FOURIER SERIES FOR FOX'S H-FUNCTION OF TWO VARIABLES

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(*Received* 8 *July* 1971 ; *revised* 17 *November* 1971)

An attempt has been made to derive a Fourier series expansion for the H-function of two variables recently defined by Verma. This series is analogous to that of other special functions such as the MacRobert's E-function, Meijer's G-function and Fox's H-function of single variable as given by MacRobert, Kesarwani, Parihar, Parashar, Kapoor & Gupta. In the end an integral has been evaluated by making use of this result.

MacRobert[1], Kesarwani[2], Parihar[3] established the Fourier series for the E- and G- function and in the recent paper Parashar[4], Kapoor & Gupta[5] has proved Fourier series for Fox's H-function of single variable. However the Fourier series expansion for H-function of two variables has not been derived so far.

The following Fourier series expansion is proposed to be established :

$$\sum_{r=0}^{\infty} \frac{(k+r)!}{k!\,r!}\, H^{n,\nu_1+1,\nu_2,m_1+2,m_2}_{p,(t+3:t'),s,(q+3:q')}\left[\begin{array}{c|l} x & (a_p,e_p) \\ & (r,h),(\gamma_t,c_t),(0,h),(-k-r-1,h);(\gamma'_{t'},c'_{t'}) \\ & (\delta_s,d_s) \\ y & \left(1+\frac{k}{2},h\right),\left(\frac{3}{2}+\frac{k}{2},h\right),(\beta_q,b_q),(1,h);(\beta'_{q'},b'_{q'}) \end{array}\right]\sin(k+2r+1)\theta$$

$$= \sqrt{\frac{\pi}{2}} \sum_{u=0}^{k} \frac{\sin\theta\,(\cos\theta-1)^u}{u!\,(k-u)!}\, H^{n,\nu_1,\nu_2,m_1+2,m_2}_{p,(t+2:t'),s,(q+2:q')}\left[\begin{array}{c|l} \frac{x}{\sin^{2h}\theta} & (a_p,e_p) \\ & (\gamma_t,c_t),(0,h),(-\tfrac{1}{2}-u,h);(\gamma'_{t'},c'_{t'}) \\ & (\delta_s,d_s) \\ y & \left(1+\frac{k+u}{2}h\right),\left(\frac{3}{2}+\frac{k+u}{2},h\right),(\beta_q,b_q); \\ & (\beta'_{q'},b'_{q'}) \end{array}\right] \quad (1)$$

where $0 \leqslant \theta \leqslant \pi$, $T \equiv \sum_1^n e_j + \sum_1^{\nu_1} c_j + \sum_1^{m_1} b_j - \sum_{n+1}^{p} e_j - \sum_1^{s} d_j - \sum_{\nu_1+1}^{t} c_j - \sum_{m_1+1}^{q} b_j > 0, |\arg x| < \frac{1}{2}T\pi,$

and

$$T' \equiv \sum_1^n e_j + \sum_1^{\nu_2} c'_j + \sum_1^{m_2} b'_j - \sum_{n+1}^{p} e_j - \sum_1^{s} d_j - \sum_{\nu_2+1}^{t'} c'_j - \sum_{m_2+1}^{q'} b'_j > 0,\ |\arg y| < \frac{1}{2}T'\pi.$$

Fox's H-function of two variables recently introduced by Verma[6] which is an extension of G-function of two variables defined by Agarwal[7]. This H-function of two variables does not only includes Fox's H-function and the Meijer's G-function of single variables as particular cases but also most of special functions of two variables, e.g., Appell's functions, the Whittaker function of two variables etc.

Thus Fox's H-function of two variables due to Verma[6] will be defined as follows:

$$H^{n,\nu_1,\nu_2,m_1,m_2}_{p,(t:t'),s,(q:q')}\left[\begin{array}{c|l} x & (a_p,e_p) \\ & (\gamma_t,c_t);(\gamma'_{t'},c'_{t'}) \\ & (\delta_s,d_s) \\ y & (\beta_q,b_q);(\beta'_{q'},b'_{q'}) \end{array}\right] = \frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty} \phi(\xi+\eta)\,\psi(\xi,\eta)\,x^{\xi}y^{\eta}\,d\xi\,d\eta, \quad (2)$$

where

$$\phi(\xi+\eta) = \frac{\prod_{j=1}^{n}\Gamma(1-a_j+e_j\xi+e_j\eta)}{\prod_{j=n+1}^{p}\Gamma(a_j-e_j\xi-e_j\eta)\prod_{j=1}^{s}\Gamma(\delta_j+d_j\xi+d_j\eta)},$$

