RECURRENCE—RELATIONS

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The main aim of this paper is to obtain some recurrence relations for the generalised H-function of two variables.

In the present paper we evaluate an integral involving the product of generalised H-function[1] of two variables and hypergeometric-function. This integral has been employed to obtain some interesting recurrence relations for the $H\,[x, y]$-function. The generalised $H\,[x, y]$-function is the generalisation of $G\,(x, y)$-function Agarwal[9], $S\,(x, y)$-function Sharma[4], Kampe De Feriet-function[10], Appell-functions (F_1, F_2, F_3, F_4), Whittaker-function and other several special-functions of two variables.

By proper choice of parameters due to general nature of $H\,[x, y]$-function, it can be reduced to Fox's H-function[5], H-function can be reduced to Whittakers function $M_{k,m}(x)$, generalised Laguerre polynomial $L_n^{\alpha}(x)$ Hermite polynomial $H_n(x)$ and regular and irregular Coulomb wave function F_L and G_L etc.; thereby providing us with such results as may be used in various problems such as boundary value problems, encountered in quantum mechanics *viz.*, collision problems of two particles with Coulomb interaction, problems.

Harmonics oscillator and the Hydrogen atom[2], which may in turn find application in certain defence

In a recent paper Munot & Kalla[1] have defined generalised H-function of two variables as follow :

$$H\left[\begin{array}{c|c|c} \left[\begin{matrix} m_1, 0 \\ p_1 - m_1, q_1 \end{matrix}\right] & \{(a_{p_1}, A_{p_1})\} ; \{(b_{q_1}, B_{q_1})\} & x \\ \left(\begin{matrix} m_2, n_2 \\ p_2 - m_2, q_2 - n_2 \end{matrix}\right) & \{(c_{p_2}, \alpha_{p_2})\} ; \{(d_{q_2}, D_{q_2})\} & \\ \left(\begin{matrix} m_3, n_3 \\ p_3 - m_3, q_3 - n_3 \end{matrix}\right) & \{(e_{p_3}, E_{p_3})\} ; \{(f_{q_3}, F_{q_3})\} & y \end{array}\right].$$

$$= \frac{1}{(2\pi i)^2} \int_{L_1} \int_{L_2} F(\xi + \eta)\, \phi_1(\xi)\, \phi_2(\eta)\, x^{\xi} y^{\eta}\, d\xi\, d\eta . \tag{1}$$

$\{(a_{p_1}, A_{p_1})\}$ stands for the set of parameters $(a_1, A_1), \ldots, (a_{p_1}, A_{p_1})$, L_1 and L_2 are suitable contour's and

$$F(\xi + \eta) = \frac{\prod_{j=1}^{m_1} \Gamma(a_j + A_j \xi + A_j \eta)}{\prod_{j=m_1+1}^{p_1} \Gamma(1 - a_j - A_j \xi - A_j \eta) \prod_{j=1}^{q_1} \Gamma(b_j + B_j \xi + B_j \eta)},$$

$$\phi_1(\xi) = \frac{\prod_{j=1}^{m_2} \Gamma(1 - c_j + \alpha_j \xi) \prod_{j=1}^{n_2} \Gamma(d_j - D_j \xi)}{\prod_{j=m_2+1}^{p_2} \Gamma(c_j - \alpha_j \xi) \prod_{j=n_2+1}^{q_2} \Gamma(1 - d_j + D_j \xi)},$$

$$\phi_2(\eta) = \frac{\prod_{j=1}^{m_3} \Gamma(1 - e_j + E_j \eta) \prod_{j=1}^{n_3} \Gamma(f_j - F_j \eta)}{\prod_{j=m_3+1}^{p_3} \Gamma(e_j - E_j \eta) \prod_{j=n_3+1}^{q_3} \Gamma(1 - f_j + F_j \eta)},$$

If $p_1 \geq m_1 \geq 0$, $p_2 \geq m_2 \geq 0$, $p_3 \geq m_3 \geq 0$, $q_1 \geq 0$, $q_2 \geq n_2 \geq 0$, $q_3 \geq n_3 \geq 0$, p, m and n are all positive integers. We shall write the L.H.S. of (1) more briefly as $H\,[x, y]$.

