MAGNETOHYDRODYNAMIC COUETTE FLOW AND HEAT TRANSFER IN A ROTATING SYSTEM

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The influence of electrode geometry for the MHD Couette flow in the presence of the Hall current and heat transfer of a conducting liquid in a rotating system has been studied.

The flow between two parallel plates with one plate moving (Couette flow) is of general interest. Pai[1], Lehnert[2] and Bleviss[3] have considered the magnetohydrodynamic Couette flow and heat transfer. Sherman and Sutton[4] extended this problem to include the influence of the loading factor. Jana, Datta and Mazumder[5] have studied the hydromagnetic Couette flow in a rotating frame of reference. However, in an ionized gas, when the strength of the magnetic field is very strong, one cannot neglect the effects of Hall current.

The object of the present study is to extend the work of Jana, Datta and Mazumder[5] by including both the effects of Hall current as in the works of Sato[6], Yaminishi[7] and Sherman and Sutton[4] and the loading factor as in Sherman and Sutton[4], but the discussion will be limited to the influence of the loading factor K only.

In general, the open or short circuited methods depending on $K = 1$ or 0 are used to analyse generators, pumps and metres in flow problems and form the groundwork for the MHD generator analysis.

BASIC EQUATIONS AND ITS SOLUTIONS

Consider the steady Couette flow of an electrically conducting fluid between two infinite parallel plates when the fluid and the plates rotate with an angular velocity Ω in unison about an axis normal to the plates. A uniform magnetic flux B_0 acts normal to the plates parallel to the z-axis i.e., it is assumed to be generated by either an air-core solenoid or an electromagnet with a saturated iron core. The xy-plane coincides with the stationary plate and the plate $z = d$ moves with a uniform velocity U_0 in the x-direction. Assuming the flow quantities are independent of x and y, the equations of motion and energy are given by

$$-2\Omega v^* = \nu \frac{d^2 u^*}{dz^2} + \frac{J_y^* B_0}{\rho} \tag{1}$$

$$2\Omega u^* = \nu \frac{d^2 v^*}{dz^2} - \frac{J_x^* B_0}{\rho} \tag{2}$$

$$0 = \kappa \frac{d^2 T}{dz^2} + \frac{\mu}{\rho c_p} \left[\left(\frac{du^*}{dz} \right)^2 + \left(\frac{dv^*}{dz} \right)^2 + \frac{J_x^{*2} + J_y^{*2}}{\rho \sigma c_p} \right. \tag{3}$$

where u^*, v, J_x^{*2}, J_y^{*2} are the x, y components of the velocity of the fluid $\vec{v}$ and the current density $\vec{J}$ respectively, ν the kinematic co-efficient of viscosity, ρ the fluid density, κ the thermal diffusivity, μ the viscosity, σ the electrical conductivity of the fluid and c_p the specific heat at constant pressure.

The current density components follow from the modified Ohm's law (See Cowling[8]), ignoring the electron pressure and the ion slip,

$$\vec{J} = \sigma \left(\vec{E} + v \times \vec{B} - \frac{1}{ne} \vec{J} \times \vec{B} \right) \tag{4}$$

where n is the number density of electrons, $-e$ the electron charge, $\vec{B}$ (B^*_x, B^*_y, B_0) the magnetic induction vector and $\vec{E}$ $\{E_x^* = C_1, \ E_y^* = C_2, \ E_z^* (z)\}$ the electric field relative to the rotating frame.

It follows from equation (4),

$$J_x^* = \frac{\sigma}{1+m^2}\left[E_x^* + v^* B_0 - m\,(E_y^* - u^* B_0)\right] \tag{5}$$

$$J_y^* = \frac{\sigma}{1+m^2}\left[E_y^* - u^* B_0 + m\,(E_x^* + v^* B_0)\right] \tag{6}$$

where

$$m = \frac{\sigma B_0}{ne} \tag{7}$$

Equations (1) and (2) with the values of J_x^* and J_y^* substituted from equations (5) and (6) assume the following forms :

$$-EV = \frac{d^2U}{d\eta^2} + \frac{M^2}{1+m^2}\left[E_y - U + m\,(E_x + V)\right] \tag{8}$$