$$\psi(\xi,\eta)=\frac{\prod_{j=1}^{m_1}\Gamma(\beta_j-b_j\xi)\prod_{j=1}^{\nu_1}\Gamma(\gamma_j+c_j\xi)\prod_{j=1}^{m_2}\Gamma(\beta'_j-b'_j\eta)\prod_{j=1}^{\nu_2}\Gamma(\gamma'_j+c'_j\eta)}{\prod_{j=m_1+1}^{q}\Gamma(1-\beta_j+b_j\xi)\prod_{j=\nu_1+1}^{t}\Gamma(1-\gamma_j-c_j\xi)\prod_{j=m_2+1}^{q'}\Gamma(1-\beta'_j+b'_j\eta)\prod_{j=\nu_2+1}^{t'}\Gamma(1-\gamma'_j-c'_j\eta)},$$

and $\quad 0\leqslant m_1\leqslant q,\ 0\leqslant m_2\leqslant q',\ 0\leqslant \nu_1\leqslant t,\ 0\leqslant \nu_2\leqslant t',\ 0\leqslant n\leqslant p$.

The sequence of parameters (β_{m_1}, b_{m_1}), (β'_{m_2}, b'_{m_2}), $(\gamma_{\nu_1}, c_{\nu_1})$, $(\gamma'_{\nu_2}, c'_{\nu_2})$ and (a_n, e_n) are such that none of the poles of integrand coincides. The paths of integration are indented, if necessary, in such a manner that all the poles of $\Gamma(\beta_j-b_j\xi)$, $j=1,2,..,m_1$ and $\Gamma(\beta'_k-b'_k\eta)$, $k=1,2,..,m_2$ lie to the right and those of $\Gamma(\gamma_j+c_j\xi)$, $j=1,2,..,\nu_1$, $\Gamma(\gamma'_k+c'_k\eta)$, $k=1,2,..,\nu_2$ and $\Gamma(1-a_j+e_j\xi+e_j\eta)$, $j=1,2,..,n$, lie to the left of imaginary axis.

The integral (2) converges if

$$T\equiv\sum_{1}^{\pi}e_j+\sum_{1}^{\nu_1}c_j+\sum_{1}^{m_1}b_j-\sum_{n+1}^{p}e_j-\sum_{1}^{s}d_j-\sum_{\nu_1+1}^{t}c_j-\sum_{m_1+1}^{q}b_j>0,\ |\arg x|<\frac{1}{2}T\pi,$$

and

$$T^*\equiv\sum_{1}^{n}e_j+\sum_{1}^{\nu_2}c_j+\sum_{1}^{mg}b'_j-\sum_{n+1}^{p}e_j-\sum_{1}^{s}d_j-\sum_{v+1}^{t'}c'_j-\sum_{m_2+1}^{q'}b'_j>0,\ |\arg y|<\frac{1}{2}T^*\pi$$

We shall give below some results left and use them later on. Askey[8] give with $\lambda=1-s$

$$(\sin\theta)^{1-2s}P_n^{1-s}(\cos\theta)=\sum_{r=0}^{\infty}\frac{2^{2s}(n+r)!\,\Gamma(n+2-2s)\,\Gamma(r+s)}{\Gamma(1-s)\,\Gamma(s)\,r!\,n!\,\Gamma(n+r+2-s)}\sin(n+2r+1)\theta, \tag{3}$$

where $s<1$ and $0\leqslant\theta\leqslant\pi$ and $P_n^{(\lambda)}(\cos\theta)$ is given by

$$(1-2r\cos\theta+r^2)^{-\lambda}=\sum_{n=0}^{\infty}P_n^{(\lambda)}(\cos\theta)\,.\,r^n, \tag{4}$$

also Rainville[9]

$$P_n^{(\lambda)}(z)=\sum_{m=0}^{n}\frac{(2\lambda)_{m+n}\left(\frac{z-1}{2}\right)^m}{m!\,(n-m)!\,(\lambda+\frac{1}{2})_m}. \tag{5}$$

The Legendre duplication formula

$$\sqrt{(\pi)}\ \Gamma(2z)=2^{2z-1}\ \Gamma(z)\ \Gamma(z+\tfrac{1}{2}). \tag{6}$$