Let us define the associated function $(m_1 = 0)$

$$H_1[x, y] = H \left[\begin{array}{c|c|c} \left[\begin{array}{c} 0, 0 \\ p_1, q_1 \end{array}\right] & \{(a_{p_1}, A_{p_1})\} ; \{(b_{q_1}, B_{q_1})\} & \\ \left(\begin{array}{c} m_2, n_2 \\ p_2 - m_2, q_2 - n_2 \end{array}\right) & \{(c_{p_2}, \alpha_{p_2})\} ; \{(d_{q_2}, D_{q_2})\} & x \\ \left(\begin{array}{c} m_3, n_3 \\ p_3 - m_3, q_3 - n_3 \end{array}\right) & \{(e_{p_3}, E_{p_3})\} ; \{(f_{q_3}, F_{q_3})\} & y \end{array} \right], \quad (2)$$

(2) converges under the following conditions:

$$\theta_2 \geqslant 0, \ \phi_2 > 0, \ |arg\, x| < \tfrac{1}{2}\phi_2 \pi, \ \theta_3 \geqslant 0, \ \phi_3 > 0, \ |arg\, y| < \tfrac{1}{2}\phi_3 \pi$$

where

$$\theta_2 \equiv \sum_1^{q_1} B_j + \sum_1^{q_2} D_j - \sum_1^{p_1} A_j - \sum_1^{p_2} \alpha_j, \quad \theta_3 \equiv \sum_1^{q_1} B_j + \sum_1^{q_3} F_j - \sum_1^{p_1} A_j - \sum_1^{p_3} E_j,$$

$$\phi_2 \equiv - \sum_1^{p_1} A_j - \sum_1^{q} B_j + \sum_1^{m_2} \alpha_j - \sum_{m_2+1}^{p_2} \alpha_j + \sum_1^{n_2} D_j - \sum_{n_2+1}^{q_2} D_j,$$

$$\phi_3 \equiv - \sum_1^{p_1} A_j - \sum_1^{q_1} B_j + \sum_1^{m_3} E_j - \sum_{m_3+1}^{p_3} E_j + \sum_1^{n_3} F_j - \sum_{n_3+1}^{q_3} F_j,$$

also

$H_1[x, y] = 0(|x|^{\lambda_2}, |y|^{\lambda_3})$ for small values of x and y, and $H_1[x, y] = 0(|x|^{\mu_2}, |y|^{\mu_3})$ for large values of x and y conditions stated with (2)

where

$$\lambda_2 \equiv min\, R\left(\frac{d_j}{D_j}\right)(j = 1, \ldots, n_2), \ \lambda_3 \equiv min\, R\left(\frac{f_j}{F_j}\right)(j = 1, \ldots, n_3),$$

$$\mu_2 \equiv max\, R\left(\frac{c_j - 1}{\alpha_j}\right)(j = 1, \ldots, m_2), \ \mu_3 \equiv max\, R\left(\frac{e_j - 1}{E_j}\right)(j = 1, \ldots, m_3).$$

for convenience the R.H.S. of (2) can be written as

$$H \left[\begin{array}{c|c|c} \cdots & \cdots & \\ \left(\begin{array}{c} m_2, n_2 \\ p_2 - m_2, q_2 - n_2 \end{array}\right) & \{(c_{p_2}, \alpha_{p_2})\} ; \{(d_{q_2}, D_{q_2})\} & x, y \\ \cdots & \cdots & \end{array} \right].$$

