# TORSIONAL VIBRATION OF A NON-HOMOGENEOUS COMPOSITE CYLINDRICAL SHELL SUBJECTED TO A MAGNETIC FIELD

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This paper investigates the propagation of torsional wave in a non-homogeneous composite cylindrical shell characterised by an aeolotropic material in the region $r_1 \leqslant r \leqslant r_2$ and visco-elastic material representing a parallel union of Kelvin and Maxwell bodies in the region $r_2 \leqslant r \leqslant r_3$. The non-homogeneity of the shell is due to the variable elastic constants $C_{ij}$, variable density $\rho$ and variable shear modulus $\mu$. Lastly, frequency equation and phase velocity of the wave have been calculated. The perturbation equations of the field and the torsional vibration of aeolotropic as well as visco-elastic shell have also been investigated.

The investigations relating to the combined effect of mechanical and electromagnetic fields in elastic and visco-elastic materials have received an impetus in recent years due to their extensive applications in various branches of science and technology, particularly in plasmatrons and aeromagnetic flutter. The significance of these investigations, derived chiefly from the behaviour of seismic wave propagation, has a reasonable bearing on many seismological problems, particularly in the detection of mechanical explosions in the interior of the earth and in radiation of electromagnetic energy into vacuum adjacent to magnetoelastic bodies. Such problems have been discussed in a series of papers by Kaliski[1,2], Sinha[3], Giri[4], Yadava[5], Narain & Verma[6], Narain[7], and many others. As a sequal to these, the present paper is an attempt to discuss the torsional vibration of a non-homogeneous composite cylindrical shell subjected to a magnetic field. The non-homogeneity of the shell is due to the variable elastic constants $c_{ij}$ $(i, j=1,2,\ldots.6)$, variable density $\rho$ and variable shear modulus $\mu$.

## PROBLEM, FUNDAMENTAL EQUATIONS AND BOUNDARY CONDITIONS

We consider a perfectly conducting non-homogeneous cylindrical shell characterized by an aeolotropic material in the region $r_1 \leqslant r \leqslant r_2$ and visco-elastic material representing a parallel union of Kelvin and Maxwell bodies in the region $r_2 \leqslant r \leqslant r_3$. The boundary of the shell is supposed to be mechanically stress free. We assume that the shell is placed in vacuum and initially there exists an axial magnetic field of intensity $H$. Since the problem considered is of magnetoelasticity, we consider the expressions connecting the component of stress and strain; the constitutive relations of material together with magnetoelastic equations supplemented by electro-magnetic equations of Maxwell. The constitutive relations of the aeolotropic bodies in the cylindrical coordinates $(r,\theta,z)$ as in Love[8] are given by

$$\left.\begin{aligned}
\sigma_{rr} &= c_{11}\, e_{rr} + c_{12}\, e_{\theta\theta} + c_{13}\, e_{zz} \\
\sigma_{\theta\theta} &= c_{21}\, e_{rr} + c_{22}\, e_{\theta\theta} + c_{23}\, e_{zz} \\
\sigma_{zz} &= c_{31}\, e_{rr} + c_{32}\, e_{\theta\theta} + c_{33}\, e_{zz} \\
\sigma_{rz} &= c_{44}\, e_{rz} \\
\sigma_{\theta z} &= c_{55}\, e_{\theta z} \\
\sigma_{r\theta} &= c_{66}\, e_{r\theta}
\end{aligned}\right\} \tag{1}$$

where $\sigma_{rr}$, $\sigma_{\theta\theta}$, .... etc and $e_{rr}$, $e_{\theta\theta}$, ...... etc are components of stress and strain respectively and $c_{ij}$ $(i, j=1, 2..6)$ are elastic constants. Assuming that the temperature remains constant the stress-strain relation for visco-elastic solid under consideration as in Nowacki[9] is

$$\left(1 + m_1 \frac{\partial}{\partial t}\right) s_{ij} = 2\mu \left(1 + m_2 \frac{\partial}{\partial t}\right) e_{ij} \tag{2}$$

where

$$\left.\begin{aligned} s_{ij} &= \sigma_{ij} - \frac{1}{3}\, s\delta_{ij} \quad (s = 3ke) \\ e_{ij} &= \epsilon_{ij} - \frac{1}{3}\, e\delta_{ij} \quad (e = \epsilon_{kk}) \end{aligned}\right\} \tag{3}$$

are deviatoric components of stress and strain tensors $\sigma_{ij}$ and $\epsilon_{ij}$; $\lambda$, $\mu$ are Lame's constants; $k = \lambda + \frac{2}{3}\mu$ is the bulk modulus, $m_1$ $m_2$ are visco-elastic moduli and $\delta_{ij}$ is Kronecker's delta. The strain displacement relation is,

$$2\epsilon_{ij} = u_{i,j} + u_{j,i} \tag{4}$$

and the stress equation of motion is

$$\sigma_{ij,j} + (J \times B)_i = \rho\, \frac{\partial^2 u_i}{\partial t^2} \tag{5}$$

