FLOW OF A DUSTY GAS BETWEEN TWO OSCILLATING PLATES

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The flow of an incompressible viscous dusty gas, induced by two infinite flat plates oscillating in their own planes, is studied. Expressions for both the gas and the dust velocities are derived using the Laplace transform. The velocity profiles are drawn for different configurations of the dust particles and the results are compared with those of the clean gas.

The contamination of air in cities by dust particles has necessitated the study of the flow of dusty gases. Saffman[1] formulated the basic equations for the flow of a dusty gas. Since then, there have been several papers in this field. Regarding the plate problems, Liu[2] and Michael & Miller[3] have studied the flow produced by the motion of an infinite plane in a dusty fluid occupying the semi-infinite space above it. Recently Khan[4] has discussed the flow of a viscous liquid between two harmonically oscillating infinite plates, when a constant body force is applied initially in the direction of motion of the plates. In this paper, the flow of a dusty gas between two infinite flat plates oscillating in their own planes is investigated. The Laplace transform technique is used in solving the problem.

GOVERNING EQUATIONS

A viscous, incompressible, dusty gas bounded by two infinite flat plates executing simple harmonic oscillations with a frequency 'ω' in their own planes $y = \pm a$ is considered. Both the gas and the particle cloud are supposed to be static at the beginning. The dust particles are assumed to be spherical in shape and uniform in size so that the conservation equations given by Saffman[1] are applicable. The number density of the particles is taken as a constant throughout the flow. Under these assumptions, the flow will be a parallel flow in which the streamlines are along the x-axis and the velocities are functions of the distance y and time t.

Now let y and t denote the non-dimensionalised distance and time, with reference to a and $\frac{1}{\omega}$ respectively. Let Re be the *Reynolds'* number. Let f be the mass concentration of the dust particles and let τ be the relaxation time of the dust particles, non-dimensionalised with respect to $\frac{1}{\omega}$. If u and v are the velocities of the gas and the dust respectively, non-dimensionalised with respect to $a\omega$, then Saffman's equations[1] reduce to the following simultaneous equations :

$$\frac{\partial u}{\partial t} = \frac{1}{Re}\frac{\partial^2 u}{\partial y^2} + \frac{f}{\tau}\left(v - u\right) , \tag{1}$$

$$\tau \frac{\partial v}{\partial t} = (u - v) \tag{2}$$

which are to be solved under the following initial and boundary conditions :

Initial Conditions

$$u = v = 0 \text{ at } t = 0 \text{ for all } y \tag{3}$$

Boundary Conditions

$$u = u_0 \sin t \text{ at } y = \pm 1 . \tag{4}$$

Since the flow is symmetrical about the plane $y=0$, only the flow in the region $0 \leqslant y \leqslant 1$ is considered and accordingly the boundary conditions (4) are equivalent to the following :

$$u = u_0 \sin t \text{ at } y = 1$$

$$\frac{\partial u}{\partial y} = 0 \qquad \text{at } y = 0 \tag{5}$$

SOLUTION OF THE EQUATIONS

Let $\bar{u}$ and $\bar{v}$ given by

$$\bar{u}(y, p) = \int_0^\infty e^{-pt} \; u(y, t)\, dt$$

and

$$\bar{v}(y, p) = \int_0^\infty e^{-pt} \; v(y, t)\, dt$$

denote the Laplace transforms of u and v respectively.

Applying Laplace transforms to (1), (2) and (5) and using (3) we have

$$p\,\bar{u} = \frac{1}{Re}\,\frac{d^2\bar{u}}{dy^2} + \frac{f}{\tau}\,(\bar{v} - \bar{u}) \qquad (6)$$

$$\tau\, p\, \bar{v} = \bar{u} - \bar{v} \qquad (7)$$

Boundary Conditions

$$\bar{u} = \frac{u_0}{1 + p^2} \quad \text{at } y = 1$$

$$\frac{\partial \bar{u}}{\partial y} = 0 \quad \text{at } y = 0 \qquad (8)$$

Solving for $\bar{u}$ and $\bar{v}$, we obtain

$$\bar{u} = \left(\frac{u_0}{1 + p^2}\right)\left(\frac{\cosh(ky)}{\cosh(k)}\right),$$

$$\bar{v} = \left(\frac{u_0}{(1 + p^2)(1 + p\tau)}\right)\left(\frac{\cosh(ky)}{\cosh(k)}\right), \qquad (9)$$

where

$$k = \left[Re\, p\left\{1 + \frac{f}{1 + p\tau}\right\}\right]^{\frac{1}{2}} \qquad (10)$$

