{k + i\infty} \frac{F(s)}{x^{1-s}} \frac{\prod_{1}^{m} \Gamma(b_j + \beta_j s) \prod_{1}^{n} \Gamma(1 - a_j - \alpha_j + \alpha_j s)}{\prod_{m+1}^{q} \Gamma(1 - b_j - \beta_j + \beta_j s) \prod_{n+1}^{p} \Gamma\, a_j + \alpha_j - \alpha_j s)}\, ds = u(x) \qquad (0 < x < 1). \tag{5}$$

As $\alpha > 0$, the integral is absolutely convergent and defines an analytic function[5] of x for all values of k. Let us take $k = N - \frac{1}{2}$. Multiplying (5) by x^{-w}. where $w = u + iv$ and $\sigma > u$, and integrating over (0, 1) and then inverting the order of integration, which is justified by absolute convergence, we have

$$\frac{1}{2\pi i} \int_{N - \frac{1}{2} - i\infty}^{N - \frac{1}{2} + i\infty} \frac{\prod_{1}^{m} \Gamma(b_j + \beta_j - \beta_j s) \prod_{1}^{n} \Gamma(1 - a_j - \alpha_j s)}{\prod_{m+1}^{q} \Gamma(1 - b_j - \beta_j + \beta_j s) \prod_{n+1}^{p} \Gamma(a_j + \alpha_j - \alpha_j s)} F(s) \frac{ds}{s - w} = U(1 - w) \qquad u < N - \tfrac{1}{2} \tag{6}$$

where $U(s)$ is the Mellin transform of the function $u(x)$.

Equation (2) can be reduced in the same way and put in a form similar to (6), thus

$$\frac{1}{2\pi i} \int_{N - \frac{1}{2} - i\infty}^{N - \frac{1}{2} + i\infty} \frac{\prod_{1}^{m'} \Gamma(b_j' + \beta_j' - \beta_j' s) \prod_{1}^{n'} \Gamma(1 - a_j' - \alpha_j' + \alpha_j' s)}{\prod_{m'+1}^{q'} \Gamma(1 - b_j' - \beta_j' + \beta_j' s) \prod_{n'+1}^{p'} \Gamma(a_j' + \alpha_j' - \alpha_j' s)} F(s) \frac{ds}{s - w} = V(1 - w) \qquad (u > N - \tfrac{1}{2}) \tag{7}$$

where $V(s)$ is the Mellin transform of $v(x)$.

Let us write

$$F(s) \prod_{1}^{m'} \Gamma(b_j' + \beta_j' - \beta_j' s) = \chi(s). \tag{8}$$

The use of the above transformation would be clear, after a few lines. $\chi(s)$ is regular and uniform in any interior strip for $\sigma < R(Q)$ and is equal to $O\, exp(-\frac{1}{2}\nu\pi|t|)|t|^{\mu}$), where $\nu = \sum_{1}^{m'} \beta_j' > 0$ and $\mu = R.$ $[(\tau - \nu)\sigma + \nu + \sum_{1}^{m'} b_j' - \frac{1}{2}m' - Q] + \epsilon$, uniformly in any interior Strip. then the equations (6) and (7) become

$$\frac{1}{2\pi i}\int_{N-\frac{1}{2}-i\infty}^{N-\frac{1}{2}+i\infty} \frac{\prod_{1}^{m}\Gamma(b_j+\beta_j-\beta_j s)\prod_{1}^{n}\Gamma(1-a_j-\alpha_j+\alpha_j s)\quad \chi(s)}{\prod_{m+1}^{q}\Gamma(1-b_j-\beta_j+\beta_j s)\prod_{n+1}^{p}\Gamma(a_j+\alpha_j-\alpha_j s)\prod_{1}^{m'}\Gamma(b_j'+\beta_j'-\beta_j' s)}\frac{ds}{s-w} = U(1-w) \tag{9}$$