Maxwell's equations governing the electromagnetic field in the body and the electromagnetic field equations in vacuum are similar to that given in the paper of Narain[7]. Since we are considering torsional vibration, displacement vector $u$ has only $v$ as its non-vanishing component which is independent of $\theta$. Thus

$$\left.\begin{aligned} u_r &= u_z = 0 \\ u_\theta &= v \end{aligned}\right\} \tag{6}$$

and the magnetic intensity $H$ has the components

$$\left.\begin{aligned} H_r &= H_\theta = 0 \\ H_z &= H \quad \text{(constant)} \end{aligned}\right\} \tag{7}$$

Using equations (1) and (6) the only non-vanishing stress equation of motion (5) for aeolotropic material of the shell comes out to be

$$c_{66}\left\{\frac{\partial^2 v_1}{\partial r^2} + \frac{1}{r}\frac{\partial v_1}{\partial r} - \frac{v_1}{r^2}\right\} + c_{55}\frac{\partial^2 v_1}{\partial z^2} - \frac{H^2}{4\pi}\frac{\partial^2 v_1}{\partial z^2} + \frac{\partial}{\partial r}(c_{66})\left(\frac{\partial v_1}{\partial r} - \frac{v_1}{r}\right) = \rho\,\frac{\partial^2 v_1}{\partial t^2} \tag{8}$$

and using equations (2), (3), (4), and (5) the non-vanishing stress equation of motion for visco-elastic material of the shell comes out to be

$$\mu\left(1 + m_2\frac{\partial}{\partial t}\right)\left\{\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v_2}{\partial r} - \frac{v_2}{r^2} + \frac{\partial^2 v_2}{\partial z^2}\right\} + \mu\left(1 + m_2\frac{\partial}{\partial t}\right)\left(\frac{\partial v_2}{\partial r} - \frac{v_2}{r}\right)\frac{\partial \mu}{dr} - \left(1 + m_1\frac{\partial}{\partial t}\right)\left(\frac{H^2}{4\pi}\frac{\partial^2 v_2}{\partial z^2} + \rho\,\frac{\partial^2 v_2}{\partial t^2}\right) = 0 \tag{9}$$

For harmonic torsional vibration we seek the solutions of the form

$$v_j = F_j(r)\, e^{i(qz+pt)} \qquad (j = 1, 2) \tag{10}$$

and consequently the equations (8) & (9) take the following forms

$$\frac{d^2F_1}{dr^2} + \frac{1}{r}\frac{dF_1}{dr} - \frac{F_1}{r^2} - \frac{1}{c_{66}}\left\{c_{55}\,q^2 - \frac{H^2q^2}{4\pi} - \rho p^2\right\}F_1(r) + \frac{1}{c_{66}}\frac{d}{dr}(c_{66})\left(\frac{dF_1}{dr} - \frac{F_1}{r}\right) = 0 \tag{11}$$

and

$$\frac{d^2F_2}{dr^2}+\frac{1}{r}\frac{dF_2}{dr}+\left\{\frac{(1+m_1\,ip)}{\mu\,(1+m_2\,ip)}\,\frac{H^2q^2}{4\pi}-q^2+\frac{(1+m_1\,ip)}{\mu(1+m_2\,ip)}\rho p^2-\right.$$

$$\left.-\frac{1}{r^2}\right\}F_2\,(r)+\left(\frac{dF_2}{dr}-\frac{F_2}{r}\right)\frac{1}{\mu}\,\frac{d\mu}{dr}=0 \qquad (12)$$

The electromagnetic field equations in vacuum take the following forms

$$\frac{d^2h^*_0}{dr^2}+\frac{1}{r}\,\frac{dh^*_0}{dr}+\omega^2\,h_0^*=0 \qquad (13)$$

and

$$\frac{d^2E^*_0}{dr^2}+\frac{1}{r}\,\frac{dE^*_0}{dr}+\omega^2\,E^*_0=0 \qquad (14)$$

where

$$\omega^2=p^2-q^2$$

If the expression for the material in the region $r_1 \leqslant r \leqslant r_2$ be denoted by the suffix 1 and for that in the region $r_2 \leqslant r \leqslant r_3$ by the suffix 2 then the boundary conditions on the surface are

$$\left.\begin{array}{l}(\sigma_{rr})_1+(T_{r\theta})_1-(T^*_{r\theta})_1=0 \text{ on } r=r_1\\ (\sigma_{rr})_2+(T_{r\theta})_2-(T^*_{r\theta})_2=0 \text{ on } r=r_3\end{array}\right\} \qquad (15)$$

and the continuity of the stress displacement and Maxwellian tensor in the shell on the surface $r = r_2$ when formulated are

$$\left.\begin{array}{l}(v)_1=(v)_2 \text{ on } r=r_2\\ (\sigma_{r\theta})_1=(\sigma_{r\theta})_2 \text{ on } r=r_2\\ (T_{r\theta})_1=(T_{r\theta})_2 \text{ on } r=r_2\end{array}\right\} \qquad (16)$$

where $T_{r\theta}$ and $T^*_{r\theta}$ are Maxwell tensors in the shell and vacuum respectively.

