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Magneto-Elastic Torsional Waves in a Composite Non Homogeneous Cylindrical Shell under Initial Stress

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Abstract. This paper investigates magneto-elastic torsional waves in a composite non homogeneous cylindrical shell under initial stress. The non homogeneous character of the shell is due to the variable elastic constants C_{ij} and variable density ρ. The composite form of the shell is due to the combination of orthotropic elastic material and visco-elastic material of general linear type. Frequency equation for the said wave has been derived.

1. Introduction

Although in the classical theory of linear elasticity solutions of the problems are numerous for materials whose elastic co-efficients are same at all points within the body in question, there are materials where these vary considerably from point to point. The recent trend of researches concerning non homogeneous elasticity may be found in the works of Olszak[1], Gibson[2] and Huston[3]. Recently, Chakravorty[4] discussed the vibration of a cylinder of transversely isotropic material. Asthana[5] studied the propagation of torsional waves in a composite aelotropic cylinder under magnetic field. Narain[6] deduced the frequency equation of magneto-elastic torsional waves in a bar under initial stress. Sequel to these, this paper is an attempt to discuss the problem of magneto-elastic torsional waves in a composite non homogeneous cylindrical shell under initial stress. The non homogeneity of the shell is due to the variable elastic constants C_{ij} and variable density ρ. The composite form is due to the combination of orthotropic elastic material and visco-elastic material of general linear type. The basic equations are formulated with the help of Biot's Incremental theory. The, frequency equation has also been derived.

2. Problem and Fundamental Equations

Let the cylindrical shell have a and c as its inner and outer radii and b be the radius of the surface of separation. The shell is composed of orthotropic material in the region $a \leqslant r \leqslant b$ and visco-elastic material of general linear type in the region $b \leqslant r \leqslant c$. In the former region the elastic constants and density are supposed to vary as $C_{ij} = a_{ij} r^2$ and $\rho = \rho_0 r^2$ and in the latter region the rigidity and density are respectively given by μ (constant) and $\rho = \rho_0' / r^2$. The stress-strain relations in the region $a \leqslant r \leqslant b$ are given by

$$\left.\begin{aligned}
\sigma_{rr} &= C_{11}e_{rr} + C_{12}e_{\theta\theta} + C_{13}e_{zz} \\
\sigma_{\theta\theta} &= C_{12}e_{rr} + C_{22}e_{\theta\theta} + C_{23}e_{zz} \\
\sigma_{zz} &= C_{13}e_{rr} + C_{23}e_{\theta\theta} + C_{33}e_{zz} \\
\sigma_{rz} &= C_{44}e_{rz} \\
\sigma_{\theta z} &= C_{55}e_{\theta z} \\
\sigma_{r\theta} &= C_{66}e_{r\theta}
\end{aligned}\right\} \tag{1}$$

and that in the region $b \leqslant r \leqslant c$ are given by

$$\left.\begin{aligned}
\left(1 + \nu_1 \frac{\partial}{\partial t}\right) \sigma_{rr} &= 2\mu \left(1 + \nu_2 \frac{\partial}{\partial t}\right) e_{rr} \\
\left(1 + \nu_1 \frac{\partial}{\partial t}\right) \sigma_{\theta\theta} &= 2\mu \left(1 + \nu_2 \frac{\partial}{\partial t}\right) e_{\theta\theta} \\
\left(1 + \nu_1 \frac{\partial}{\partial t}\right) \sigma_{zz} &= 2\mu \left(1 + \nu_2 \frac{\partial}{\partial t}\right) e_{zz} \\
\left(1 + \nu_1 \frac{\partial}{\partial t}\right) \sigma_{rz} &= 2\mu \left(1 + \nu_2 \frac{\partial}{\partial t}\right) e_{rz} \\
\left(1 + \nu_1 \frac{\partial}{\partial t}\right) \sigma_{\theta z} &= 2\mu \left(1 + \nu_2 \frac{\partial}{\partial t}\right) e_{\theta z} \\
\left(1 + \nu_1 \frac{\partial}{\partial t}\right) \sigma_{r\theta} &= 2\mu \left(1 + \nu_2 \frac{\partial}{\partial t}\right) e_{r\theta}
\end{aligned}\right\} \tag{2}$$

where C_{11}, C_{12} ... etc. are elastic constants and ν_1, ν_2 are visco-elastic constants.