The following results will be required in the sequel (See Erdelyi[3]):

$$\int_0^\infty x^{s-1}\, {}_2F_1(\alpha, \beta; Y; -x)\, dx = \frac{\Gamma(Y)\,\Gamma(s)\,\Gamma(\alpha - s)\,\Gamma(\beta - s)}{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(Y - s)},$$

$$0 < R(s) < min\, R(\alpha, \beta) \quad (3)$$

With the help of the definitions of $S[x, y]$-function[4] Fox's $H[x]$-function[5] and Meijer's $G(x)$-function[6], the following equalities follows from (1):

$$H \left[\begin{array}{c|c|c} \left[\begin{array}{c} m_1, 0 \\ p_1 - m_1, q_1 \end{array}\right] & \{(a_{p_1}, 1)\} ; \{(b_{q_1}, 1)\} & \\ \left(\begin{array}{c} m_2, n_2 \\ p_2 - m_2, q_2 - n_2 \end{array}\right) & \{(c_{p_2}, 1)\} ; \{(d_{q_2}, 1)\} & x \\ \left(\begin{array}{c} m_3, n_3 \\ p_3 - m_3, q_3 - n_3 \end{array}\right) & \{(e_{p_3}, 1)\} ; \{(f_{q_3}, 1)\} & y \end{array} \right]$$

$$= S \left[\begin{array}{c|c|c} \begin{bmatrix} m_1, 0 \\ p_1 - m_1, q_1 \end{bmatrix} & \{ (a_{p_1}) \} ; \{ (b_{q_1}) \} & \\ \begin{pmatrix} m_2, n_2 \\ p_2 - m_2, q_2 - n_2 \end{pmatrix} & \{ (c_{p_2}) \} ; \{ (d_{q_2}) \} & x \\ \begin{pmatrix} m_3, n_3 \\ p_3 - m_3, q_3 - n_3 \end{pmatrix} & \{ (e_{p_3}) \} ; \{ (f_{q_3}) \} & y \end{array} \right], \tag{4}$$

$$H \left[\begin{array}{c|c|c} \begin{bmatrix} 0, 0 \\ 0, 0 \end{bmatrix} & \text{----} ; & x \\ \begin{pmatrix} m_2, n_2 \\ p_2 - m_2, q_2 - n_2 \end{pmatrix} & \{ (c_{p_2}, \alpha_{p_2}) \} ; \{ (d_{q_2}, D_{q_2}) \} & \\ \begin{pmatrix} m_3, n_3 \\ p_3 - m_3, q_3 - n_3 \end{pmatrix} & \{ (e_{p_3}, E_{p_3}) \} ; \{ (f_{q_3}, F_{q_3}) \} & y \end{array} \right]$$

$$= H^{n_2, m_2}_{p_2, q_2} \left[x \,\middle|\, \begin{array}{c} \{ (c_{p_2}, \alpha_{p_2}) \} \\ \{ (d_{q_2}, D_{q_2}) \} \end{array} \right] H^{n_3, m_3}_{p_3, q_3} \left[y \,\middle|\, \begin{array}{c} \{ (e_{p_3}, E_{p_3}) \} \\ \{ (f_{q_3}, F_{q_3}) \} \end{array} \right], \tag{5}$$

$$H \left[\begin{array}{c|c|c} \begin{bmatrix} 0, 0 \\ 0, 0 \end{bmatrix} & \text{----} ; & x \\ \begin{pmatrix} m_2, n_2 \\ p_2 - m_2, q_2 - n_2 \end{pmatrix} & \{ (c_{p_2}, 1) \} ; \{ (d_{q_2}, 1) \} & \\ \begin{pmatrix} m_3, n_3 \\ p_3 - m_3, q_3 - n_3 \end{pmatrix} & \{ (e_{p_3}, 1) \} ; \{ (f_{q_3}, 1) \} & y \end{array} \right]$$

$$= G^{n_2, m_2}_{p_2, q_2} \left(x \,\middle/\, \begin{array}{c} \{ (c_{p_2}) \} \\ \{ (d_{q_2}) \} \end{array} \right) G^{n_3, m_3}_{p_3, q_3} \left(y \,\middle/\, \begin{array}{c} \{ (e_{p_3}) \} \\ \{ (f_{q_3}) \} \end{array} \right). \tag{6}$$