## METHOD OF SOLUTION

We suppose that the elastic constants $c_{ij}$, density $\rho$ and the shear modulus $\mu$ of the shell vary as

$$\left.\begin{array}{l}c_{ij}=\mu_{ij}\,r^2\\ \rho=\rho_0\,r^2\\ \mu=\mu_0\,r^2\end{array}\right\}\quad (i,j=1,2,\ldots .6) \qquad (17)$$

where $\mu_{ij}$, $\rho_0$ and $\mu_0$ are constants and $r$ is the radius vector. The solutions of the equations (11) and (12) with help of (17) are given by (c.f. Narain[7]).

$$v_1=\frac{1}{r}\left\{A_1\,J_{\nu_1}\,(\lambda_1 r)+B_1\,Y_{\nu_1}\,(\lambda_1 r)\right\}e^{i(qz+pt)} \qquad (18a)$$

and

$$v_2=\frac{1}{r}\left\{A_2\,J\nu_2\,(\lambda_2 r)+B_2\,Y\nu_2\,(\lambda_2 r)\right\}e^{i(qz+pt)} \qquad (18b)$$

where

$$\left.\begin{aligned}
\lambda^2{}_1 &= \frac{\rho_0 p^2}{\mu_{66}} - \frac{\mu_{55}}{\mu_{66}} q^2 \\
\mu^2{}_1 &= 3 - \frac{H^2 q^2}{4\pi \; \mu_{66}} \\
\lambda^2{}_2 &= \frac{(1 + m_1 ip)\, \rho_0 \, p^2}{\mu_0 (1 + m_2 ip)} - q^2 \\
\mu^2{}_2 &= 3 - \frac{(1 + m_1 ip)}{\mu_0 (1 + m_2 ip)} \; \frac{H^2 q^2}{4x} \\
\nu^2{}_1 &= \mu^2{}_1 + 1 \\
\nu^2{}_2 &= \mu^2{}_2 + 1
\end{aligned}\right\} \tag{19}$$

and $A_1$, $B_1$, $A_2$, $B_2$ are constants and $J\nu_1$, $J\nu_2$ and $Y\nu_1$, $Y\nu_2$ are Bessel functions of first and second kind respectively. Using the recurrence formulae

$$\begin{aligned}
J'_p(x) &= J_{p-1}(x) - \frac{p}{x} J_p(x) \\
&= \frac{p}{x} J_p(x) - J_{p+1}(x)
\end{aligned} \tag{20}$$

$(\sigma_{rr})_1$, $(\sigma_{rr})_2$ are given by

$$(\sigma_{rr})_1 = \mu_{66} \Big[ A_1 \Big\{ \lambda_1 r \, J_{\nu_1 - 1}(\lambda_1 r) - (\lambda_1 \nu_1 + 2) \, J_{\nu_1}(\lambda_1 r) \Big\} + + B_1 \Big\{ \lambda_1 r Y_{\nu_1 - 1}(\lambda_1 r) - (\lambda_1 \nu_1 + 2) \, Y_{\nu_1}(\lambda_1 r) \Big\} \Big] e^{i(qz + pt)} \tag{21}$$

$$\text{for } r_1 \leqslant r \leqslant r_2$$

and

$$\left[ \left( 1 + m_1 \frac{d}{dt} \right) \sigma_{rr} \right]_2 = \mu_0 (1 + m_2 ip) \Big[ A_2 \Big\{ \lambda_2 r \, J_{\nu_2 - 1}(\lambda_2 r) - - (\lambda_2 \nu_2 + 2) \, J_{\nu_2}(\lambda_2 r) \Big\} + B_2 \Big\{ \lambda_2 r Y_{\nu_2 - 1}(\lambda_2 r) - - (\lambda_2 \nu_2 + 2 \, Y_{\nu_2})(\lambda_2 r) \Big\} \Big] e^{i(qz + pt)} \tag{22}$$

$$\text{for } r_2 \leqslant r \leqslant r_3$$

Since $T_{rr} = T^*_{rr} = 0$, the boundary conditions (15) and (16) yield

$$A_1 \Big\{ \lambda_1 r_1 \, J_{\nu_1 - 1}(\lambda_1 r_1) - (\lambda_1 \nu_1 + 2) \, J_{\nu_1}(\lambda_1 r_1) \Big\} + + B_1 \Big\{ \lambda_1 r_1 Y_{\gamma_1 - 1}(\lambda_1 r_1) - (\lambda_1 \nu_1 + 2) \, Y_{\nu_1}(\lambda_1 r_1) \Big\} = 0 \tag{23}$$