Inverting (see Appendix), we have

$$u = \frac{u_0}{(E^2 + F^2)}\left[E_y\,(E \sin t - F \cos t) + F_y\,(E \cos t + F \sin t)\right] +$$

$$+ \frac{u_0\,\pi}{Re}\sum_{n=0}^{\infty}\left[(-1)^n\,(2n + 1)\left\{\cos\frac{(2n + 1)\pi y}{2}\right\}\right.\cdot$$

$$\left.\cdot\left\{\frac{e^{p_1 t}\,(1 + p_1\tau)^2}{(1 + p_1^2)\,[(1 + p_1\tau)^2 + f]} + \frac{e^{p_2 t}\,(1 + p_2\tau)^2}{(1 + p_2^2)\,[(1 + p_2\tau)^2 + f]}\right\}\right]$$

$$\text{for } 0 \leqslant y < 1, \qquad (11)$$

$$v = \frac{u_0}{(1+\tau^2)(E^2+F^2)} \Big[E_y \Big\{ (E - F\tau) \sin t - (F + E\tau) \cos t \Big\} +$$

$$+ F_y \Big\{ (F + E\tau) \sin t + (E - F\tau) \cos t \Big\} \Big] +$$

$$+ \frac{u_0 \pi}{Re} \sum_{n=0}^{\infty} \Big[(-1)^n (2n+1) \Big\{ \cos \Big(\frac{(2n+1)\pi y}{2} \Big) \Big\} \cdot$$

$$\cdot \Big\{ \frac{e^{p_1 t} (1 + p_1 \tau)}{(1+p_1^2 [(1+p_1\tau)^2 + f]} + \frac{e^{p_2 t}(1+p_2\tau)}{(1+p_2^2)[(1+p_2\tau)^2+f]} \Big\} \Big] \tag{12}$$

for $0 \lessapprox y < 1$,

$$v = \frac{u_0}{1+\tau^2} \Big[\sin t - \tau \cos t + \tau^2 e^{-t/\tau} \Big]$$

at $y = 1$, (13)

where

$$E = \cosh(\sqrt{X}) \cos(\sqrt{Y}), F = \sinh(\sqrt{X}) \sin(\sqrt{Y}),$$

$$E_y = \cosh(y\sqrt{X}) \cos(y\sqrt{Y}), F_y = \sinh(y\sqrt{X}) \sin(y\sqrt{Y}), \tag{14}$$

where X and Y are given by

$$X = \frac{1}{2} \Big\{ x \pm (x^2 + z^2)^{\frac{1}{2}} \Big\}, \quad Y = \frac{z^2}{2 \Big\{ x \pm (x^2 + z^2)^{\frac{1}{2}} \Big\}} \tag{15}$$

with

$$x = \frac{Re}{1+\tau^2} (f\tau), z = \frac{Re}{1+\tau^2} (1 + f + \tau^2) \tag{16}$$

and

$$p_1 = A + B, \qquad p_2 = A - B \tag{17}$$

where

$$A = -\Big[\frac{1+f}{2\tau} + \frac{(2n+1)^2 \pi^2}{8\,Re} \Big]$$

$$B = \frac{1}{8\,Re\,\tau} \Big\{ 16\,Re^2 (1+f)^2 + 8\,Re\,\tau (2n+1)^2 \pi^2 (f-1) + (2n+1)^4 \pi^4 \tau^2 \Big\}^{\frac{1}{2}} \tag{18}$$

For a clean gas (i.e., for a gas without dust particles), the velocity of the gas is given by[4]

$$u_c = 16\, u_0\, \pi\, Re \sum_{n=0}^{\infty} \Bigg\{ \frac{(-1)^n (2n+1) \exp\Big[\frac{-(2n+1)^2 \pi^2}{4\,Re} t \Big]}{\Big[16\,Re^2 + (2n+1)^4 \pi^4 \Big]} \cos \Big(\frac{(2n+1)\pi y}{2} \Big) \Bigg\} +$$

$$+ \Big[u_0 \Big/ \Big\{ \Big(\cosh\sqrt{Re/2} \cos\sqrt{Re/2} \Big)^2 + \Big(\sinh\sqrt{Re/2} \sin\sqrt{Re/2} \Big)^2 \Big\} \Big] \cdot$$

$$\cdot \Big[\sin t \Big\{ \cosh\Big(y\sqrt{Re/2} \Big) \cosh\Big(\sqrt{Re/2} \Big) \cos\Big(y\sqrt{Re/2} \Big) \cos\Big(\sqrt{Re/2} \Big) +$$