$$(u < N - \tfrac{1}{2})$$

and

$$\frac{1}{2\pi i}\int_{N-\frac{1}{2}-i\infty}^{N-\frac{1}{2}+i\infty} \frac{\prod_{1}^{n'}\Gamma(1-a_j'-\alpha_j'+\alpha_j' s)\quad \chi(s)}{\prod_{m'+1}^{q'}\Gamma(1-b_j'-\beta_j'+\beta_j' s)\prod_{n'+1}^{p'}\Gamma(a_j'+\alpha_j'-\alpha_j' s)}\frac{ds}{s-w} = -V(1-w) \tag{10}$$

$$(u > N - \tfrac{1}{2})$$

Now let

$\triangle = \max R[(a_i + \alpha_i - 1)/\alpha_i, (a_j' + \alpha_j' - 1)/\alpha_j', (b_k + \beta_k - 1)/\beta_k] < c < N - \frac{1}{2} < c' < N,$

where $i = 1, \ldots, n, j = 1, \ldots, n', k = m+1, \ldots, q$, and let $c < u < N - \frac{1}{2}$

in (9). Since the integrand in (9) is regular in the strip $\triangle < \sigma < N$ except for a simple pole at $s = w$, and since it is $O(e^{-\frac{1}{2}\alpha\pi|t|}|t|^{-\lambda-1+\epsilon})$, where $\alpha > 0$ and $\lambda = R(\sum_{1}^{p} a_j - \sum_{1}^{q} b_j - (p-q)/2 + (Q-1)\tau)$,

we may move the line of integration to $\sigma = c$, and obtain

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\prod_{1}^{m}\Gamma(b_j+\beta_j-\beta_j s)\prod_{1}^{n}\Gamma(1-a_j-\alpha_j+\alpha_j s)\quad \chi(s)}{\prod_{m+1}^{q}\Gamma(1-b_j-\beta_j+\beta_j s)\prod_{n+1}^{p}\Gamma(a_j+\alpha_j-\alpha_j s)\prod_{1}^{m'}\Gamma(b_j'+\beta_j'-\beta_j' s)}\frac{ds}{s-w}$$

$$= U(1-w) - \frac{\prod_{1}^{m}\Gamma(b_j+\beta_j-\beta_j w)\prod_{1}^{n}\Gamma(1-a_j-\alpha_j+\alpha_j w)\quad \chi(w)}{\prod_{m+1}^{q}\Gamma(1-b_j-\beta_j+\beta_j w)\prod_{n+1}^{p}\Gamma(a_j+\alpha_j-\alpha_j w)\prod_{1}^{m'}\Gamma(b_j'+\beta_j'-\beta_j' w)} \tag{11}$$

The L.H.S. of (11) is a regular function of w for $u > c$ and so also is the R.H.S. When we multiply the R.H.S. of (11) by

$$\Psi = \frac{\prod_{1}^{n'}\Gamma(1-a_j'-\alpha_j'+\alpha_j' w)\prod_{m+1}^{q}\Gamma(1-b_j-\beta_j+\beta_j w)}{\prod_{1}^{n}\Gamma(1-a_j-\alpha_j+\alpha_j w)\prod_{m'+1}^{q'}\Gamma(1-b_j'-\beta_j'+\beta_j' w)}, \tag{12}$$

where either

$$b_i\,,\,\beta_i\ (\,i=1,\,\ldots,\,m\,)\qquad = b_i',\ \beta_i'\ \ (\,i=1,\,\ldots,\,m';\ m=m'\,)$$
$$a_i\,,\,\alpha_i\ (\,i=n+1,\,\ldots,\,p\,) = a_i',\ \alpha_i'\ \ (\,i=n'+1,\,\ldots,\,p';\ n-p=n'-p'\,)$$

or

$$a_i\,,\,\alpha_i\ (\,i=n+1,\,\ldots,\,p\,) = b_i,\ \ \beta_i\ \ (\,i=1,\,\ldots,\,m;\ p-n=m\,)$$

and

$$a_i',\,\alpha_i'\ (\,i=n'+1,\,\ldots,p'\,) = b_i',\ \beta_i'\ \ (\,i=1,\,\ldots,m';\ p'-n'=m'\,),$$

which is regular for $u > \triangle$, therefore for $u > c$, we see that the coefficient of x in the aforesaid product reduces to the integrand in (10). This is the use of the transformation stated in (8) above.