$$A_2 \left\{ \lambda_2 r_3 J_{\nu_2-1} ( \lambda_2 r_3 ) - ( \lambda_2 \nu_2 + 2 ) J_{\nu_2} ( \lambda_2 r_3 ) \right\} +$$

$$+ B_2 \left\{ \lambda_2 r_3 Y_{\nu_2-1} ( \lambda_2 r_3 ) - ( \lambda_2 \nu_2 + 2 ) J_{\nu_2} ( \lambda_2 r_3 ) \right\} = 0 \tag{24}$$

$$A_1 J_{\nu_1} ( \lambda_1 r_2 ) + B_1 Y_{\nu_1} ( \lambda_1 r_2 ) = A_2 J_{\nu_2} ( \lambda_2 r_2 ) + B_2 Y_{\nu_2} ( \lambda_2 r_2 ) \tag{25}$$

and

$$\mu_{66} \Big[ A_1 \left\{ \lambda_1 r_2 J_{\nu_1-1} ( \lambda_1 r_2 ) - ( \lambda_1 \nu_1 + 2 ) J_{\nu_1} ( \lambda_1 r_2 ) \right\} +$$

$$+ B_1 \left\{ \lambda_1 r_2 Y_{\nu_1-1} ( \lambda_1 r_2 ) - ( \lambda_1 \nu_1 + 2 ) Y_{\nu_1} ( \lambda_1 r_2 ) \right\} \Big] =$$

$$= \mu_0 ( 1 + m_2 ip ) \Big[ A_2 \left\{ \lambda_2 r_2 J_{\nu_2-1} ( \lambda_2 r_2 ) - ( \lambda_2 \nu_2 + 2 ) J_{\nu_2} ( \lambda_2 r_2 ) \right\} +$$

$$+ B_2 \left\{ \lambda_2 r_2 Y_{\nu_2-1} ( \lambda_2 r_2 ) - ( \lambda_2 \nu_2 + 2 ) Y_{\nu_2} ( \lambda_2 r_2 ) \right\} \Big] . \tag{26}$$

Thus we have four linear equations (23) to (26) to determine four constants $A_1$, $B_1$, $A_2$, $B_2$ in forms of material constants of the problem. Eliminating these constants from (23) to (26) the frequency equation is obtained as

$$\Big[ \left\{ \lambda_2 r_3 J_{\gamma_2-1} ( \lambda_2 r_3 ) - ( \lambda_2 \nu_2 + 2 ) J_{\nu_2} ( \lambda_2 r_3 ) \right\} Y_{\nu_2} ( \lambda_2 r_2 ) -$$

$$- J_{\nu_2} ( \lambda_2 r_2 ) \left\{ \lambda_2 r_3 Y_{\nu_2-1} ( \lambda_2 r_3 ) - ( \lambda_2 \nu_2 + 2 ) Y_{\nu_2} ( \lambda_2 r_3 ) \right\} \Big] \times$$

$$\Big[ \mu_{66} \left\{ \lambda_1 r_2 J_{\nu_1-1} ( \lambda_1 r_2 ) - ( \lambda_1 \nu_1 + 2 ) J_{\nu_1} ( \lambda_1 r_2 ) \right\} \left\{ \lambda_1 r_1 Y_{\nu_1-1} ( \lambda_1 r_1 ) - \right.$$

$$\left. - ( \lambda_1 \nu_1 + 2 ) Y_{\nu_1} ( \lambda_1 r_1 ) \right\} - \left\{ \lambda_1 r_1 J_{\nu_1-1} ( \lambda_1 r_1 ) - ( \lambda_1 \nu_1 + 2 ) J_{\nu_1} ( \lambda_1 r_1 ) \right\} \times$$

$$\left\{ \lambda_1 r_2 Y_{\nu_1-1} ( \lambda_1 r_2 ) - ( \lambda_1 \nu_1 + 2 ) Y_{\nu_1} ( \lambda_1 r_2 ) \right\} \mu_{66} \Big] = \Big[ \mu_0 ( 1 + m_2 ip )$$

$$\left\{ \lambda_2 r_2 Y_{\nu_2-1} ( \lambda_2 r_2 ) - ( \lambda_2 \nu_2 + 2 ) Y_{\nu_2} ( \lambda_2 r_2 ) \right\} \left\{ \lambda_2 r_3 J_{\nu_2-1} ( \lambda_2 r_3 ) - ( \lambda_2 \nu_2 + 2 ) \right.$$

$$\left. J_{\nu_2} ( \lambda_2 r_3 ) \right\} \Big] \Big[ J_{\nu_1} ( \lambda_1 r_2 ) \left\{ \lambda_1 r_1 Y_{\nu_1-1} ( \lambda_1 r_1 ) - ( \lambda_1 \nu_1 + 2 ) Y_{\nu_1} ( \lambda_1 r_1 ) \right\} -$$

$$- \left\{ \lambda_1 r_1 J_{\nu_1-1} ( \lambda_1 r_1 ) - ( \lambda_1 \nu_1 + 2 ) J_{\nu_1} ( \lambda_1 r_1 ) \right\} \left\{ Y_{\nu_1} ( \lambda_1 r_2 ) \right\} \Big] . \tag{27}$$

