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Finite Bending of Blocks

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Abstract. The problem of bending of a circular block with concentric hole into ellipsoidal shell is considered, and the earlier results have been obtained as particular cases.

1. Introduction

The problems of bending of rectangular and circular blocks have been considered recently[1,2,3,4]. The solution of the problem of bending of an aelotropic circular block into ellipsoidal shell was considered by S. Ram Rao, *et al.*[5]. In this paper, the problem of bending of a circular block with concentric hole is considered. The earlier results[5] are obtained as particular cases. When the strain energy function is specified for a particular material, the results of this paper can be applied to determine the stresses required to bend the block into the particular form. We adopt the notation and formula of Green & Adkins[2].

2. Bending of an Incompressible Aelotropic Circular Block with Concentric Circular Hole into Ellipsoidal Shell

Suppose that in the undeformed state of the body it is a circular block with a concentric circular hole bounded by the planes $x_3 = a_1$, $x_3 = a_2 (a_2 > a_1)$ and the cylinders $b_1^2 \leqslant x_1^2 + x_2^2 \leqslant b_2^2$. The block is bent symmetrically about the x_3-axis into a part of an ellipsoidal shell, whose inner and outer boundaries are the ellipsoides of revolution obtained by revolving the confocal ellipses

$$x_3 = c \cosh \xi \cos \eta, \; x = c \sinh \xi \sin \eta, \; \xi = \xi_i \qquad i = 1, 2 \tag{1}$$

about the x_3-axis respectively and the edge $\eta = \eta_2$. The edges $\eta = \eta_i$ in the deformed state corresponds to the circles $x_1^2 + x_2^2 = b_i^2$ in the undeformed state. It is assumed that the circumference of the circular hole before deformation becomes part of the ellipsoidal shell after deformation and corresponds to $\eta = \eta_1$. Let the y_i-axis coincide with the x_i-axis, and the curvilinear coordinates θ_i in the deformed state be a system of orthogonal coordinates (ξ, η, ϕ). Then we have

$$y_1 = c \sinh \xi \sin \eta \cos \phi,\ y_2 = c \sinh \xi \sin \eta \sin \phi,$$

$$y_3 = c \cosh \xi \cos \eta \tag{2}$$

Since the deformation is symmetrical about the x_3-axis, we see that

(i) the planes x_3 = constant in the undeformed state become the ellipsoidal surfaces ξ = constant in the deformed state,

(ii) the curves $x_1^2 + x_2^2$ = constant in the undeformed state become the circles η = constant in the deformed state, and

(iii) $\tan^{-1} \dfrac{x_2}{x_1} = \phi$.

These imply that

$$x_3 = f(\xi),\ (x_1^2 + x_2^2)^{1/2} = F(\eta),\ x_1 \tan \phi = x_2 \tag{3}$$

which give

$$x_1 = F(\eta) \cos \phi,\ x_2 = F(\eta) \sin \phi,\ x_3 = f(\xi). \tag{4}$$

The metric tensors for the strained and unstrained state of the body are given by

$$G_{ij} = \begin{bmatrix} c^2(\cosh^2 \xi - \cos^2 \eta) & 0 & 0 \\ 0 & c^2(\cosh^2 \xi - \cos^2 \eta) & 0 \\ 0 & 0 & c^2 \sinh^2 \xi \sin^2 \eta \end{bmatrix} \tag{5}$$

$$g_{ij} = \begin{bmatrix} f^{2\prime} & 0 & 0 \\ 0 & F'^2 & 0 \\ 0 & 0 & F^2 \end{bmatrix} \tag{6}$$

where $\quad f' = \dfrac{df}{d\xi}$ and $F' = \dfrac{dF}{d\eta}$

The condition of incompressibility $I_3 = 1$ gives

$$\frac{c^3(\cosh^2 \xi - \cos^2 \eta) \sinh \xi}{f'} = \frac{FF'}{\sin \eta} \tag{7}$$

With the assumption that η is small[4], we get

$$\frac{c^3 \sinh^3 \xi}{f'} = \frac{FF'}{\eta} = A \tag{8}$$

where A is an arbitrary constant.

From this, we get

$$x_3 = f(\xi) = \frac{c^3}{A}\int \sinh^3 \xi \, d\xi + B \tag{9}$$

$$= \frac{c^3}{A}\left[\frac{\cosh^3 \xi}{3} - \cosh \xi\right] + B \tag{10}$$

and

$$x_1^2 + x_2^2 = F^2(\eta) = A\eta^2 + D \tag{11}$$

where B and D are constants.