THE INTEGRAL

The integral to be evaluated is

$$\int_0^\infty x^\lambda \, {}_2F_1 \left(\begin{array}{c} \beta_1, \beta_2 ; \\ Y \quad ; \end{array} - sx \right) H_1 \left[\begin{array}{c} \sigma \\ U x, V \end{array} \right] dx = s^{-\lambda - 1} \frac{\Gamma(Y)}{\Gamma(\beta_1) \Gamma(\beta_2)} \,.$$

$$\cdot H \left[\begin{array}{c|c|c} \begin{bmatrix} 0, 0 \\ p_1, q_1 \end{bmatrix} & \{ (a_{p_1}, A_{p_1}) \} ; \{ (b_{q_1}, B_{q_1}) \} & U/s^\sigma \\ \begin{pmatrix} 1 + m_2, 2 + n_2 \\ 1 + p_2 - m_2, q_2 - n_2 \end{pmatrix} & \begin{array}{c} (-\lambda, \sigma), \{ (c_{p_2}, \alpha_{p_2}) \}, (Y - \lambda - 1, \sigma) ; \\ (\beta_1 - \lambda - 1, \sigma), (\beta_2 - \lambda - 1, \sigma), \{ (d_{q_2}, D_{q_2}) \} \end{array} & \\ \begin{pmatrix} m_3, n_3 \\ p_3 - m_3, q_3 - n_3 \end{pmatrix} & \{ (e_{p_3}, E_{p_3}) \} ; \{ (f_{q_3}, F_{q_3}) \} & V \end{array} \right] \tag{7}$$

valid for $\sigma > 0$, $\theta_2 > 0$, $\phi_2 > 0$, $| \arg U | < \frac{1}{2} \phi_2 \pi$, $\theta_3 > 0$, $\phi_3 > 0$, $| \arg V | < \frac{1}{2} \phi_3 \pi$, $| \arg s | < \pi$, $R [\lambda + 1 + \sigma \lambda_2] > 0$ and $R [\lambda + 1 + \sigma \mu_2 - \beta_i] < 0$ $(i = 1, 2)$.

Proof

On substituting the value of the generalised H-function of two variables from (2) in the integrand of (7) and inverting the order of integration which is justified under the conditions stated above due to the absolute convergence of the integrals involved in the process, we get

$$\frac{1}{(2\pi i)^2} \int_{L_1} \int_{L_2} \frac{\phi_1(\xi) \phi_2(\eta) U^\xi V^\eta}{\prod_{j=1}^{p_1} \Gamma(1 - a_j - A_j \xi - A_j \eta) \prod_{j=1}^{q_1} \Gamma(b_j + B_j \xi + B_j \eta)} \,.$$

$$\cdot \left[\int_0^\infty x^{\lambda + \sigma \xi} \, {}_2F_1 \left(\begin{array}{c} \beta_1 \, \beta_2 ; \\ Y \quad ; \end{array} - sx \right) dx \right] d\xi \, d\eta.$$

putting $sx = x$ and evaluating the inner integral with the help of the result (3), (7) is obtained on interpreting the result thus obtained with the help of (2).