Introducing the wave lenght $\lambda = \frac{2\pi}{q}$ and the phase velocity $c_1 = \frac{p}{q}$ of the torsional wave inside the shell we can determine $c_1$ from equation

$$c_1 = \beta \left\{ \xi^2 \left( \frac{\lambda_1}{2\pi r_1} \right)^2 + \frac{\mu_{55}}{\mu_{66}} \right\}^{1/2} \tag{28}$$

where

$$\beta^2 = \frac{\mu_{66}}{\rho_0}$$

and $\xi$ is a root of the transcendental equation

$$\Big[ \Big\{ x_2\, \xi\, J_{\nu_2-1}(x_2\xi) - (\lambda_2\nu_2 + 2)\, J_{\nu_2}(x_2\,\xi) \Big\}\, Y_{\nu_2}(x_1\xi) -$$

$$- J_{\nu_2}(x_1\xi) \Big\{ x_2\xi\, Y_{\nu_2-1}(x_2\xi) - (\lambda_2\,\nu_2 + 2)\, Y_{\nu_2}(x_2\xi) \Big\} \Big] \Big[ \mu_{66} \Big\{ r\xi\, J_{\nu_1-1}(r\xi) -$$

$$- (\lambda_1\,\nu_1 + 2)\, J_{\nu_1}(r\xi) \Big\} \Big\{ \xi\, Y_{\nu_1-1}(\xi) - (\lambda_1\,\nu_1 + 2)\, Y_{\nu_1}(\xi) \Big\} - \Big\{ \xi J_{\nu_1-1}(\xi) -$$

$$- (\lambda_1\,\nu_1 + 2)\, J_{\nu_1}(\xi) \Big\} \Big\{ r\xi\, Y_{\nu_1-1}(r\xi) - (\lambda_1\,\nu_1 + 2)\, Y_{\nu_1}(r\xi) \Big\}\, \mu_{66} \Big] =$$

$$= \Big[ \mu_0\,(1 + m_2\, ip) \Big\{ x_2\xi\, Y_{\nu_2-1}(x_2\xi) - (\lambda_2\,\nu_2 + 2)\, Y_{\nu_2}(x_2\xi) \Big\} -$$

$$- \mu_0\,(1 + m_2\, ip) \Big\{ x_1\xi\, J_{\nu_2-1}(x_1\xi) - (\lambda_2\,\nu_2 + 2)\, J_{\nu_2}(x_1\xi) \Big\} \Big] \times$$

$$\Big[ J_{\nu_1}(r\xi) \Big\{ \xi Y_{\nu_1-1}(\xi) - (\lambda_1\,\nu_1 + 2)\, Y_{\nu_1}(\xi) \Big\} - \Big\{ \xi J_{\nu_1-1}(\xi) - (\lambda_1\,\nu_1 + 2)$$

$$J_{\nu_1}(\xi) \Big\}\, Y_{\nu_1}(r\xi) \Big] \tag{29}$$

where

$$\left. \begin{array}{ll} x_1 = \dfrac{\lambda_2\, r}{\lambda_1}, & x_2 = \dfrac{\lambda_2\, r^2}{\lambda_1} \\ \xi = \lambda_1\, r_1, & r = \dfrac{r_2}{r_1} = \dfrac{r_3}{r_2} \end{array} \right\} \tag{30}$$

The non-dimensional phase velocity $c^*$ $\left( = \dfrac{c_1}{\beta} \right)$ is given by

$$c^* = \left\{ \frac{\xi^2}{n^2} + \frac{\mu_{55}}{\mu_{66}} \right\}^{1/2} \tag{31}$$

where

$$n = \frac{2\pi r_1}{\lambda}$$

is the wave number. For perfectly elastic material $\mu_{55} = \mu_{66} = \mu_0$ and hence from (31), we have

$$c^* = \left\{ \frac{\xi^2}{n^2} + 1 \right\}^{1/2} \tag{32}$$

We use the following results of Watson[10] to find the value for small values of $x$

$$\lim_{x\to 0} J_n(x) \cong \frac{x^n}{2^n \pi\,(n)} \quad \text{and} \lim_{x\to 0} Y_n(x) \cong \frac{1}{x^n} \quad \text{if } n \neq 0 \tag{33}$$