Now integrating the product of (12) and the R.H.S. of (11) round a large rectangle to the right of $u = c'$, we get

$$\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty}\Bigg[\ \chi\,(w)\ \frac{\prod_{1}^{n'}\Gamma\,(\,1-a_j'-\alpha_j'+\alpha_j'\,w\,)}{\prod_{m'+1}^{q'}\Gamma\,(\,1-b_j'-\beta_j'+\beta_j'\,w\,)\prod_{n'+1}^{p'}\Gamma\,(a_j'+\alpha_j'-\alpha_j'\,w\,)}$$

$$-\,U\,(\,1-w\,)\ \frac{\prod_{1}^{n'}\Gamma\,(1-a_j'-\alpha_j'+\alpha_j'\,w)\prod_{m+1}^{q'}\Gamma\,(\,1-b_j-\beta_j+\beta_j w)}{\prod_{1}^{n}\Gamma\,(\,1-a_j-\alpha_j+\alpha_j w\,)\prod_{m'+1}^{q'}\Gamma\,(\,1-b_j'-\beta_j'+\beta_j'\,w\,)}\Bigg]\frac{dw}{w-z}\ =\ 0\,, \tag{13}$$

with $x < c'$, provided that the integrand is of suitable order at infinity. To prove this we use the lemma given by Busbridge[7] and following the notation therefore we take

$$\varphi\ (w)\ =\ \chi\ (w)\ \frac{\prod_{1}^{m}\Gamma\,(\,b_j+\beta_j-\beta_j\,w\,)\prod_{1}^{n}\Gamma\,(\,1-a_j-\alpha_j+\alpha_j\,w\,)}{\prod_{m+1}^{q}\Gamma\,(1-b_j-\beta_j+\beta_j\,w\,)\prod_{n+1}^{p}\Gamma\,(a_j+\alpha_j-\alpha_j\,w\,)\prod_{1}^{m'}\Gamma\,(\,b_j'+\beta_j'-\beta_j'\,w\,)} \tag{14}$$

so that

$$\varphi\,(\,w\,) = \begin{cases} O\left(e^{-\frac{1}{2}\alpha\pi|v|}\ \ |v|^{-\lambda+\epsilon}\right) & (\,\lambda < 1\,)\\[2mm] O\left(e^{-\frac{1}{2}\alpha\pi|v|}\ \ |v|^{-1+\epsilon}\right) & (\,\lambda \geqslant 1\,), \end{cases}$$

where $\alpha > 0$ and $\lambda = R\left[\ \sum_{1}^{p} a_j - \sum_{1}^{q} b_j - \tau - (p-q)/_2 + Q\,\tau\ \right]$.

Taking $\beta = \gamma - \epsilon$, where $\gamma = \min\,(\,\lambda,\,1\,)$ and applying the lemma, we see that the product of (12) and the R.H.S. of (11) is less than

$$A\,M^{u}\ |\,u-c\,|^{-\delta}\ \ |\,v\,|^{-\gamma+\epsilon+\delta}\ (\,u^2+v^2\,)^{(B-1)/2}\qquad (\,0<\delta<\gamma-\epsilon\,)$$

where A depends on z and δ,

$$M = \prod_{1}^{n'}\alpha_j'^{\,\alpha_j'}\ \prod_{m+1}^{q}\beta_j^{\,\beta_j}\ \prod_{m'+1}^{q'}\beta_j'^{\,-\beta_j'}\ \prod_{1}^{n}\alpha_j^{\,-\alpha_j'}$$

$$B = R\left[\tfrac{1}{2}(q-q'+n'-n) - \sum_{1}^{n'} a_j' - \sum_{m+1}^{q} b_j + \sum_{m'+1}^{q'} b_j' + \sum_{1}^{n} a_j\right], \tag{15}$$

and

$$0 = \sum_{1}^{n'} \alpha_j' + \sum_{m+1}^{q} \beta_j - \sum_{m'+1}^{q'} \beta_j' - \sum_{1}^{n} \alpha_j = L \qquad \text{(say)}.$$