RECURRENCE RELATIONS

(i)

$$(d_1-d_2)\,H\left[\begin{pmatrix} m_2, n_2 \\ p_2-m_2, q_2-n_2 \end{pmatrix}\middle| \begin{matrix} (c_1-1,\alpha_1), (c_2,\alpha_2), \ldots, (c_{p_2},\alpha_{p_2}); \\ (d_1-1,\alpha_1), (d_2-1,\alpha_1), (d_3,D_3),\ldots,(d_{q_2},D_{q_2}) \end{matrix} \middle| x, y\right]$$

$$= (c_1-d_1)\,H\left[\begin{pmatrix} m_2, n_2 \\ p_2-m_2; q_2-n_2 \end{pmatrix}\middle| \begin{matrix} (c_1,\alpha_1), (c_2,\alpha_2), \ldots, (c_{p_2},\alpha_{p_2}); \\ (d_1-1,\alpha_1), \{(d_2,\alpha_1)\}, (d_3,D_3),\ldots,(d_{q_2},D_{q_2}) \end{matrix} \middle| x, y\right] -$$

$$-(c_1-d_2)\,H\left[\begin{pmatrix} m_2, n_2 \\ p_2-m_2, q_2-n_2 \end{pmatrix}\middle| \begin{matrix} (c_1,\alpha_1), (c_2,\alpha_2), \ldots, (c_{p_2},\alpha_{p_2}); \\ (d_1,\alpha_1), (d_2-1,\alpha_1), (d_3,D_3),\ldots,(d_{q_2},D_{q_2}) \end{matrix} \middle| x, y\right]$$

where $m_2 \geqslant 1$ and $n_2 \geqslant 2$. (8)

(ii)

$$(1+d_1+d_2-c_1-c_{p_2})\,H\left[\begin{pmatrix} m_2, n_2 \\ p_2-m_2, q_2-n_2 \end{pmatrix}\middle| \begin{matrix} (c_1,\alpha_1), (c_2,\alpha_2), \ldots, \{(c_{p_2-1},\alpha_{p_2-1})\}\,(c_{p_2},\alpha_1); \\ (d_1,\alpha_1), (d_2,\alpha_1), (d_3,D_3),\ldots,(d_{q_2},D_{q_2}) \end{matrix} \middle| x, y\right]$$

$$= H\left[\begin{pmatrix} m_2, n_2 \\ p_2-m_2, q_2-n_2 \end{pmatrix}\middle| \begin{matrix} (c_1,\alpha_1), (c_2,\alpha_2), \ldots, \{(c_{p_2-1},\alpha_{p_2-1})\}, (c_{p_2},\alpha_1); \\ (d_1+1,\alpha_1), (d_2,\alpha_1), (d_3,D_3),\ldots,(d_{q_2},D_{q_2}) \end{matrix} \middle| x, y\right] + (d_2-c_{p_2})\cdot$$

$$\cdot\,(d_2-c_1)\,H\left[\begin{pmatrix} m_2, n_2 \\ p_2-m_2, q_2-n_2 \end{pmatrix}\middle| \begin{matrix} (c_1,\alpha_1), (c_2,\alpha_2), \ldots, \{(c_{p_2-1},\alpha_{p_2-1})\}, (c_{p_2},\alpha_1); \\ (d_1,\alpha_1), (d_2-1,\alpha_1), (d_3,D_3),\ldots,(d_{q_2},D_{q_2}) \end{matrix} \middle| x, y\right] +$$

$$+ H\left[\begin{pmatrix} m_2, n_2 \\ p_2-m_2, q_2-n_2 \end{pmatrix}\middle| \begin{matrix} (c_1-1,\alpha_1), (c_2,\alpha_2), \ldots, \{(c_{p_2-1},\alpha_{p_2-1})\}, (c_{p_2}-1\,\alpha_1); \\ (d_1,\alpha_1), (d_2-1,\alpha_1), (d_3,D_3),\ldots,(d_{q_2},D_{q_2}) \end{matrix} \middle| x, y\right]$$

where $n_2 \geqslant 2$ and $1 \leqslant m_2 \leqslant p_2-1$. (9)