$$+ \sinh\Big(y\sqrt{Re/2} \Big) \sinh\Big(\sqrt{Re/2} \Big) \sin\Big(y\sqrt{Re/2} \Big) \sin\Big(\sqrt{Re/2} \Big) \Big\} +$$

$$+ \cos t \Big\{ \sinh\Big(y\sqrt{Re/2} \Big) \cosh\Big(\sqrt{Re/2} \Big) \sin\Big(y\sqrt{Re/2} \Big) \cos\Big(\sqrt{Re/2} \Big) -$$

$$- \cosh\Big(y\sqrt{Re/2} \Big) \sinh\Big(\sqrt{Re/2} \Big) \cos\Big(y\sqrt{Re/2} \Big) \sin\Big(\sqrt{Re/2} \Big) \Big\} \Big] \tag{19}$$

As $t \to \infty$, $e^{p_1 t}$ and $e^{p_2 t}$ vanish (For, p_1 and p_2 are the roots of the quadratic equation

$$(4\, Re\, \tau) p^2 + \left\{ 4\, Re\, (1 + f) + (2\, n + 1)^2\, \pi^2\, \tau \right\} p + \left\{ (2\, n + 1)^2\, \pi^2 \right\} = 0$$

with no change of sign and are obviously negative) so that there is no contribution to the velocities from the summation terms. Hence the solutions for the asymptotic case $t \to \infty$ are:

$$u = \frac{u_0}{E^2 + F^2} \left[E_y\, (E \sin t - F \cos t) + F_y (E \cos t + F \sin t) \right] \quad \text{for } 0 \leqslant y < 1, \tag{20}$$

$$v = \frac{u_0}{(1 + \tau^2)\,(E^2 + F^2)} \left[E_y \left\{ (E - F\tau) \sin t - (F + E\tau) \cos t \right\} + F_y \left\{ (F + E\tau) \sin t + (E - F\tau) \cos t \right\} \right] \quad \text{for } 0 \leqslant y < 1, \tag{21}$$

$$v = \frac{u_0}{(1 + \tau^2)} \left[\sin t - \tau \cos t \right] \text{ at } y = 1, \tag{22}$$

which are periodic functions.

DISCUSSION AND CONCLUSIONS

The velocities u_c, u and v are calculated for times $t = 50$, 100, 500 taking the parameters characterizing the flow as follows:

$$Re = 10^3;\ u_0 = 0{\cdot}1;\ f = 0{\cdot}1,\ 0{\cdot}2;\ \text{and } \tau = 0{\cdot}05,\ 0{\cdot}1.$$

Figs. 1-3 represent the variations of u_c, u and v with y at different times. The velocity profiles for u are similar to that of the clean gas showing that the presence of dust does not have any effect on the manner in which u varies with y. Furthermore, the dust velocity v varies in the same fashion as u does (except for $t = 100$, for which there is a discrepancy between the shapes of the u and v profiles which may be due to the