$$EU = \frac{d^2V}{d\eta^2} + \frac{M^2}{1+m^2}\left[-E_x - V + m\,(E_y - U)\right] \tag{9}$$

in terms of the non-dimensional variables defined by

$$\left.\begin{aligned}
&\eta = z/d,\ (U, V, 0) = (u^*, v^*, 0)/U_0\\
&\{E_x, E_y, E_z(\eta)\} = (E_x^*, E_y^*, E_z^*)/U_0 B_0\\
&M^2 \text{ (Hartmann number)} = B_0^2\, d^2\,(\sigma/\rho\nu)\\
&E \text{ (Ekman number)} = 2\Omega d^2/\nu\\
&(b_x, b_y, 1) = (B_x^*, B_y^*, B_0)/B_0\\
&(j_x, j_y, 0) = (J_x^*, J_y^*, 0)/\sigma U_0 B_0\\
&R_m \text{ (magnetic Reynolds number)} = \sigma\,\mu_e\,U_0\,d
\end{aligned}\right\} \tag{10}$$

where μ_e is the magnetic permeability.

Equations (7) and (8) are to be solved subject to the boundary conditions :

$$U = V = 0 \text{ at } \eta = 0 \text{ and } U = 1,\ V = 0 \text{ at } \eta = 1 \tag{11}$$

For simplicity, it is assumed $E_x = 0$ and $E_y = K$ (constant), the loading factor. Physically there is a different potential over each electrode and the electrodes are infinitely far apart so that there are no variations in the x-direction and that there is a constant potential over each electrode, supposed to be made of materials of infinite electrical conductivity and the electrodes are infinitely far apart so that there are no variations in the y-direction.

It should be noted that $E_z(\eta)$ can be calculated after the solutions for flow velocity and induced magnetic field are obtained. Further, it must be noted that when E_z is a function of η, this implies that the lines of constant potential in the plane of the electrodes are curved. Accordingly, the finely segmented electrodes mentioned earlier must be considered curved, the curvature being calculable once the solution has been derived.

Under these assumptions and in terms of $q = U + iV$, equations (8-9) can be written as

$$\frac{d^2 q}{d\eta^2} - \left\{\alpha_0 + i\,(E + \beta_0)\right\} q = -\alpha_0\, K\,(1 + m) \tag{12}$$

The boundary conditions of equation (11) become

$$q = 0, \text{ at } \eta = 0, \text{ and } q = 1, \text{ at } \eta = 1 \tag{13}$$

where

$$\alpha_0 = M^2/1 + m^2,\quad \beta_0 = M^2\, m/1 + m^2 \tag{14}$$

Solving equation (12), subject to equation (13), we get

$$q = \frac{\sinh(h_1\eta)}{\sinh(h_1)} - \frac{M^2 K}{(1+m^2)\,h_1^2}\,(1 + im)\left\{\frac{\sinh(h_1\eta) + \sinh\{h_1(1-\eta)\}}{\sinh(h_1)} - 1\right\} \tag{15}$$

where

$$h_1 = \alpha + i\beta$$
$$\alpha = \left[\frac{\sqrt{\alpha_0^2 + (E + \beta_0)^2} + \alpha_0}{2} \right]^{1/2}$$
$$\beta = \left[\frac{\sqrt{\alpha_0^2 + (E + \beta_0)^2} - \alpha_0}{2} \right]^{1/2} \tag{16}$$

RESULTS AND DISCUSSION

Since we are primarily interested to bring the influence of the loading factor K, we have taken the other parameters as fixed. Moreover, their influence on the flow problems has already been reported in literature by Jana, Datta and Mazumder[5] and Sato[6] etc. Throughout the discussion we have taken $M^2 = 7$, $m = 1$, $E = 3$ and $K = 0, 0.5$ and 1.0.

Separating equation (15) into real and imaginary parts, the expressions for the primary velocity U and the secondary velocity V have been obtained. Since the expressions are too unwieldy, their graphical solutions are presented in Figs. 1-2. It is observed from these two figures that as the loading parameter K increases, both the primary and secondary velocities increase at any point of the fluid in the channel. Further, Fig. 2 shows that there is a flow reversal for the secondary flow for $K = 1.0$. The critical value at which such a reversal takes place is estimated to be $K = 0.82612$ for $M^2 = 7$.