(iii)

$$(c_{p_2}-c_1)\,H\left[\begin{pmatrix} m_2, n_2 \\ p_2-m_2, q_2-n_2 \end{pmatrix}\middle| \begin{matrix} (c_1,\alpha_1), (c_2,\alpha_2), \ldots, \{(c_{p_2-1},\alpha_{p_2-1})\}, (c_{p_2},\alpha_1); \\ (d_1,\alpha_1), (d_2,\alpha_1), (d_3,D_3),\ldots,(d_{q_2},D_{q_2}) \end{matrix} \middle| x, y\right]$$

$$= H\left[\begin{pmatrix} m_2, n_2 \\ p_2-m_2, q_2-n_2 \end{pmatrix}\middle| \begin{matrix} (c_1,\alpha_1), (c_2,\alpha_2), \ldots, \{(c_{p_2-1},\alpha_{p_2-1})\}, (c_{p_2}-1,\alpha_1); \\ (d_1,\alpha_1), (d_2,\alpha_1), (d_3,D_3),\ldots,(d_{q_2},D_{q_2}) \end{matrix} \middle| x, y\right] +$$

$$+(c_{p_2}-d_1)(c_{p_2}-d_2)\; H\left[\begin{pmatrix} \cdots\; m_2, n_2 \\ p_2-m_2, q_2-n_2\; \cdots \end{pmatrix} \left| \begin{matrix} (c_1-1, \alpha_1), (c_2, \alpha_2), \ldots, \{(c_{p_2-1}, \alpha_{p_2-1})\}, (c_{p_2}, \alpha_1); \\ (d_1-1, \alpha_1), (d_2-1, \alpha_1), (d_3, D_3), \ldots, (d_{q_2}, D_{q_2}) \end{matrix} \right| x, y \right] +$$

$$+(1+d_1+d_2-2c_{p_2})\; H\left[\begin{pmatrix} \cdots\; m_2, n_2 \\ p_2-m_2, q_2-n_2\; \cdots \end{pmatrix} \left| \begin{matrix} (c_1-1, \alpha_1), (c_2, \alpha_2), \ldots, \{(c_{p_2-1}, \alpha_{p_2-1})\}, (c_{p_2}-1, \alpha_1); \\ (d_1-1, \alpha_1), (d_2-1, \alpha_1), (d_3, D_3), \ldots, (d_{q_2}, D_{q_2}) \end{matrix} \right| x, y \right] +$$

$$+H\left[\begin{pmatrix} \cdots\; m_2, n_2 \\ p_2-m_2, q_2-n_2\; \cdots \end{pmatrix} \left| \begin{matrix} (c_1-1, \alpha_1), (c_2, \alpha_2), \ldots, \{(c_{p_2-1}, \alpha_{p_2-1})\}, (c_{p_2}-2, \alpha_1); \\ (d_1-1, \alpha_1), (d_2-1, \alpha_1), (d_3, D_3), \ldots, (d_{q_2}, D_{q_2}) \end{matrix} \right| x, y \right],$$

where $n_2 \geqslant 2$ and $1 \leqslant m_2 \leqslant p_2 - 1$. **(10)**

(iv)

$$(d_1-c_1+1)\; H\left[\begin{pmatrix} \cdots\; m_2, n_2 \\ p_2-m_2, q_2-n_2\; \cdots \end{pmatrix} \left| \begin{matrix} (c_1, \alpha_1), (c_2, \alpha_2), \ldots, \{(c_{p_2-1}, \alpha_{p_2-1})\}, (c_{p_2}, \alpha_1); \\ (d_1, \alpha_1), (d_2, \alpha_1), (d_3, D_3), \ldots, (d_{q_2}, D_{q_2}) \end{matrix} \right| x, y \right]$$

$$= H\left[\begin{pmatrix} \cdots\; m_2, n_2 \\ p_2-m_2, q_2-n_2\; \cdots \end{pmatrix} \left| \begin{matrix} (c_1, \alpha_1), (c_2, \alpha_2), \ldots, \{(c_{p_2-1}, \alpha_{p_2-1})\}, (c_{p_2}, \alpha_1); \\ (d_1, \alpha_1), (d_2, \alpha_1), (d_3, D_3), \ldots, (d_{q_2}, D_{q_2}) \end{matrix} \right| x, y \right] +$$