As a consequence of the result (33) the equation (29) takes the form

$$\left[ r^{\nu_2} \left\{ 2\nu_2 - (\lambda_2 \nu_2 + 2) \right\} - r^{-\nu_2} \left\{ x^2_2 \, \xi^2 - (\lambda_2 \nu_2 + 2) \right\} \right] \cdot \mu_{66} \cdot$$
$$\left[ r^{\nu_1} \left\{ 2\nu_1 - (\lambda_1 \nu_1 + 2) \right\} \left\{ \xi^2 - (\lambda_1 \nu_1 + 2) \right\} - r^{-\nu_1} \left\{ 2\nu_1 - (\lambda_1 \nu_1 + 2) \right\} \right.$$
$$\left. \left\{ r^2 \xi^2 - (\lambda_1 \nu_1 + 2) \right\} \right] = \left\{ 2\nu_2 - (\lambda_2 \nu_2 + 2) \right\} \mu_0 (1 + m_2 \, ip) \left[ r^{-\nu_2} \right.$$
$$\left. \left\{ x^2_2 - (\lambda_2 \nu_2 + 2) \right\} + \left\{ x^2_1 \, \xi^2 - (\lambda_2 \nu_2 + 2) \right\} r^{\nu_2} \right] \left[ r^{\nu_1} \left\{ \xi^2 - (\lambda_1 \nu_1 + 2) \right\} - \right.$$
$$\left. - r^{-\nu_1} \left\{ 2\nu_1 - (\lambda_1 \nu_1 + 2) \right\} \right] \tag{34}$$

If there were no magnetic field, i.e. $H=0$ then from equation (19) we have $\nu_1 = \nu_2 = 2$ and hence the equation (34) becomes

$$\left[ r^4 \left\{ 2(1 - \lambda_2) \right\} - \left\{ x^2_2 \, \xi^2 - 2 (\lambda_2 + 1) \right\} \right] \left[ r^4 \left\{ 2 (1 - \lambda_1) \right\} \right.$$
$$\left. \left\{ \xi^2 - 2 (\lambda_1 + 2) \right\} - 2 (1 - \lambda_1) \left\{ r^2 \xi^2 - 2 (\lambda_1 + 1) \right\} \right] = \frac{2 (1 - \lambda_2) \, \mu_0 (1 + m_2 \, ip)}{\mu_{66}} \times$$
$$\times \left[ \left\{ x^2_2 \, \xi^2 - 2 (\lambda_2 + 1) \right\} + \left\{ x^2_1 \, \xi^2 - 2 (\lambda_2 + 1) \right\} r^4 \right] \left[ r^4 \left\{ \xi^2 - 2 (\lambda_1 + 2) \right\} - \right.$$
$$\left. - \left\{ 2 (1 - \lambda_1) \right\} \right]. \tag{35}$$

## NUMERICAL RESULTS

For $r=2$ the equation (35) takes the form

$$\left[ 32 (1 - \lambda_2) - \left\{ x^2_2 \, \xi^2 - 2 (\lambda_2 + 1) \right\} \right] \left[ 32 (1 - \lambda_2) \left\{ \xi^2 - 2 (\lambda_1 + 2) \right\} - \right.$$
$$\left. - 2 (1 - \lambda_1) \left\{ 4 \xi^2 - 2 (\lambda_1 + 2) \right\} \right] = 2 (1 - \lambda_2) \, k \left[ \left\{ x^2_2 \, \xi^2 - 2 (\lambda_2 + 1) \right\} + \right.$$
$$\left. + \left\{ x^2_1 \, \xi^2 - 2 (\lambda_2 + 1) \right\} \right] \left[ 16 \left\{ \xi^2 - 2 (\lambda_1 + 2) \right\} - 2 (1 - \lambda_1) \right] \tag{36}$$

where

$$k = \frac{\mu_0 (1 + m_2 \, ip)}{\mu_{66}}.$$

Taking $\lambda_1 = 1.3$, $\lambda_2 = 1.6$ and $k=1$, we get $\xi^2 = 1.58$ and $-0.33$ approximately. Thus for one set of values of $\lambda_1$, $\lambda_2$, $k$ and $r$ we get four values of $\xi$ corresponding to four modes of vibration. From equation (32) we can obtain different values of $c^*$ for different wave numbers.