Integrating the above function round the rectangle whose corners are the points $c'-iT$, $T-iT$, $T+iT$ and $c'+iT$ $(T > |y|)$, we see that, for $M \leqslant 1$, $B \leqslant 1$,

$$\left|\int_{T-iT}^{T+iT}\right|, \quad \left|\int_{c'+iT}^{T+iT}\right|, \quad \left|\int_{c'-iT}^{T-iT}\right|$$

are all of $O(T^{-\gamma+\epsilon+B})$ and tend to zero as $T \to \infty$, provided $\max(B, 0) < \gamma$.

Now let us consider (10) when $N - \frac{1}{2} < u < c'$. This can also be transformed in a manner similar to that of (9). Since the integrand in (10) is regular for $\triangle < \sigma < N$ except for a simple pole at $s = w$ and it is $O(e^{-\frac{1}{2}\pi\alpha'|t|}\,|t|^{-\beta'-1})$, where $\alpha' > o$ and $\beta' = R\left[\tau'(\gamma-) + \sum_{1}^{p'} a_j' - \sum_{1}^{q'} b'_j - (p'-q')/2 - \tau\gamma + Q'\right] - \epsilon$, therefore by moving the line of integration to $\sigma = c'$, we obtain

$$\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} \frac{\prod_{1}^{m'}\Gamma(1-a_j'-\alpha_j'+\alpha_j' s)\quad \chi(s)}{\prod_{m'+1}^{q'}\Gamma(1-b_j'-\beta_j'+\beta_j' s)\prod_{n'+1}^{p'}\Gamma(a_j'+\alpha_j'-\alpha_j' s)}\,\frac{ds}{s-w}$$

$$= -V(1-w) - \chi(w)\,\frac{\prod_{1}^{n'}\Gamma(1-a_j'-\alpha_j'+\alpha_j' w)}{\prod_{m'+1}^{q'}\Gamma(1-b_j'-\beta_j'+\beta_j' w)\prod_{n'+1}^{p'}\Gamma(a_j'+\alpha_j'-\alpha_j' w)}. \tag{16}$$

Substituting from (16) into (13), we obtain $\chi(w)$, which on using (8) gives

$$F(w) = \frac{\prod_{m'+1}^{q'}\Gamma(1-b_j'-\beta_j'+\beta_j' w)\prod_{n'+1}^{p'}\Gamma(a_j'+\alpha_j'-\alpha_j' w)}{\prod_{1}^{n'}\Gamma(1-a_j'-\alpha_j'+\alpha_j' w)\prod_{1}^{m'}\Gamma(b_j'+\beta_j'-\beta_j' w)}\,.$$

$$\cdot\left[-V(1-w) + \frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} U(1-s)\frac{\prod_{1}^{n'}\Gamma(1-a_j'-\alpha_j'+\alpha_j' s)\prod_{m+1}^{q}\Gamma(1-b_j-\beta_j+\beta_j s)}{\prod_{1}^{n}\Gamma(1-a_j-\alpha_j+\alpha_j s)\prod_{m'+1}^{q'}\Gamma(1-b_j'-\beta_j'+\beta_j' s)}\,\frac{ds}{s-w}\right]$$

$$(u < c') \tag{17}$$

whence the Mellin inversion formula gives the solution.