As the internal and external boundaries of the ellipsoidal shell are given by $\xi = \xi_i$ $(i = 1, 2)$ respectively which were initially the planes $x_3 = a_1$, $x_3 = a_2$, Eqn. (10) gives

$$a_i = \frac{c^3}{A}\,\frac{(\cosh^3 \xi_i - 3 \cosh \xi_i)}{3} + B, \; (i = 1, 2) \tag{12}$$

which when solved give the values of A and B.

Since the bending is symmetric about the x_3-axis, we must have

$$x_1^2 + x_2^2 = b_i^2, \quad \text{when } \eta = \eta_i \; (i = 1, 2).$$

Then Eqn. (11) gives

$$A\eta_1^2 + D = b_1^2 \text{ and } A\eta_2^2 + D = b_2^2 \tag{13}$$

we have $$D = (b_2^2\,\eta_1^2 - b_1^2\,\eta_2^2)/(\eta_1^2 - \eta_2^2) \tag{14}$$

Now, the expressions corresponding to Eqns. (5) and (6) become

$$G_{ij} = \begin{bmatrix} c^2 \sinh^2 \xi & 0 & 0 \\ 0 & c^2 \sinh^2 \xi & 0 \\ 0 & 0 & \eta^2 c^2 \sinh^2 \xi \end{bmatrix} \tag{15}$$

$$g_{ij} \quad \begin{bmatrix} \dfrac{c^6 \sinh^6 \xi}{A^2} & 0 & 0 \\ 0 & \dfrac{A^2\eta^2}{A\eta^2 + D} & 0 \\ 0 & 0 & A\eta^2 + D \end{bmatrix} \tag{16}$$

The strain components are given by

$$2e_{11} = c^2 \sinh^2 \xi \left[\frac{(A\eta^2 + D)\cos^2\phi}{A^2\eta^2} + \frac{\eta^2 \sin^2\phi}{A\eta^2 + D}\right] - 1,$$

$$2e_{22} = c^2 \sinh^2 \xi \left[\frac{A\eta^2 + D}{A^2\eta^2}\sin^2\phi + \frac{\eta^2 \cos^2\phi}{A\eta^2 + D}\right] - 1 \tag{17}$$

$$2e_{33} = \frac{A^2}{c^4 \sinh^4 \xi} - 1,$$

$$e_{ij} = 0 \text{ for } i \neq j$$

The stress tensors are given by

$$T^{11} = \frac{A^2}{c^6 \sinh^6 \xi} \frac{\partial W}{\partial e_{33}} + \frac{P}{c^2 \sinh^2 \xi}$$

$$T^{22} = \frac{A\eta^2 + D}{A^2\eta^2} \left(\frac{\partial w}{\partial e_{11}} \cos^2 \phi + \frac{\partial w}{\partial e_{22}} \sin^2 \phi \right) + \frac{p}{\eta^2 c^2 \sinh^2 \xi}$$

$$T^{33} = \frac{1}{A\eta^2 + D} \left(\frac{\partial w}{\partial e_{11}} \sin^2 \phi + \frac{\partial w}{\partial e_{22}} \cos^2 \phi \right) + \frac{p}{\eta^2 c^2 \sinh^2 \xi} \tag{18}$$

The non-zero Christoffel symbols are given by

$$\Gamma^1_{11} = \Gamma^2_{21} = \Gamma^3_{31} = \coth \xi, \qquad \Gamma^1_{22} = -\coth \xi$$

$$\Gamma^1_{33} = \eta^2 \coth \xi \tag{19}$$

Then the equations of equilibrium reduce to

$$\frac{\partial T^{11}}{\partial \xi} + \coth \xi (4T^{11} - T^{22} - \eta^2 T^{33}) = 0 \tag{20}$$

This gives

$$\frac{\partial p}{\partial \xi} = \frac{\partial W}{\partial \xi} + \frac{4A^2 \cosh \xi}{c^4 \sinh^5 \xi} \left(\frac{\partial W}{\partial e_{33}} \right) - \frac{A^2}{c^4 \sinh^4 \xi} \frac{\partial}{\partial \xi} \left(\frac{\partial W}{\partial e_{33}} \right) \tag{21}$$

which, on integration, gives

$$p = W + W_0 - \frac{A^2}{c^4 \sinh^4 \xi} \left(\frac{\partial W}{\partial e_{33}} \right) \tag{22}$$

where W_0 is a constant.