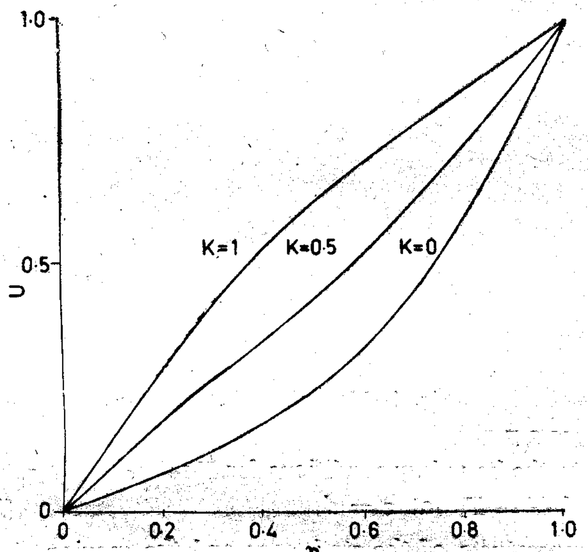

Fig. 1—Velocity of the primary flow.

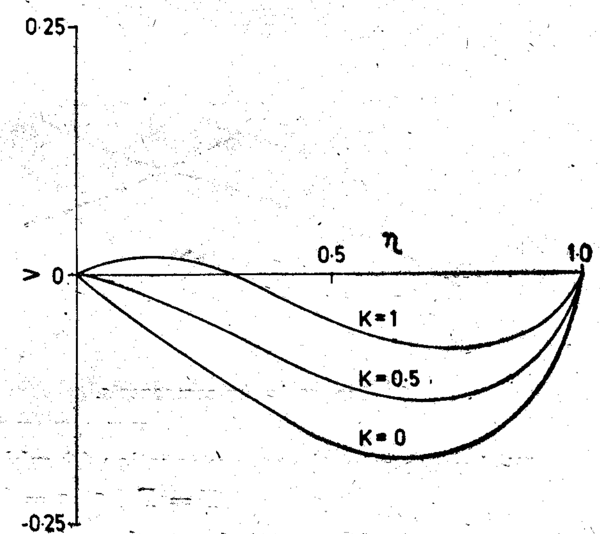

Fig. 2—Velocity of the secondary flow.

The non-dimensional shear stress at the plate $\eta = 1$ is given by

$$\tau = \frac{\sqrt{\alpha^2 \times \beta^2}}{[\cosh(2\alpha) - \cos(2\beta)]^{1/2}} \left[\cosh(2\alpha) + \cos(2\beta) + \right.$$
$$\left. + 2a^2(b^2 + c^2)(\cosh\alpha - \cos\beta)^2 - 4ac\sinh\alpha\sin\beta \right]^{\frac{1}{2}} \tag{17}$$

where

$$a = \alpha_0 K \Big/ \left[\alpha^2_0 + (E + \beta_0)^2 \right], \quad b = \alpha_0 + m(E + \beta_0)$$
$$c = E + \beta_0 - m\alpha_0 \tag{18}$$

The values of τ from the above equation have been entered in Table 1. It is concluded that the resultant shear stress increases with an increase in the loading factor.

The current density components are given by

$$j_x + i\,j_y = (q - K)\,(m - i)/(1 + m^2) \tag{19}$$

and the magnetic induction vector components follow from

$$b_y + i\,b_x = R_m \int (j_x + i\,j_y)\,d\eta + C \tag{20}$$

where C can be chosen to maintain $b_x = b_y = 0$ at $\eta = 1$ as in the solenoid model.