When the displacement currents are neglected, the electromagnetic field equations are

$$\left.\begin{aligned}
&\text{curl } \vec{H} = 4\pi\vec{J}, \qquad \text{curl } \vec{E} = -\frac{1}{C}\frac{\partial \vec{B}}{\partial t} \\
&\text{div } \vec{B} = 0, \qquad \vec{B} = \mu_e \vec{H}
\end{aligned}\right\} \tag{3}$$

The electromagnetic field equations in vacuum, are

$$\left(\nabla^2 - \frac{1}{C^2}\frac{\partial^2}{\partial t^2}\right)\vec{E}^* = 0$$

(equation continued on p. 127)

$$\left.\begin{aligned}&\left(\nabla^2-\frac{1}{C^2}\frac{\partial^2}{\partial t^2}\right)\vec{h}^*=0\\&\text{curl}\,\vec{E}^*=-\frac{1}{C}\frac{\partial\vec{h}^*}{\partial t}\\&\text{curl}\,\vec{h}^*=\frac{1}{C}\frac{\partial\vec{E}^*}{\partial t}\end{aligned}\right\}\tag{4}$$

and the generalized Ohm's law in the deformable medium is

$$\vec{J}=\sigma\left\{\vec{E}+\frac{1}{C}\frac{\partial\vec{u}}{\partial t}\times\vec{B}\right\}\tag{5}$$

where $\vec{J}, \vec{H}, \vec{E}, \vec{B}$ denote respectively current density vector, magnetic intensity vector, electric intensity vector, and magnetic induction vector, $\vec{u}$ is the displacement vector in the strained solid and μ_e, σ denote respectively the magnetic permeability and electric conductivity, $\vec{h}$ is the perturbation in the magnetic field. If the cylinder is under initial stress $\bar{\sigma}_{33}$ $(=P)$ along the axis of z, then the equilibrium equations are given by Narain[6] as

$$\left.\begin{aligned}&\frac{\partial}{\partial r}\sigma_{rr}+\frac{1}{r}\frac{\partial}{\partial\theta}\sigma_{r\theta}+\frac{\partial}{\partial z}\sigma_{rz}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}\\&\quad+\bar{\sigma}_{33}\frac{\partial\bar{w}_\theta}{\partial z}+F_r=\frac{\rho\partial^2u_r}{\partial t^2}\\&\times\frac{\partial}{\partial r}\sigma_{r\theta}+\frac{1}{r}\frac{\partial}{\partial\theta}\sigma_{\theta\theta}+\frac{\partial}{\partial z}\sigma_{\theta z}+\frac{2\sigma_{r\theta}}{r}\\&\quad-\bar{\sigma}_{33}\frac{\partial w_r}{\partial z}+F_\theta=\frac{\rho\partial^2u_0}{\partial t^2}\\&\times\frac{\partial}{\partial z}\sigma_{rz}+\frac{1}{r}\frac{\partial}{\partial\theta}\sigma_{\theta z}+\frac{\partial}{\partial z}\sigma_{zz}+\frac{\sigma_{rz}}{r}\\&\quad+\bar{\sigma}_{33}\left(\frac{\partial w_\theta}{\partial r}-\frac{1}{r}\frac{\partial w_r}{\partial\theta}\right)+F_z=\frac{\rho\partial^2u_z}{\partial t^2}\end{aligned}\right\}\tag{6}$$

where u_r, u_θ, u_z are components of displacement and F_r, F_θ, F_z are components of Lorentz's force $\vec{F}\,(=\vec{J}\times\vec{B})$ per unit volume due to the axial magnetic field.

This components of rotation are given by

$$\left.\begin{aligned}&w_r=\tfrac{1}{2}\left(\frac{1}{r}\frac{\partial u_z}{\partial\theta}-\frac{\partial u_\theta}{\partial z}\right)\\&w_\theta=\tfrac{1}{2}\left(\frac{\partial u_r}{\partial z}-\frac{\partial u_z}{\partial r}\right).\end{aligned}\right\}\tag{7}$$