THEOREM 1

If (*i*) $\tau, \tau', \alpha, \alpha' > 0$, (*ii*) $M \leqslant 1$, (*iii*) $L = 0$ and either (*iv*) $\max(B, 0) <$

$$(v)\quad (b_i,\ \beta_i)_{(i=1,\ldots,m)} = (b'_i,\ \beta'_i)_{(i=1,\ldots,m',\ m=m')}$$

$$(a_i,\ \alpha_i)_{(i=n+1,\ldots,p)} = (a'_i,\ \alpha'_i)_{(i=n'+1,\ldots,p',\ n-p=n'-p')}.$$

or $(iv)'$ $\max(B_1, 0) < \gamma$ where $\beta_1 = R\left[\tfrac{1}{2}(p^1 - p + q - q^1) - \sum_1^{n'} a_j' - \sum_{m+1}^{q} b_j + \sum_1^{n} a_j + \sum_{m'+1}^{q'} b_j'\right]$

$$(v)'\quad (a_i,\ \alpha_i)_{(i=n+1,\ldots,p)} = (b_i,\ \beta_i)_{(i=1,\ldots,m,\ p-n=m)}$$

$$(a'_j,\ \alpha'_j)_{(i=n'+1,\ldots,p')} = (b_j',\ \beta_j')_{(i=1,\ldots,m',\ p'-n'=m',)}$$

then the solution of (1) and (2) is given by

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(w)\, x^{-w}\, dw, \tag{18}$$

where $F(w)$ is given by (17) and c is determined by

$$\triangle < c < N - \tfrac{1}{2}.$$

A Special Case $v(x) = 0$

When $v(x) = 0$ the solution obtained is more general than given in the above, the restrictions (v) and (v') of theorem 1 are not necessary in this case. The problem is exactly similar to that considered by Busbridge[7] and her method of proof is fairly applicable.

Taking $v(x) = 0$ in (2),

$$\zeta(s) = \frac{\prod_1^{m} \Gamma(b_j + \beta_j - \beta_j s) \prod_1^{n'} \Gamma(1 - a_j' - \alpha_j' + \alpha_j' s)}{\prod_{m'+1}^{q'} \Gamma(1 - b_j' - \beta_j' + \beta_j' s) \prod_{n+1}^{p} \Gamma(a_j' + \alpha_j - \alpha_j s)}\ F(s) \tag{19}$$

and $N' = \min R\,(Q,\ b_i/\beta_i + 1,\ b_j'/\beta'_j + 1,\ a'_k/\alpha'_k + 1),\ (1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant m',\ n'+1 \leqslant k \leqslant p')$, (1) and (2) transform into

$$\frac{1}{2\pi i}\int_{N'-\frac{1}{2}-i\infty}^{N'-\frac{1}{2}+i\infty} \frac{\prod_1^{n} \Gamma(1 - a_j - \alpha_j + \alpha_j s) \prod_{m'+1}^{q'} \Gamma(1 - b'_j - \beta_j^1 + \beta_j' s)}{\prod_{m+1}^{q} \Gamma(1 - b_j - \beta_j + \beta_j s) \prod_1^{n'} \Gamma(1 - a_j' - \alpha_j' + \alpha_j' s)}\ \zeta(s)\ \frac{ds}{s-w} = U(1-w)$$

$$(u < N' - \tfrac{1}{2}) \tag{20}$$

and

$$\frac{1}{2\pi i}\int_{N'-\frac{1}{2}-\infty}^{N'-\frac{1}{2}+i\infty} \frac{\prod_1^{m'} \Gamma(b_j' + \beta_j' - \beta_j' s) \prod_{n+1}^{p} \Gamma(a_j + \alpha_j - \alpha_j s)}{\prod_{n'+1}^{p'} \Gamma(a_j' + \alpha_j' - \alpha_j' s) \prod_1^{m} \Gamma(b_j + \beta_j - \beta_j s)}\ \zeta(s)\ \frac{ds}{s-w} = 0 \tag{21}$$

$$(u > N' - \tfrac{1}{2}).$$

Setting $\Delta' = \max_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant n'}} \left(\frac{a_i - 1}{\alpha_i} + 1,\ \frac{a_j' - 1}{\alpha_j'} + 1\right) < c < N' - \tfrac{1}{2} < c' < N'$ and