Carrying out the integration yields the following result :

$$\frac{b_y + i\,b_x}{R_m} = \frac{(m - i)}{1 + m^2}\left[\frac{\cosh(h_1\eta) - \cosh(h_1)}{h_1 \sinh h_1} - \frac{M^2 K (1 + i\,m)}{(1 + m^2)\,h_1^2} \cdot \left\{\frac{\cosh(h_1\eta) - \cosh(h_1) - \cosh[h_1(1 - \eta)] + 1}{h_1 \sinh(h_1)} - \eta + 1\right\} - K(\eta - 1)\right] \tag{21}$$

From Figs. 3-4, showing the current density and magnetic field components respectively, it is concluded as K increases (i) both j_x and b_x decrease and (ii) both j_y and b_y increase. Unless the magnetic Reynolds number R_m is of the order of unity or larger, it can be seen that b_x will be small compared to the applied field.

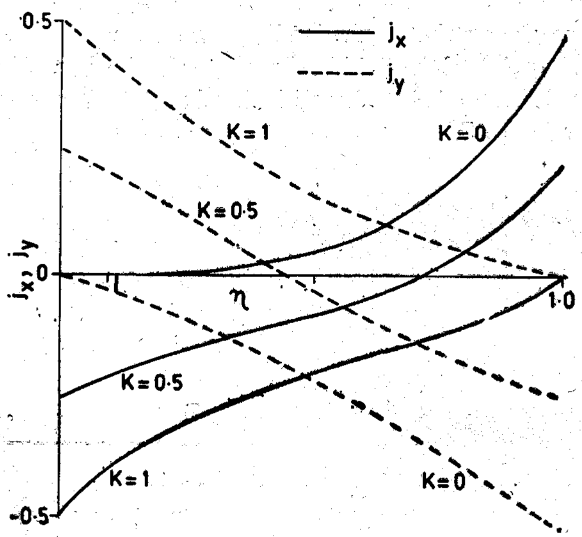

Fig. 3—Current density components.

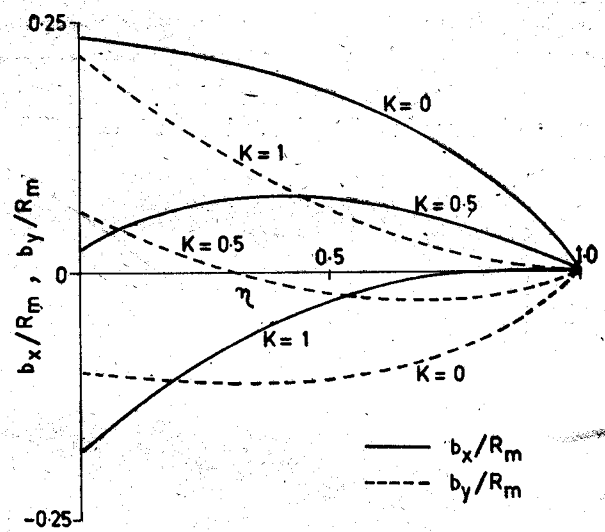

Fig. 4—Magnetic field components.

HEAT TRANSFER

We have to solve the energy equation (3) subject to the boundary conditions for T as

$$T = T_0 \text{ at } z = 0 \text{ and } T = T_1 \text{ at } z = d \tag{22}$$

where T_0 and T_1 ($T_1 > T_0$) denote the uniform temperatures of the stationary and the moving plates respectively.

Introducing

$$\left.\begin{aligned} \theta(\eta) &= \frac{T - T_0}{T_1 - T_0}, \\ Pr \text{ (Prandtl number)} &= \nu/\kappa \\ Ec \text{ (Eckert number)} &= U_0^2/c_p\,(T_1 - T_0) \end{aligned}\right\} \tag{23}$$

in equation (3), we get on using equations (5) and (6).

$$\frac{d^2\theta}{d\eta^2} = -\,Pr\,Ec\left[\frac{dq}{d\eta} \cdot \frac{d\bar{q}}{d\eta} + \alpha_0\left\{q\,\bar{q} - K(q + \bar{q}) + K^2\right\}\right] \tag{24}$$

where

$$\bar{q} = U - i\,V$$

The boundary conditions in equation (22) become

$$\theta(0) = 0, \quad \theta(1) = 1. \tag{25}$$

Using the equation (25), the equation (24) has been solved. The expression for θ is quite lengthy and hence we present a graphical solution in Fig. 5 for fixed values of M^2, m, E mentioned earlier and for $Pr = 0.7$, $Ec = 0.5$. It shows that close to the plate $\eta = 0$ the temperature increases as K increases while it decreases close to $\eta = 1$ as K increases.