## SOLUTION OF THE PERTURBATION FIELD EQUATIONS

The electromagnetic field equations are solved under the boundary conditions

$$E = \overset{*}{E} \text{ on } r = r_1 \text{ and } E = \overset{*}{E} \text{ on } r = r_3 \tag{37}$$

and

$$\frac{d\overset{*}{h}}{dt} = - \frac{c}{r} \frac{dE}{dz} \text{ on } r = r_1 \text{ and } r = r_3 \tag{38}$$

and also the radiation condition as $r \to \infty$. As given by Narain[7]

$$E = \left[ -\frac{H}{c}\frac{dv}{dt}, 0, 0 \right] \tag{39}$$

$$h = \left[ 0, H\frac{dv}{dz}, 0 \right] \tag{40}$$

and the solution of the equation (13) and (14) are taken so as to satisfy the radiation condition in the form.

$$h_0^* = CH_0^{(2)}(\omega r) \text{ for } r > r_3$$

$$= DH_0^{(1)}(\omega r) \text{ for } r < r_1 \tag{41}$$

$$\left. \begin{aligned} E_0^* &= C_1 H_0^{(2)}(r) \text{ for } r > r_3 \\ &= D_1 H_0^{(1)}(r) \text{ for } r < r_1 \end{aligned} \right\} \tag{42}$$

where $H_0^{(1)}$, $H_0^{(2)}$ are Hankel functions of zero order and of first and second kind. $C, D, C_1, D_1$ are constants. The boundary condition (37) with (42) gives

$$D_1 = \frac{H}{iCr_1} \frac{\{A_1 J\nu_1(\lambda_1 r_1) + B_1 Y\nu_1(\lambda_1 r_1)\}}{H_0^{(1)}(\omega r_1)} \tag{43a}$$

$$C_1 = \frac{H}{iCr_3} \left\{ A_2 J\nu_2(\lambda_2 r_3) + B_2 Y\nu_2(\lambda_2 r_3) \right\} \tag{43b}$$

The boundary condition (38) with the help of (41) and (42) gives

$$D = \frac{Hiq}{r_1^2} \frac{\{A_1 J\nu_1(\lambda_1 r_1) + B_1 Y\nu_1(\lambda_1 r_1)\}}{H_0^{(1)}(\omega r_1)} \tag{44a}$$

and

$$C = \frac{Hiq}{r_3^2} \left\{ A_2 J\nu_2(\lambda_2 r_3) + B_2 Y\nu_2(\lambda_2 r_3) \right\} \Big/ H_0^{(2)}(\omega r_3). \tag{44b}$$

Hence, the perturbed fields are given by

$$E^* = \frac{Hp}{icr_1} \frac{\{A_1 J\nu_1(\lambda_1 r_1) + B_1 Y\nu_1(\lambda_1 r_1)\}}{H_0^{(1)}(\omega r_1)} H_0^{(1)}(\omega r)\, e^{i(qz + pt)} \quad \text{for } r < r_1 \tag{45a}$$

$$E^* = \frac{Hp}{icr_3} \frac{\{A_2 J\nu_2(\lambda_2 r_3) + BY\nu_2(\lambda_2 r_3)\}}{H_0^{(2)}(\omega r_3)} H_0^{(2)}(\omega r)\, e^{i(qz + pt)} \quad \text{for } r > r_3 \tag{45b}$$

$$h^* = \frac{Hiq}{r_1^2} \frac{\{A_1 J\nu_1(\lambda_1 r_1) + B_1 Y\nu_1(\lambda_1 r_1)\}}{H_0^{(1)}(\omega r_1)} H_0^{(1)}(\omega r)\, e^{i(qz + pt)} \quad \text{for } r < r_1 \tag{46a}$$

$$h^* = \frac{Hiq}{r_3^2} \left( \frac{\left\{ A_2 J\nu_2(\lambda_2 r_3) + B_1 Y_{\nu_2}(\lambda_2 r_3) \right\}}{H_0^{(2)}(\omega r_3)} \right) H_0^{(2)}(\omega r) e^{i(qz + pt)} \quad \text{for } r > r_3 \tag{46b}$$

## TORSIONAL VIBRATION OF NON-HOMOGENEOUS AEOLOTROPIC SHELL

Suppose $r = R_1$ and $r = R_2$ be the boundaries of the aeolotropic shell which separates the solid from vacuum. In this case the boundary conditions are $\sigma_{r\theta} = 0$ on $r = R_1$ and $r = R_2$ hence

$$A_1 \left\{ \lambda_1 R_1 J_{\nu_1 - 1}(\lambda_1 R_1) - (\lambda_{1\nu_1} + 2) J_{\nu_1}(\lambda_1 R_1) \right\} + $$
$$+ B_1 \left\{ \lambda_1 R_1 Y_{\nu_1 - 1}(\lambda_1 R_1) - (\lambda_1 \nu_1 + 2) Y\nu_1(\lambda_1 R_1) \right\} = 0 \tag{47}$$

and

$$A_1 \left\{ \lambda_1 R_2 J\nu_{1-1}(\lambda_1 R_2) - (\lambda_1\nu_1+2) J\nu_1(\lambda_1 R_2) \right\} + + B_1 \left\{ \lambda_1 R_2 Y\nu_{1-1}(\lambda_1 R_2) - (\lambda_1\nu_1+2) Y\nu_1(\lambda_1 R_2) \right\} = 0 \tag{48}$$