applying an analysis similar to the previous case, (20) leads to

$$\int_{c'-i\infty}^{c'+i\infty} \left\{ \frac{\prod_{m+1}^{q} \Gamma(1-b_j-\beta_j+\beta_j w) \prod_{1}^{n'} \Gamma(1-a_j'-\alpha_j'+\alpha_j' w)}{\prod_{1}^{n} \Gamma(1-a_j-\alpha_j+\alpha_j w) \prod_{m'+1}^{q'} \Gamma(1-b'_j-\beta_j'+\beta_j' w)} U(1-w)-\zeta(w) \right\} \frac{dw}{w-xi\zeta} = 0, \quad (22)$$

valid for $x < c'$, $M \leqslant 1$, $L = 0$ and max $(B_2, 0) < \gamma$, where $B_2 = B+\frac{1}{2}(m'-m)$.

Now let us consider (21) when $N'-\frac{1}{2} < u < c'$. We deduce in a similar manner as in the case of equation (9),

$$\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} \zeta(s) \frac{\prod_{1}^{m'} \Gamma(b_j'+\beta_j'-\beta_j' s) \prod_{n+1}^{p} \Gamma(a_j+\alpha_j-\alpha_j s)}{\prod_{n'+1}^{p'} \Gamma(a_j'+\alpha_j'-\alpha_j' s) \prod_{1}^{m} \Gamma(b_j+\beta_j-\beta_j s)} \frac{ds}{s-w}$$

$$= \zeta(w) \frac{\prod_{1}^{m'} \Gamma(b_j'+\beta_j'-\beta_j' w) \prod_{n+1}^{p} \Gamma(a_j+\alpha_j-\alpha_j w)}{\prod_{n'+1}^{p'} \Gamma(a_j'+\alpha_j'-\alpha_j' w) \prod_{1}^{m} \Gamma(b_j+\beta_j-\beta_j w)} = \phi_1(w) \quad (say). \quad (23)$$

The L.H.S. of this equation is regular for $u < c'$, hence so is the R.H.S. For $\tau = \tau'$,

$$|\phi_1(w)| = \begin{cases} O\left(e^{-\frac{1}{2}\pi\alpha'|t|} |t|^{-\lambda'+\epsilon} \right) & (\lambda' < 1) \\ O\left(e^{-\frac{1}{2}\pi\alpha'|t|} |t|^{-1+\epsilon} \right) & (\lambda' \geqslant 1), \end{cases}$$

where $\alpha' > 0$ and $\lambda' = R\left(\sum_{1}^{p'} a_j' - \sum_{1}^{q'} b_j' - \tau' - \frac{1}{2}(p'-q') + Q\tau' \right)$ on the line $\sigma = c'$.

Multiplying $\phi_1(w)$ by inverse of the coefficient of $\zeta(w)$ and using Busbridge's lemma[7] with $\beta = \gamma' - \epsilon$, where $\gamma' = \min(\lambda', 1)$, we see that

$$\left| \frac{\zeta(w)}{w-z} \right| \leqslant A M_1^{u} |u-c'|^{-\delta} |v|^{-\gamma+\epsilon+\delta} (u^2+v^2)^{(B_3-1)/2},$$

where $u < x$, (A depending upon z and δ), $0 < \delta < \gamma^1 - \epsilon$,

$$M_1 = \prod_{1}^{m'} B'_j{}^{B'_j} \prod_{n+1}^{p} \alpha_j^{\alpha_j} \prod_{n'+1}^{p'} \alpha'_j{}^{-\alpha'_j} \prod_{1}^{m} \beta_j^{-\beta_j} \leqslant 1$$

$$B_3 = R\left[\tfrac{1}{2}(p-p'+n'-n+m'-m) + \sum_{n'+1}^{p'} a_j' + \sum_{1}^{m} b_j - \sum_{1}^{m'} b_j' - \sum_{n+1}^{p} a_j \right]$$

and

$$L_1 = \sum_{n'+1}^{p'} \alpha_j' + \sum_{1}^{m} \beta_j - \sum_{1}^{m'} \beta_j' - \sum_{n+1}^{p} \alpha_j = 0.$$