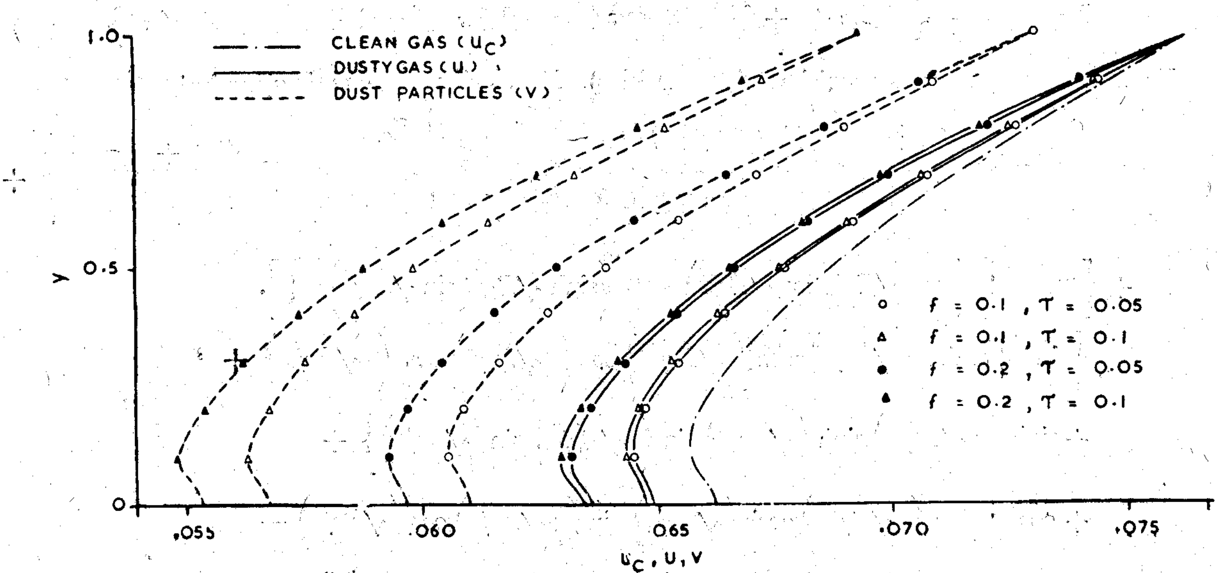

Fig. 1—Velocity distributions at $t = 50$.

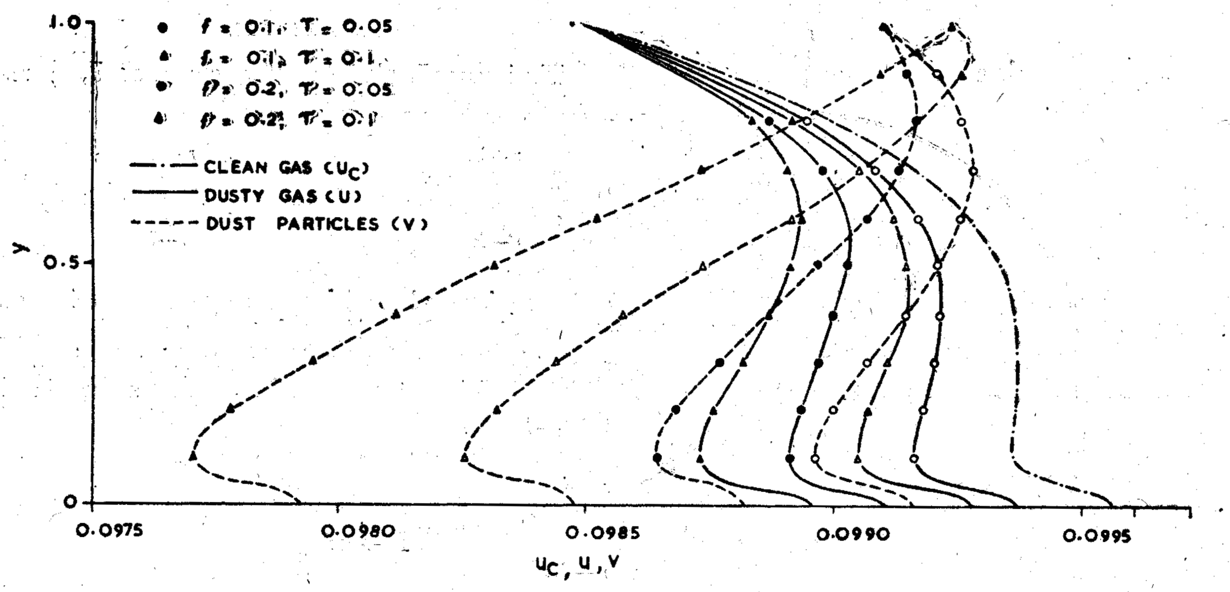

Fig. 2—Velocity distributions at $t=100$.