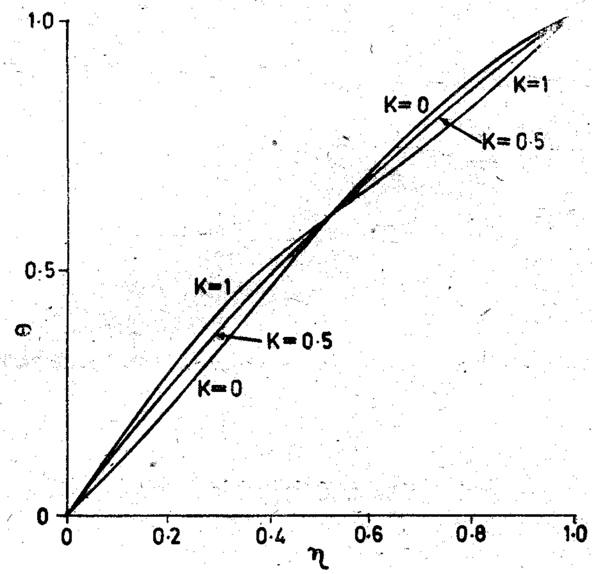

Fig. 5—Temperature distribution.

The rate of heat transfer at the plate $\eta = 1$ is given by,

$$h = 1 - \frac{Pr\, Ec}{\cosh(2\alpha) - \cos(2\beta)} \Big[2\alpha\beta BDLH_3 + 4\alpha\beta BD(\alpha A_1 + \beta A_2) + 2\alpha_0\alpha\beta BLH_2 + 4\alpha_0\alpha\beta B(\beta A_2 - \alpha A_1) + \alpha_0(N+Q)\left\{\cosh(2\alpha) - \cos(2\beta)\right\} + \frac{2\alpha_0}{D^2} H_1(2\alpha\beta S + \alpha_0 R) + \frac{2\alpha_0}{D^2} H_2(2\alpha\beta R - \alpha_0 S) + \frac{4\alpha_0}{D^2} E_3(2\alpha\beta KP - \alpha_0 A) + \frac{4\alpha_0}{D^2} E_4(2\alpha\beta A + \alpha_0 KP) - \Big\{ GBDL + \alpha_0 HBL + \frac{\alpha_0}{2}(N+Q) . \left\{\cosh(2\alpha) - \cos(2\beta)\right\} + \frac{2\alpha_0}{D^2}(2\alpha\beta S + \alpha_0 R) \left\{\cosh(2\alpha) - \cos(2\beta)\right\} \Big\} + GBDN + \alpha_0 HBN - \frac{2\alpha_0}{D^2}(2\alpha\beta KP - \alpha_0 A) \left\{\cosh(2\alpha) - \cos(2\beta)\right\} \Big] \tag{26}$$

where

$$B = \frac{1}{4\alpha^2\beta^2},\ D = \alpha^2 + \beta^2,\ L = (1 - ab)^2 + a^2c^2$$
$$N = a^2(b^2 + c^2),\ Q = K^2 - 2abK,\ S = ac(1 - K)$$
$$R = ab(1 + K) - K - a^2(b^2 + c^2),\ P = ac,\ A = Kab - a^2(b^2 + c^2)$$
$$G = \beta^2\cosh(2\alpha) - \alpha^2\cos(2\beta),\ H = \beta^2\cosh(2\alpha) + \alpha^2\cos(2\beta),$$
$$H_3 = \beta\sinh(2\alpha) + \alpha\sin(2\beta),\ A_1 = C\cosh\alpha\sin\beta + P\sinh\alpha\cos\beta,$$
$$A_2 = C\sinh\alpha\cos\beta - P\cosh\alpha\sin\beta,$$
$$H_2 = \beta\sinh(2\alpha) - \alpha\sin(2\beta),\ H_1 = \alpha\sinh(2\alpha) + \beta\sin(2\beta),$$
$$E_3 = \alpha\sinh\alpha\cos\beta + \beta\cosh\alpha\sin\beta,$$
$$E_4 = \alpha\cosh\alpha\sin\beta - \beta\sinh\alpha\cos\beta,$$
$$C = ab - a^2(b^2 + c^2). \tag{27}$$