$$+H\left[\begin{pmatrix} \cdots\; m_2, n_2 \\ p_2-m_2, q_2-n_2\; \cdots \end{pmatrix} \left| \begin{matrix} (c_1-1, \alpha_1), (c_2, \alpha_2), \ldots, \{(c_{p_2-1}, \alpha_{p_2-1})\}, (c_{p_2}-1, \alpha_1); \\ (d_1, \alpha_1), (d_2-1, \alpha_1), (d_3, D_3), \ldots, (d_{q_2}, D_{q_2}) \end{matrix} \right| x, y \right] +$$

$$+(c_{p_2}-d_1)(c_{p_2}-d_2)\; H\left[\begin{pmatrix} \cdots\; m_2, n_2 \\ p_2-m_2, q_2-n_2\; \cdots \end{pmatrix} \left| \begin{matrix} (c_1-1, \alpha_1), (c_2, \alpha_2), \ldots, \{(c_{p_2-1}, \alpha_{p_2-1})\}, (c_{p_2}, \alpha_1); \\ (d_1-1, \alpha_1), (d_2-1, \alpha_1), (d_3, D_3), \ldots, (d_{q_2}, D_{q_2}) \end{matrix} \right| x, y \right] +$$

$$+(d_2-c_{p_2})\; H\left[\begin{pmatrix} \cdots\; m_2, n_2 \\ p_2-m_2, q_2-n_2\; \cdots \end{pmatrix} \left| \begin{matrix} (c_1-1, \alpha_1), (c_2, \alpha_2), \ldots, \{(c_{p_2-1}, \alpha_{p_2-1})\}, (c_{p_2}-1, \alpha_1); \\ (d_1-1, \alpha_1), (d_2-1, \alpha_1), (d_3, D_3), \ldots, (d_{q_2}, D_{q_2}) \end{matrix} \right| x, y \right].$$

where $n_2 \geqslant 2$ and $1 \leqslant m_2 \leqslant p_2 - 1$. **(11)**

(v)

$$(d_1-c_1)\; H\left[\begin{pmatrix} \cdots\; m_2, n_2 \\ p_2-m_2, q_2-n_2\; \cdots \end{pmatrix} \left| \begin{matrix} (c_1, \alpha_1), (c_2, \alpha_2), \ldots, \{(c_{p_2-1}, \alpha_{p_2-1})\}, (c_{p_2}, \alpha_1); \\ (d_1, \alpha_1), (d_2, \alpha_1), (d_3, D_3), \ldots, (d_{q_2}, D_{q_2}) \end{matrix} \right| x, y \right]$$

$$= H\left[\begin{pmatrix} \cdots\; m_2, n_2 \\ p_2-m_2, q_2-n_2\; \cdots \end{pmatrix} \left| \begin{matrix} (c_1, \alpha_1), (c_2, \alpha_2), \ldots, \{(c_{p_2-1}, \alpha_{p_2-1})\}, (c_{p_2}-1, \alpha_1); \\ (d_1, \alpha_1), (d_2, \alpha_1), (d_3, D_3), \ldots, (d_{q_2}, D_{q_2}) \end{matrix} \right| x, y \right] +$$

$$+(d_1-c_{p_2})(d_1-c_1)\; H\left[\begin{pmatrix} \cdots\; m_2, n_2 \\ p_2-m_2, q_2-n_2\; \cdots \end{pmatrix} \left| \begin{matrix} (c_1, \alpha_1), (c_2, \alpha_2), \ldots, \{(c_{p_2-1}, \alpha_{p_2-1})\}, (c_{p_2}, \alpha_1); \\ (d_1-1, \alpha_1), (d_2, \alpha_1), (d_3, D_3), \ldots, (d_{q_2}, D_{q_2}) \end{matrix} \right| x, y \right] +$$