Taking $x > c$ and integrating the above function round the rextangle whose corners are the point $c-iT$, $c+iT$, $-T+iT$, $-T-iT$ $(T > |y|)$, we see that

$$\left| \int_{c+iT}^{-T+iT} \right|, \quad \left| \int_{-T+iT}^{-T-iT} \right|, \quad \left| \int_{-T-iT}^{c-iT} \right|$$

are all $O(T^{-(\gamma-\epsilon-B_3)})$, and therefore, for max $(B_3,0) < \gamma$,

$$\lim_{T\to\infty} \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \zeta(w)\frac{dw}{w-z} = 0 , \qquad (x > c) . \tag{24}$$

Let $c < x < c'$. Since $\zeta(w)$ is regular for $\triangle' < u < N'$ and is $O(e^{-\frac{1}{2}\pi a|v|}|v|^{p})$, where $\alpha > 0$ and

$$P = R[\sum_{1}^{m} b_j - \sum_{1}^{n'} a_j' + \sum_{m'+1}^{q'} b_j' - \sum_{n+1}^{p} \alpha_j + \tau + \tfrac{1}{2}(p - q' + n' - n + m' - m) - \tau Q] + \epsilon$$

We may move the line of integration in (24) to $u=c'$ and get

$$\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} \zeta(w)\cdot\frac{dw}{w-\zeta} = \zeta(z) . \tag{25}$$

Substituting from (25) into (22) we have

$$\zeta(z) = \frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} \frac{\prod_{m+1}^{q}\Gamma(1-b_j-\beta_j+\beta_j w)\prod_{1}^{n'}\Gamma(1-a_j'-\alpha_j'+\alpha'_j w)}{\prod_{1}^{n}\Gamma(1-\alpha_j-\alpha_j+\alpha_j(w)\prod_{m'+1}^{q'}\Gamma(1-b'_j-\beta'_j+\beta_j' w)} U(1-w)\frac{dw}{w-z} \qquad (x < c') \tag{26}$$

and then the value of $F(w)$ is determined from (19) and the solution $f(x)$ can be obtained by using Mellin' inversion formula.

THEOREM 2

If (*i*) $\tau = \tau' > 0$, $\alpha > 0$, $\alpha' > 0$, (*ii*) $M' \leqslant 1$, (*iii*) $L_1 = 0$ and (*iv*) max $(B_2, B_3, 0) < \gamma$, then the solution is given by (18) in which F(w) has the value given in (19) through (26) and $\triangle' < c < N' - \frac{1}{2}$.

Particular Case

Taking $m = m' = 1$, $n = p = n' = p' = 0$, $q = q' = 2$,

$a_i = a_i' = 0$, $b'_1 = \nu/2 + \alpha/2$, $b_2 = -\nu/2 + \alpha/2$, $b'_1 = \nu/2$, $b'_2 = -\nu/2$,

$\alpha_i = \beta_i = \alpha'_i = \beta'_i = 1$, $u(x) = 2^{-(a+1)} x^{a1/2} g(x^{1/2})$, $v(x) = 0$

and $f(2\sqrt{x})/2\sqrt{x}$ for $f(x)$ and using the relation

$$x^{\mu} J_\nu(x) = 2^{\mu} H^{1\,0}_{0\,2}\left(\frac{x^2}{4}\middle|(\nu/_2 + \mu/_2, 1)(-\nu/_2 + \mu/_2, 1)\right),$$

We see, after a change of variable, that the equations (1) and (2) reduce to the dual integral equations considered by Titchmarsh[4]. The solution presented by equations (19) and (26) with an application of Mellin's inversion formula reduces to the solution given by Busbridge[7] (equations (1·3) and (1·4))

EXAMPLES

Let $u(x) = x^{\mu}$ and $v(x) = x^{\nu}$.