3. Solution of the Problem

Since the cylinder is under the initial stress P along the axis of z and we are considering the case of torsional waves, we have

$$u_r = 0,\ u_z = 0,\ u_\theta = v(r, z) \tag{8}$$

and

$$\left.\begin{aligned} & e_{rr} = 0,\ e_{\theta\theta} = 0,\ e_{zz} = 0,\ e_{rz} = 0 \\ & e_{\theta z} = \frac{\partial v}{\partial z},\quad e_{r\theta} = \left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) \\ & w_r = -\tfrac{1}{2}\frac{\partial v}{\partial z},\ w_z = \tfrac{1}{2}\left(\frac{\partial v}{\partial r} + \frac{v}{r}\right) \end{aligned}\right\} \tag{9}$$

where e_{ij}'s are components of incremental strain. If we suppose that $\vec{H}_0$ is the initial magnetic field parallel to the z-axis and $\vec{h}$ is small perturbation of the field then

$$\vec{H} = \vec{H}_0 + \vec{h}$$

Again, if the rod considered is a perfect conductor ($\sigma \to \infty$) then from Eqn. (5), we have

$$\vec{E} = -\frac{1}{C}\frac{\partial \vec{u}}{\partial t} \times \vec{B} = \left(-\frac{H}{C}\frac{\partial v}{\partial t}, 0, 0\right) \tag{10}$$

where $H = |\vec{H}_0|$. Using Eqns. (3) and (10), we have

$$\vec{h} = \left[0, H\frac{\partial v}{\partial z}, 0\right] \tag{11}$$

Also we have

$$\vec{F} = \vec{J} \times \vec{B} = \left[0, -\frac{H^2 \partial^2 v}{4\pi\, \partial z^2}, 0\right]. \tag{12}$$

Using Eqns. (8), (9) and (12), we find that the first and third of Eqn. (6) are identically satisfied and the remaining second equation takes the form

$$\frac{\partial}{\partial r}\sigma_{r\theta} + \frac{\partial}{\partial z}\sigma_{\theta z} + \frac{2}{r}\sigma_{r\theta} + \frac{P}{2}\frac{\partial^2 v}{\partial z^2} - \frac{H^2}{4\pi}\frac{\partial^2 v}{\partial z^2} = \rho\frac{\partial^2 v}{\partial t^2}. \tag{13}$$

If v_1 be the displacement in the region $a \leqslant r \leqslant b$ and the elastic constants and the density vary according to the rule,

$$C_{ij} = a_{ij}\, r^2,\ \rho = \rho_0\, r^2 \tag{14}$$

in this region, then using Eqns. (1), (9), (13) and (14), we have

$$\frac{\partial^2 v_1}{\partial r^2} + \frac{3}{r}\frac{\partial v_1}{\partial r} - \frac{3v_1}{r^2} + \frac{a_{55}}{a_{66}}\frac{\partial^2 v_1}{\partial z^2} + \frac{P'}{a_{66}}\frac{1}{r^2}\frac{\partial^2 v_1}{\partial z^2} = \frac{\rho_0}{a_{66}}\frac{\partial^2 v_1}{\partial t^2} \tag{15}$$

where

$$P' = \frac{P}{2} - \frac{H^2}{4\pi}.$$

Assuming the solution of the Eqn. (15) as

$$v_1 = V_1(r)\, e^{i(qz+pt)}. \tag{16}$$

The Eqn. (15) together with Eqn. (16) gives

$$\frac{\partial^2 V_1}{\partial r^2} + \frac{3}{r}\frac{\partial V_1}{\partial r} + \left(\lambda^2 - \frac{x^2}{r^2}\right) V_1 = 0 \tag{17}$$

where

$$\lambda^2 = \frac{\rho_0 p^2}{a_{66}} - \frac{a_{55}}{a_{66}} q^2$$

$$x^2 = 3 + \frac{P' q^2}{a_{66}}.$$

Putting $V_1 = \frac{1}{r}\psi(r)$ the Eqn. (17) becomes

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \left(\lambda^2 - \frac{\nu^2}{r^2}\right)\psi = 0 \tag{18}$$

where

$$\nu^2 = x^2 + 1.$$

The solution of the Eqn. (18) is given by

$$\psi = A_1 J_\nu(\lambda r) + B_1 Y_\nu(\lambda r)$$

so that, we get

$$v_1 = \frac{1}{r}\left[A_1 J_\nu(\lambda r) + B_1 Y_\nu(\lambda r)\right] e^{i(qz+pt)} \tag{19}$$

where A_1, B_1 are constants and J_ν, Y_ν are Bessel's function of order ν and of first and second kind respectively.