Verma[6] gives

$$H^{n,\nu_1,\nu_2,m_1,m_2}_{p,(t:t'),s,(q:q')}\left[\begin{array}{c|l} x & (a_p,e_p) \\ & (\gamma_t,c_t);(\gamma'_{t'},c'_{t'}) \\ & (\delta_s,d_s) \\ y & (\beta_q,b_q);(\beta'_{q'},b'_{q'})\end{array}\right]=x^{-r}H^{n,\nu_1,\nu_2,m_1,m_2}_{p,(t:t'),s,(q:q')}\left[\begin{array}{c|l} x & (a_p+e_p r,e_p) \\ & (\gamma_t-c_t r,c_t);(\gamma'_{t'},c'_{t'}) \\ & (\delta_s-d_s r,d_s) \\ y & (\beta_q+b_q r,b_q);(\beta'_{q'},b'_{q'})\end{array}\right] \tag{7}$$

and

$$H^{n,\nu_1,\nu_2,m_1,m_2}_{p,(t:t'),s,(q:q')}\left[\begin{array}{c|l} x & (a_p,1) \\ & (\gamma_t,1);(\gamma'_{t'},1) \\ & (\delta_s,1) \\ y & (\beta_q,1);(\beta'_{q'},1)\end{array}\right]=G^{n,\nu_1,\nu_2,m_1,m_2}_{p,(t:t'),s,(q:q')}\left[\begin{array}{c|l} x & (a_p) \\ & (\gamma_t);(\gamma'_{t'}) \\ & (\delta_s) \\ y & (\beta_q);(\beta'_{q'})\end{array}\right]. \tag{8}$$

Proof of equation (1) : On expressing the H-function of two variables as Mellin-Barnes type of double integral in L.H.S. of (1) and changing the order of summation and integration as permissible by absolute convergence for stated conditions in (1), the series becomes

$$\frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty}\phi(\xi+\eta)\,\psi(\xi,\eta)\left[\sum_{r=0}^{\infty}\frac{(k+r)!\,\Gamma\left(1+\frac{k}{2}-h\xi\right)\Gamma\left(\frac{3}{2}+\frac{k}{2}-h\xi\right)\Gamma(r+h\xi)}{k!\,r!\,\Gamma(h\xi)\,\Gamma(1-h\xi)\,\Gamma(k+r+2-h\xi)}\cdot\sin(k+2r+1)\theta\right]x^{\xi}\,y^{\eta}\,d\xi\,d\eta\,.$$

Using the result (3) and (6), we have

$$\frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty}\phi(\xi+\eta)\,\psi(\xi,\eta)\left[\frac{\sqrt{(\pi)}}{2^{k+1}}\,(\sin\theta)^{(1-2h)}\,P_k^{(1-h\xi)}(\cos\theta)\right]x^{\xi}\,y^{\eta}\,d\xi\,d\eta\,.$$

Now substituting the value of $P_k^{(1-h\xi)}(\cos\theta)$ from (5) and then using the result (6), we get

$$\sum_{u=0}^{k}\frac{\sqrt{(\pi)}}{2}\frac{\sin\theta\,(\cos\theta-1)^u}{u!\,(k-u)!}\left[\frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty}\phi(\xi+\eta)\,\psi(\xi,\eta)\,\frac{\Gamma\left(1+\frac{k+u}{2}-h\xi\right)\Gamma\left(\frac{3}{2}+\frac{k+u}{2}-h\xi\right)}{\Gamma(1-h\xi)\,\Gamma\left(\frac{3}{2}+u-h\xi\right)}\cdot\left(\frac{x}{\sin^{2h}\theta}\right)^{\xi}y^{\eta}\,d\xi\,d\eta\right],$$

by definition of H-function of two variables (2), we get R.H.S. of (1), which completes the proof.

We shall derive here other Fourier series by applying the property of the H-function or by specializing the parameters.

(*i*) The following Fourier series for H-function is arrived by using the result (7) :

$$\sum_{r=0}^{\infty}\frac{(k+r)!}{k!\,r!}\,H^{n,\nu_1+1,\nu_2,m_1+2,m_2}_{p,(t+3:t'),s,(q+3:q')}\left[\begin{matrix}x\\ y\end{matrix}\left|\begin{matrix}(a_p+e_p r,\,e_p)\\ (r-hr,h),(\gamma_t-c_t r,\,c_t),(-hr,h),(-k-r-hr-1,h);\\ (\gamma'_{t'},\,c'_{t'})\\ (\delta_s-d_s r,\,d_s)\\ \left(1+\frac{k}{2}+hr,h\right),\left(\frac{3}{2}+\frac{k}{2}+hr,h\right),(\beta_q+b_q r,\,b_q),(1+hr,h);\\ (\beta'_{q'},\,b'_{q'})\end{matrix}\right.\right]$$

$$\cdot\,x^{-r}\sin(k+2r+1)\theta=\frac{\sqrt{(\pi)}}{2}\sum_{u=0}^{k}\frac{\sin\theta\,(\cos\theta-1)^u}{u!\,(k-u)!}\,H^{n,\nu_1,\nu_2,m_1+2,m_2}_{p,(t+2:t'),s,(q+2:q')}$$

$$\left[\begin{matrix}\frac{x}{\sin^{2h}\theta}\\ y\end{matrix}\left|\begin{matrix}(a_p,\,e_p)\\ (\gamma_t,\,c_t),(0,h),\left(-\tfrac{1}{2}-u,h\right);(\gamma'_{t'},\,c'_{t'})\\ (\delta_s,\,d_s)\\ \left(1+\frac{k+u}{2},h\right),\left(\frac{3}{2}+\frac{k+u}{2},h\right),(\beta_q,\,b_q);(\beta'_{q'},\,b'_{q'})\end{matrix}\right.\right]\qquad(9)$$