It is seen from equation (26) that when $Ec = Ec^*$, where

$$\frac{\cosh(2\alpha) - \cos(2\beta)}{Pr\, Ec^*} = 2\alpha\,\beta\,B\,D\,L\,H_3 + 4\,\alpha\,\beta\,B\,D\,(\alpha\,A_1 + \beta\,A_2) + 2\,\alpha\,\beta\,\alpha_0\,B\,L\,H_2 +$$

$$+ 4\,\alpha\,\beta\,\alpha_0\,B\,(\beta\,A_2 - \alpha\,A_1) + \alpha_0\,(N+Q)\left\{\cosh(2\alpha) - \cos(2\beta)\right\} + \frac{2\alpha_0}{D^2}\,H_1\,(2\,\alpha\,\beta\,S + \alpha_0\,R) +$$

$$+ \frac{2\alpha_0}{D^2}\,H_2\,(2\,\alpha\,\beta\,R - \alpha_0\,S) + \frac{4\alpha_0}{D^2}\,E_3\,(2\,\alpha\,\beta\,P\,K - \alpha_0\,A) + \frac{4\alpha_0}{D^2}\,E_4\,(2\,\alpha\,\beta\,A + \alpha_0\,K\,P) -$$

$$- \left\{G\,B\,D\,L + \alpha_0\,H\,B\,L + \frac{1}{2}\,\alpha_0\,(N+Q)\left\{\cosh(2\alpha) - \cos(2\beta)\right\} + \frac{2\alpha_0}{D^2}\,(2\,\alpha\,\beta\,S + \alpha_0\,R)\right.$$

$$\left.\left\{\cosh(2\alpha) - \cos(2\beta)\right\}\right\} + G\,B\,D\,N + \alpha_0\,H\,B\,N - \frac{2\alpha_0}{D^2}\,(2\,\alpha\,\beta\,K\,P - \alpha_0\,A)$$

$$\left\{\cosh(2\alpha) - \cos(2\beta)\right\} \qquad (28)$$

then there is no flow of heat either from the plate to the fluid or from the fluid to the plate.

Further, for $Ec < Ec^*$, the heat flows from the upper plate to the fluid, while it flows from the fluid to the upper plate for $Ec > Ec^*$.

The rate of heat transfer at the moving plate and the critical Eckert number have been entered in Tables 2 and 3 respectively for fixed values of M^2, m, E, Pr and Ec mentioned earlier. As K increases, it is concluded that both the rate of heat tiansfer at the moving plate and the critical Eckert number increase.

TABLE 1

RESULTANT SHEAR STRESS τ AT THE MOVING PLATE

K	0	0.5	1.0
τ	2.669499	2.763505	3.114170

TABLE 2

RATE OF HEAT TRANSFER h AT THE MOVING PLATE

K	0	0.5	1.0
h	0.369755	0.755465	0.801731

TABLE 3

CRITICAL ECKERT NUMBER Ec^*

K	0	0.5	1.0
Ec^*	0.793300	2.044401	2.521352

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REFERENCES

1. PAI, S. I., 'Magnetogasdynamics and Plasma Dynamics' (Springer-Verlag), 1962.
2. LEHNERT, O. B., *Ark. Fys.*, **5** (1952), 69.
3. BLEVISS, Z. O., *J. Aero/Space Sci.*, **25** (1958), 601.
4. SHERMAN, A. & SUTTON, G. W., 'Magnetohydrodynamics' (Evanston, Illinois), 1961, p. 173.
5. JANA, R. N., DATTA, N. & MAZUMDER, B. S., *J. Phys., Soc. Japan*, **42** (1977), 1034.
6. SATO, H., *J. Phys. Soc. Japan*, **16** (1961), 1427.
7. YAMINISHI, T., *J. Phys. Soc. Japan*, **5** (1962), 29.
8. COWLING, T. G., 'Magnetohydrodynamics' (Interscience Publishers, Inc., New York), 1957, p. 101.