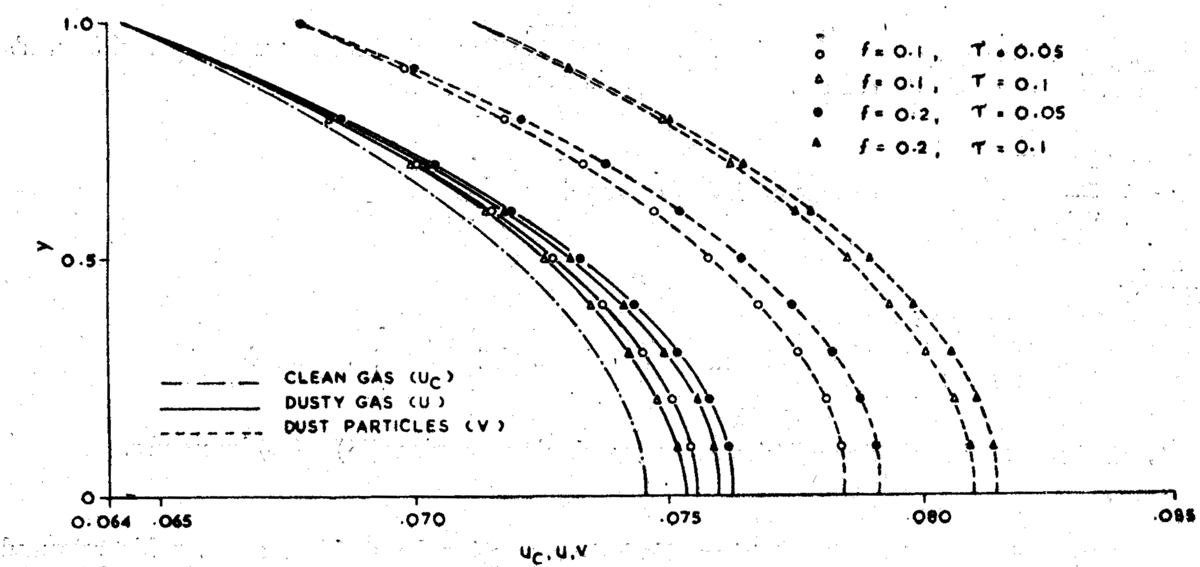

Fig. 3—Velocity distributions at $t=500$.

capability, of the scale chosen there to represent velocities correct to five decimal places so that the change in the behaviours of u and v is seen only after the fourth decimal place).

It is evident from the graphs that, as we go away from the plate, the change in f has an appreciable effect on u compared to the very small effect due to a change in τ, whereas a change in τ has much more influence on v than a change in f has. On the other hand, very near the boundary, the gas velocity is unaffected by the presence of dust while the dust velocity is the same for all f but depends on τ.

Hence it may be concluded that the gas velocity depends more on the mass concentration of the dust particles than on their size. On the contrary, the size of the dust particles has more influence on the velocity of the dust particles than their concentration itself has. After a long time, the velocities become periodic but are still dependent on the size and concentration of dust particles in the gas.

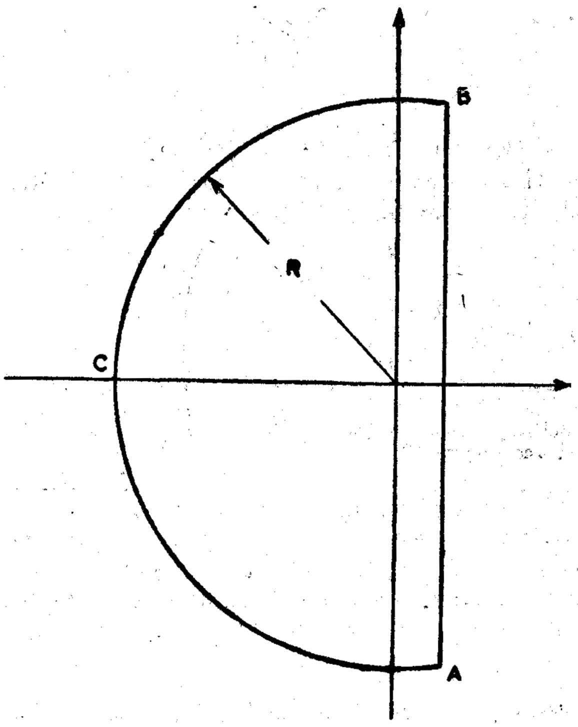

Fig. 4—Contour formed by line $x=r$ and a semi circle C with origin as centre and radius R.

APPENDIX

The inverse Laplace transforms u, v of $\bar{u}$, $\bar{v}$ respectively are given by the integrals

$$\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{pt}\,\bar{u}\,dp \quad \text{and} \quad \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{pt}\,\bar{v}\,dp$$

which can be evaluated by means of contour integration. Since there is no branch point, the contour chosen is the closed curve ABC formed by the line $x=r$ and a semicircle C with origin as centre and radius R (Fig. 4) so that

$$\int_{r-i\infty}^{r+i\infty} e^{pt}\,\bar{u}\,dp = \underset{R\to\infty}{Lt}\int_{A}^{B} e^{pt}\,\bar{u}\,dp$$

$$= \underset{R\to\infty}{Lt}\left[\oint_{ABC} e^{pt}\,\bar{u}\,dp - \int_{C} e^{pt}\,\bar{u}\,dp\right]$$

Using Cauchy's theorem of residues and Jordan's lemma, we have

$$u = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{pt}\,\bar{u}\,dp = \text{sum of the residues of } \left\{ e^{pt}\,\bar{u} \right\} \text{ at its poles}$$

Similarly,

$$v = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{pt}\,\bar{v}\,dp = \text{sum of the residues of } \left\{ e^{pt}\,\bar{v} \right\} \text{ at its poles}$$

Calculating the residues and simplifying further, we obtain expressions (11) to (13) for u and v.

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