(*ii*) On putting $h = 1 = (e_p) = (c_t) = (c'_{t'}) = (d_s) = (b_q) = (b'_{q'})$ in (1), and using (8), we get the following result for Meijer G-function of two variables :

$$\sum_{r=0}^{\infty} \frac{(k+r)!}{k!\, r!}\, G^{n,\, \nu_1+1,\, \nu_2,\, m_1+2,\, m_2}_{p,\, (t+3:t'),\, s,\, (q+3:q')} \left[\begin{array}{c|l} x & (a_p) \\ & r, (\gamma_t), 0, -k-r-1 ; (\gamma'_{t'}) \\ & (\delta_s) \\ y & 1+\frac{k}{2}, \frac{3}{2}+\frac{k}{2}, (\beta_q), 1 ; (\beta'_{q'}) \end{array} \right] \sin (k+2r+1)\, \theta$$

$$= \frac{\sqrt{(\pi)}}{2} \sum_{u=0}^{k} \frac{\sin \theta (\cos \theta - 1)^u}{u!\, (k-u)!}\, G^{n,\, \nu_1,\, \nu_2,\, m_1+2,\, m_2}_{p,\, (t+2:t'),\, s,\, (q+2:q')} \left[\begin{array}{c|l} \frac{x}{\sin^2 \theta} & (a_p) \\ & (\gamma_t), 0, -\frac{1}{2}-u ; (\gamma'_{t'}) \\ & (\delta_s) \\ y & 1+\frac{k+u}{2}, \frac{3}{2}+\frac{k+u}{2}, (\beta_q), (\beta'_{q'}) \end{array} \right]. \quad (10)$$

From (1), we can easily deduce the following integral :

$$\int_0^{\pi} \sum_{u=0}^{k} H^{n,\, \nu_1,\, \nu_2,\, m_1+2,\, m_2}_{p,\, (t+2:t'),\, s,\, (q+2:q')} \left[\begin{array}{c|l} \frac{x}{\sin^{2h} \theta} & (a_p, e_p) \\ & (\gamma_t, c_t), (0, h), (-\frac{1}{2}-u, h) ; (\gamma'_{t'}, c'_{t'}) \\ & (\delta_s d_s) \\ y & \left(1+\frac{k+u}{2}, h\right), \left(\frac{3}{2}+\frac{k+u}{2}, h\right), (\beta_q, b_q) ; (\beta'_{q'}, b'_{q'}) \end{array} \right].$$

$$\cdot \frac{\sin \theta .\, \sin (k+2r+1)\, \theta .\, (\cos \theta - 1)^u}{u!\, (k-u)!}\, d\theta = \sum_{r=0}^{\infty} \frac{\sqrt{(\pi)}\, (k+r)!}{k!\, r!}\, H^{n,\, \nu_1+1,\, \nu_2,\, m_1+2,\, m_2}_{p,\, (t+3:t'),\, s,\, (q+3:q')}$$

$$\left[\begin{array}{c|l} x & (a_p, e_p) \\ & (r, h), (\gamma_t, c_t), (0, h), (-k-r-1, h) ; (\gamma'_{t'}, c'_{t'}) \\ & (\delta_s, d_s) \\ y & \left(1+\frac{k}{2}, h\right), \left(\frac{3}{2}+\frac{k}{2}, h\right), (\beta_q, b_q), (1, h) ; (\beta'_{q'}, b'_{q'}) \end{array} \right], \quad (11)$$

where $0 \leqslant \theta \leqslant \pi$, $T \equiv \sum_1^n e_j + \sum_1^{\nu_1} c_j + \sum_1^{m_1} b_j - \sum_{n+1}^{p} e_j - \sum_1^{s} d_j - \sum_{\nu_1+1}^{t} c_j - \sum_{m_1+1}^{q} b_j > 0$, $|\arg x| < \frac{1}{2} T \pi$,

and

$$T' \equiv \sum_1^n e_j + \sum_1^{\nu_2} c'_j + \sum_1^{m_2} b'_j - \sum_{n+1}^{p} e_j - \sum_1^{s} d_j - \sum_{\nu_2+1}^{t'} c'_j - \sum_{m_2+1}^{q'} b'_j > 0, \ |\arg y| < \frac{1}{2} T' \pi.$$

ACKNOWLEDGEMENT

I am highly grateful to Prof. P. M. Gupta for encouragement and guidance during the preparation of this paper.

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