$$+(1+d_2-c_{p_2})\; H\left[\begin{pmatrix} \cdots\; m_2, n_2 \\ p_2-m_2, q_2-n_2\; \cdots \end{pmatrix} \left| \begin{matrix} (c_1-1, \alpha_1), (c_2, \alpha_2), \ldots, \{(c_{p_2-1}, \alpha_{p_2-1})\}, (c_{p_2}-1, \alpha_1); \\ (d_1-1, \alpha_1), (d_2-1, \alpha_1), (d_3, D_3), \ldots, (d_{q_2}, D_{q_2}) \end{matrix} \right| x, y \right] +$$

$$+H\left[\begin{pmatrix} \cdots\; m_2, n_2 \\ p_2-m_2, q_2-n_2\; \cdots \end{pmatrix} \left| \begin{matrix} (c_1-1, \alpha_1), (c_2, \alpha_2), \ldots, \{(c_{p_2-1}, \alpha_{p_2-1})\}, (c_{p_2}-2, \alpha_1); \\ (d_1-1, \alpha_1), (d_2-1, \alpha_1), (d_3, D_3), \ldots, (d_{q_2}, D_{q_2}) \end{matrix} \right| x, y \right].$$

where $n_2 \geqslant 2$ and $1\;\; m_2 \leqslant p_2 - 1$, **(12)** *Proof*

Since Rainville[7]

$$ {}_2F_1\left(\begin{matrix}\beta_1-1,\ \beta_2;\\ Y\ ;\end{matrix} -sx\right) = {}_2F_1\left(\begin{matrix}\beta_1,\ \beta_2-1\ ;\\ Y\ ;\end{matrix} -sx\right) + \frac{(\beta_2-\beta_1)}{Y}\, s\,x\, {}_2F_1\left(\begin{matrix}\beta_1,\ \beta_2;\\ Y+1\ ;\end{matrix} -sx\right), \quad (13) $$

we have

$$ \frac{(\beta_1-1)\,\Gamma(\beta_1-1)\,\Gamma(\beta_2)}{\Gamma(Y)} \int_0^\infty x^\lambda f(x)\, {}_2F_1\left(\begin{matrix}\beta_1-1,\ \beta_2;\\ Y\ ;\end{matrix} -sx\right) dx = \frac{(\beta_2-1)\,\Gamma(\beta_1)\,\Gamma(\beta_2-1)}{\Gamma(Y)} $$

$$ \int_0^\infty x^\lambda f(x)\, {}_2F_1\left(\begin{matrix}\beta_1,\ \beta_2-1;\\ Y;\end{matrix} -sx\right) dx + \frac{(\beta_2-\beta_1)\,\Gamma(\beta_1)\,\Gamma(\beta_2)}{\Gamma(Y+1)}\, s \int_0^\infty x^{\lambda+1} f(x), $$

$$.\ {}_2F_1\left(\begin{matrix}\beta_1,\ \beta_2;\\ Y+1\ ;\end{matrix} -sx\right) dx \quad (14) $$

provided that the integrals involved exist.

Now if we take

$$ f(x) = H_1\left[\, U x^\sigma,\ V \right] $$

in (14) and evaluate the integrals involved therein with the help of (7), we get the recurrence relation (8) after slight changes in the parameters and arguments.

The remaining recurrence relations can be proved in a similar manner, if we start with Erdelyi[3] respectively instead of relation (13).

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If we take all the A's, B's, α's, D's, E's and F's equal to unity in the recurrence relations (8) to (12), we get, by virtue of (4), corresponding recurrence relations for the S-function[4].

If we put $p_1 = q_1 = 0$ and use (5) in (8) to (12), we get the recurrence relations recently obtained by Jain[8].

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