Then

$$U(1-s)=\int_0^1 x^{-s+\mu}\,dx=\frac{\Gamma(\mu-s)}{\Gamma(1+\mu-s)} \qquad (0<\mu+1)$$

$$V(1-s)=\int_1^\infty x^{-s+\nu}\,dx=\frac{\Gamma(\nu-s)}{\Gamma(1+\nu-s)} \qquad (\sigma>\nu+1)$$

and (17) gives

$$F(w)=\frac{\prod_{m'+1}^{q'}\Gamma(1-b_j'-\beta_j'+\beta_j' w)\prod_{n'+1}^{p'}\Gamma(a_j'+\alpha_j'-\alpha_j' w)}{\prod_1^{n'}\Gamma(1-a_j'-\alpha_j'+\alpha_j' w)\prod_1^{m'}\Gamma(b_j'+\beta_j'-\beta_j' w)}\left[\frac{\Gamma(\nu-w)}{\Gamma(1+\nu-w)}\right.$$

$$\left.+\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty}\frac{\Gamma(\mu-s)\prod_1^{n'}\Gamma(1-a_j'-\alpha_j'+\alpha_j' s)\prod_{m+1}^{q}\Gamma(1-b_j-\beta_j+\beta_j s)}{\Gamma(1+\mu-s)\prod_{m'+1}^{q'}\Gamma(1-b_j'-\beta_j'+\beta_j' s)\prod_1^{n}\Gamma(1-a_j-a_j+\alpha_j s)}\,ds\int_0^1 \eta^{s-w-1}\,d\eta\right].$$

Therefore, under the assumptions of thelorem 1, solution is given by

$$f(x)=x^{-1}H_1(x^{-1})+\int_0^1 (yx)^{-1}H_2(y^{-1}x^{-1})\,H_3(y)\,dy,$$

where

$$H_1(x)=H^{p'-n'+1,\,q'-m'}_{q'+1,\,p'+1}\left(x\,\middle|\,\begin{matrix}(b'_{m'+1},\beta'_{m'+1}),\ldots,(b'_{q'},\beta'_{q'}),(\nu,1)\{(b'_{m'},\beta'_{m'})\}\\(\nu-1,1),(a'_{n'+1},\alpha'_{n'+1}),\ldots,(a'_{p'},\alpha'_{p'}),\{a'_{n'},\alpha'_{n}\}\end{matrix}\right)$$

$$H_2(x)=H^{p'-n'\,q'-m'}_{q',\,p'}\left(x\,\middle|\,\begin{matrix}b'_{m'+1},(\beta'_{m+1},),\ldots,(b'_{q'},\beta'_{q'}),\{(b'_{m'},\beta'_{m'},)\}\\a'_{n'+1},\alpha'_{n'+1}),\ldots,(a'_{p'},\alpha'_{p}),\{(a'_{n'})\}\end{matrix}\right)$$

and

$$H_3(x)=H^{1\;q-m+n'}_{q-m+n'+1,\,q'-m'+n+1}\left(x\,\middle|\,\begin{matrix}\{a'_{n'},\alpha^1{}_{n^1})\},(b_{m+1},\beta_{m+1}),\ldots,(b_q,\beta_q),(\mu,1)\\(\mu-1,1)\;(b'_{m'+1},\beta'_{m'+1})\ldots,(b'_{q'},\beta'_{q'}),\{(a_n,\alpha_n)\}\end{matrix}\right)$$

Also, with the same values of $u(x)$ and $v(x)$, theorem 2 gives the solution

$$f(x)=\int_0^1 (yx)^{-1}H_4(y^{-1}x^{-1})\,H_3(y)\,dy,$$

where

$$H_4(x)=^{p-n,\,q'-m'}_{q'-m'+m,\,p-n+n'}\left(x\,\middle|\,\begin{matrix}(b'_{m'+1},\beta'_{m'+1}),\ldots,(b'_{q'},\beta'_{q'}),\{(b_m,\beta_m)\}\\(a_{n+1},\alpha_{n+1}),\ldots,(a_p,\alpha_p),\{(a'_{n'}\,\alpha'_{n'})\}\end{matrix}\right).$$

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