If v_2 be the displacement in the region $b \leqslant r \leqslant c$ and that the density in this region be varying as

$$\rho = \frac{\rho_0'}{r^2} \tag{20}$$

where ρ_0' is constant, then using Eqns. (2), (8), (9), (13) and (20), we get

$$2\mu\left(1 + \nu_2 \frac{\partial}{\partial t}\right)\frac{\partial^2 v_2}{\partial r^2} + \frac{2\mu}{r}\left(1 + \nu_2 \frac{\partial}{\partial t}\right)\frac{\partial v_2}{\partial r} - 2\mu\left(1 + \nu_2 \frac{\partial}{\partial t}\right)\frac{v_2}{r^2}$$

(*equation continued on p.* 130)

$$+ 2\mu \left(1 + \nu_2 \frac{\partial}{\partial t}\right) \frac{\partial^2 v_2}{\partial z^2} + \left(\frac{P}{2} - \frac{H^2}{4\pi}\right) \left(1 + \nu_1 \frac{\partial}{\partial t}\right) \frac{\partial^2 v_2}{\partial z^2}$$

$$= \frac{\rho_0'}{r^2} \left(1 + \nu_1 \frac{\partial}{\partial t}\right) \frac{\partial^2 v_2}{\partial t^2}. \tag{21}$$

If we assume the solution of the Eqn. (21) as

$$v_2 = V_2(r) e^{i(qz+pt)} \tag{22}$$

then, from Eqns. (21) and (22), we find

$$\frac{\partial^2 V_2}{\partial r^2} + \frac{1}{r} \frac{\partial V_2}{\partial r} + \left(\lambda_1^2 - \frac{\mu_1^2}{r^2}\right) V_2 = 0 \tag{23}$$

where

$$\lambda_1^2 = q^2 \left\{-1 + \frac{1}{2\mu} \left(\frac{H^2}{4\pi} - \frac{P}{2}\right) \left(\frac{1 + \nu_1 ip}{1 + \nu_2 ip}\right)\right\}$$

$$\mu_1^2 = 1 - \frac{\rho_0' p^2 (1 + \nu_1 ip)}{2\mu(1 + \nu_2 ip)}$$

The solution of the Eqn. (23) is given by

$$V_2 = A_2 J_{\mu_1}(\lambda_1 r) + B_2 Y_{\mu_1}(\lambda_1 r) \tag{24}$$

where A_2, B_2 are constants and J_{μ_1}, Y_{μ_1} are Bessel functions of first and second kind and of order μ_1.

The Eqn. (22) together with the Eqn. (24) gives

$$v_1 = [A_2 J_{\mu_1}(\lambda_1 r) + B_2 Y_{\mu_1}(\lambda_1 r)] (r) e^{i(qz+pt)}. \tag{25}$$

If the expression for the material in the region $a \leqslant r \leqslant b$ be denoted by suffix 1 and for that in the region $b \leqslant r \leqslant c$ by the suffix 2, then the boundary conditions on the surface are

$$\left.\begin{aligned} (\sigma_{r\theta})_1 + (T_{r\theta})_1 - (T^*_{r\theta})_1 &= 0 \qquad \text{on } r = a \\ (\sigma_{r\theta})_2 + (T_{r\theta})_2 - (T^*_{r\theta})_2 &= 0 \qquad \text{on } r = a \end{aligned}\right\} \tag{26}$$

and the continuity of the stress-displacement and Maxwellian tensor in the shell on the surface $r = b$ when formulated are

$$\begin{aligned} v_1 &= v_2 \qquad \text{on } r = b \\ (\sigma_{r\theta})_1 &= (\sigma_{r\theta})_2 \quad \text{on } r = b \\ \text{and} \quad (T_{r\theta})_1 &= (T_{r\theta})_2 \quad \text{on } r = b \end{aligned} \tag{27}$$

where $T_{r\theta}$ and $T^*_{r\theta}$ are Maxwellian tensors in the shell and in vacuum respectively.