From Eqns. (18) and (22), the physical components of stress are given by

$$\sigma_{11} = W + W_0$$

$$\sigma_{22} = W + W_0 + \frac{A\eta^2 + D}{A^2\eta^2} c^2 \sinh^2 \xi \left(\frac{\partial W}{\partial e_{11}} \cos^2 \phi + \frac{\partial W}{\partial e_{22}} \sin^2 \phi \right) - \frac{A^2}{c^4 \sinh^4 \xi} \frac{\partial W}{\partial e_{33}}$$

$$\sigma_{33} = W + W_0 + \frac{\eta^2 (c \sinh \xi)^2}{A\eta^2 + D} \left(\frac{\partial W}{\partial e_{11}} \sin^2 \phi + \frac{\partial W}{\partial e_{22}} \cos^2 \phi \right) - \frac{A^2}{c^4 \sinh^4 \xi} \frac{\partial W}{\partial e_{33}} \tag{23}$$

Boundary Conditions

If the inner boundary of the shell $\xi = \xi_1$ is free from traction, we must have

$$\sigma_{11} = 0 \text{ when } \xi = \xi_1$$

which on substitution in Eqn. (23) gives

$$W_0 = - W(\xi_1)$$

On the outer surface of the shell $\xi = \xi_2$, we have to apply a normal traction R given by

$$R = \sigma_{11}(\xi_2) = W(\xi_2) - W(\xi_1) \tag{24}$$

On the edges $\eta = \eta_i (i = 1, 2)$ the distribution of tractions between ϕ and $\phi + d\phi$ give rise to forces F_i and couples of moments M_i about the origin given by

$$F_i = \eta_i \int_{\xi_1}^{\xi_2} \sigma_{22} \, (c^2 \sinh^2 \xi) \, d\xi, \tag{25}$$

$$M_i = \eta_i \int_{\xi_1}^{\xi_2} \sigma_{22}(c^2 \sinh^2 \xi) \, (c \cosh \xi) \, d\xi, \quad i = 1, 2 \tag{26}$$

3. Particular Case I

Bending of an Aelotropic Incompressible Circular Block with Concentric Circular Hole into a Spherical Shell

If $c \cosh \xi_i = c \sinh \xi_i$ in Eqn. (1), we get the case of a circular block bent into a spherical shell, so that $\xi \to \infty$, $c \to 0$, and $c \cosh \xi$, $c \sinh \xi \to r$, and consequently the orthogonal curvilinear coordinates (ξ, η, ϕ) are replaced by the spherical polar coordinates (r, θ, ϕ).

Then, the Eqn. (23) reduce to

$$\sigma_{11} = W + W_0$$

$$\sigma_{22} = W + W_0 + \frac{A\theta^2 + D}{A^2\theta^2} r^2 \left(\frac{\partial W}{\partial e_{11}} \cos^2 \phi + \frac{\partial W}{\partial e_{22}} \sin^2 \phi \right) - \frac{A^2}{r^4} \frac{\partial W}{\partial e_{33}}$$

$$\sigma_{33} = W + W_0 + \frac{\theta^2 r^2}{A\theta^2 + D} \left(\frac{\partial W}{\partial e_{11}} \sin^2 \phi + \frac{\partial W}{\partial e_{22}} \cos^2 \phi \right) - \frac{A^2}{r^4} \frac{\partial W}{\partial e_{33}} \tag{27}$$

From these, the forces F_i and couples of moments M_i can be determined.

Thus, to bend an aelotropic circular block with concentric circular hole into a part of an ellipsoidal shell, we require forces F_i and couples M_i on the edges $\eta = \eta_i$, $(i = 1, 2)$, together with a normal force R on its outer surface.

4. Particular Case II

The problem of bending of a circular block into ellipsoial shell without the hole is obtained as a particular case by taking $\eta_1 = 0$, when $b_1 = 0$ from Section 3.

5. Particular Case III

The problem of bending of a circular block into a spherical shell without the concentric hole is obtained as a particular case from Section 4 by taking $\eta_1 = 0$ when $b_1 = 0$.

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