Eliminating $A_1$, $B_1$ from (48), we get the frequency equation as

$$\left\{ \lambda_1 R_1 J\nu_{1-1}(\lambda_1 R_1) - (\lambda_1\nu_1+2) J\nu_1(\lambda_1 R_1) \right\}\left\{ \lambda_1 R_2 Y\nu_{1-1}(\lambda_1 R_2) + (\lambda_1\nu_1+2) Y\nu_1(\lambda_1 R_2) \right\} - - \left\{ \lambda_1 R_1 Y\nu_{1-1}(\lambda_1 R_1) - (\lambda_1\nu_1+2) Y\nu_1(\lambda_1 R_1) \right\} \times \left\{ \lambda_1 R_2 J\nu_{1-1}(\lambda_1 R_2) - (\lambda_1\nu_1+2) J\nu_1(\lambda_1 R_2) \right\} = 0 \tag{49}$$

The phase velocity $c_2 = \frac{p}{q}$ of the torsional waves are given by

$$c_2 = \left(\frac{\mu_{66}}{\rho_0}\right)^{\frac{1}{2}} \left\{ \xi^2{}_1 \left(\frac{\lambda_1}{2\pi R_1}\right)^2 + \frac{\mu_{55}}{\mu_{66}} \right\}^{\frac{1}{2}} \tag{50}$$

where $\xi_1$ is a root of the equation

$$\left\{ \xi_1 J\nu_{1-1}(\xi_1) - (\lambda_1\nu_1+2) J\nu_1(\xi_1) \right\}\left\{ x\xi_1 Y\nu_{1-1}(x\xi_1) + + (\lambda_1\nu_1+2) Y\nu_1(x\xi_1) \right\} - \left\{ \xi_1 Y\nu_{1-1}(\xi_1) - (\lambda_1\nu_1+2) Y\nu_1(\xi_1) \right\} \cdot \cdot \left\{ x\xi_1 J\nu_1 (x\xi_1) - (\lambda_1\nu_1+2) J\nu_1(x\xi_1) \right\} = 0 \tag{51}$$

where

$$x = \frac{R_2}{R_1} \text{ and } \xi_1 = \lambda_1 R_1 \tag{52}$$

For pure elastic solids $\mu_{55} = \mu_{66} = \mu_0$ and hence

$$c_2 = \sqrt{\frac{\mu_0}{\rho_0}} \left\{ \xi^2{}_1 \left(\frac{\lambda_1}{2\pi R_1}\right)^2 + 1 \right\}^{1/2} \tag{53}$$

## TORSIONAL VIBRATION OF NON-HOMOGENEOUS VISCO-ELASTIC SHELL

Let $r = R'_1$ and $r = R'_2$ be the boundaries of the visco-elastic shell then proceeding exactly similar to the previous case of aeolotropic shell we can find the phase velocity $c_3 = \frac{p}{q}$ of the torsional waves as

$$c_3 = \left\{ \frac{\mu_0(1+m_2 ip)}{\rho_0(1+m_1 ip)} \right\}^{\frac{1}{2}} \left\{ \xi^2{}_2 \left(\frac{\lambda_2}{2\pi R'_1}\right)^2 + 1 \right\}^{\frac{1}{2}} \tag{54}$$

where $\xi_2$ is the root of the equation

$$\left\{ \xi_2 J\nu_{2-1}(\xi_2) - (\lambda_2\nu_2+2) J\nu_2(\xi_2) \right\}\left\{ x'\xi_2 Y\nu_{2-1}(x'\xi_2) + + (\lambda_2\nu_2+2) Y\nu_2 \; x'\xi_2) \right\} - \left\{ \xi_2 Y\nu_{2-1}(\xi_2) - (\lambda_2\nu_2+2) Y\nu_2(\xi_2) \right\} \cdot \cdot \left\{ x'\xi_2 J\nu_{2-1}(x'\xi_2) - (\lambda_2\nu_2+2) J\nu_2(x'\xi_2) \right\} = 0 \tag{55}$$

where
$$x' = \frac{R'_2}{R'_1} \text{ and } \xi_2 = \lambda_2 R'_1 \tag{56}$$

For Kelvin Voigt soldi, we have $m_1 = 0$ and hence

$$c_3 = \left\{ \frac{\mu_0 (1 + m_2 ip)}{\rho_0} \right\}^{\frac{1}{2}} \left\{ \xi^2_2 \left( \frac{\lambda_2}{2\pi R'_1} \right)^2 + 1 \right\}^{\frac{1}{2}} \tag{57}$$

For pure elastic solids we have $m_2 = 0$ and hence

$$c_3 = \left( \frac{\mu_0}{\rho_0} \right)^{\frac{1}{2}} \left\{ \xi^2_2 \left( \frac{\lambda_2}{2\pi R'_1} \right)^2 + 1 \right\}^{\frac{1}{2}} . \tag{58}$$

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