Using recurrence formulae Pipes & Harvill[7],

$$\left.\begin{aligned} J'_p(x) &= J_{p-1}(x) - \frac{p}{x} J_p(x) \\ &= \frac{p}{x} J_p(x) - J_{p+1}(x) \end{aligned}\right\} \tag{28}$$

$(\sigma_{r\theta})_1$ and $(\sigma_{r\theta})_2$ are given by

$$\left.\begin{aligned} (\sigma_{r\theta})_1 = a_{66}[A_1\{\lambda r J_{\nu-1}(\lambda r) - (\nu + 2) J_\nu(\lambda r)\} + B_1\{\lambda r Y_{\nu-1}(\lambda r) \\ - (\nu + 2) J_\nu(\lambda r)\}] e^{i(qz+pt)} \text{ for } a \leqslant r \leqslant b \end{aligned}\right\} \tag{29}$$

and

$$\left[\left(1 + \nu_1 \frac{\partial}{\partial t}\right) \sigma_{r\theta}\right]_2 = 2\mu(1 + \nu_2 ip)\, r^{-1} [A_2\{\lambda_1 r J_{\mu_1-1}(\lambda_1 r) - (\mu_1 + 1) J_{\mu_1}(\lambda_1 r)\} + B_2\{\lambda_1 r Y_{\mu_1-1}(\lambda_1 r) - (\mu_1 + 1) Y_{\mu_1}(\lambda_1 r)\}] e^{i(qz+pt)}$$

$$\text{for } b \leqslant r \leqslant c.$$

Since $T_{r\theta} = T^*_{r\theta} = 0$, the boundary conditions (26) and (27) give

$$A_1\{\lambda a J_{\nu-1}(\lambda a) - (\nu + 2) J_\nu(\lambda a)\} + B_1\{\lambda a Y_{\nu-1}(\lambda a) - (\nu + 2) Y_\nu(\lambda a)\} = 0 \tag{30}$$

$$A_2\{\lambda_1 c J_{\mu_1-1}(\lambda_1 c) - (\mu_1 + 1) J_{\mu_1}(\lambda_1 c)\} + B_2\{\lambda_1 c Y_{\mu_1-1}(\lambda_1 c) - (\mu_1 + 1) Y_{\mu_1}(\lambda_1 c)\} = 0 \tag{31}$$

$$A_1 J_\nu(\lambda b) + B_1 Y_\nu(\lambda b) - A_2\{b J_{\mu_1}(\lambda_1 b)\} - B_2\{b Y_{\mu_1}(\lambda_1 b)\} = 0 \tag{32}$$

$$\begin{aligned} &A_1[b a_{66}\{\lambda b J_{\nu-1}(\lambda b) - (\nu + 2) J_\nu(\lambda b)\}] \\ &\quad + B_1 [b a_{66}\{\lambda b Y_{\nu-1}(\lambda b) - (\nu + 2) Y_\nu(\lambda b)\}] \\ &\quad - A_2 \left[2\mu \frac{(1 + \nu_2 ip)}{(1 + \nu_1 ip)} \{\lambda_1 b J_{\mu_1-1}(\lambda_1 b) - (\mu_1 + 1) J_{\mu_1}(\lambda_1 b)\}\right] - B_2 \left[2\mu \frac{(1 + \nu_2 ip)}{(1 + \nu_1 ip)} \{\lambda_1 b\, Y_{\mu_1-1}(\lambda_1 b) - (\mu_1 + 1) Y_{\mu_1}(\lambda_1 b)\}\right] = 0 \end{aligned} \tag{33}$$

Thus we have four linear Eqns. (30) to (33) to determine four constants A_1, B_1, A_2, B_2 in the form of material constants. From Eqns. (30) to (33) the frequency equation is obtained as

$$\begin{vmatrix} X_{11} & X_{12} & 0 & 0 \\ 0 & 0 & X_{23} & X_{24} \\ X_{31} & X_{32} & X_{33} & X_{34} \\ X_{41} & X_{42} & X_{43} & X_{44} \end{vmatrix} = 0 \tag{34}$$

where

$$X_{11} = \lambda a\, J_{\nu-1}(\lambda a) - (\nu + 2)\, J_\nu(\lambda a)$$

$$X_{12} = \lambda a\, Y_{\nu-1}(\lambda a) - (\nu + 2) Y_\nu(\lambda a)$$

$$X_{23,4} = \lambda_1 c J,\ Y_{\mu_1-1}(\lambda_1 c) - (\mu_1 + 1) J,\ Y_{\mu_1}(\lambda_1 C)$$

$$X_{31,2} = J,\ Y_\nu(\lambda b)$$

$$X_{33,4} = -\, b\, [J,\ Y_{\mu_1}(\lambda_1 b)]$$

$$X_{41,2} = b a_{66}\, [\lambda b J,\ Y_{\nu-1}(\lambda b) - (\nu + 2)\, J,\ Y_\nu(\lambda b)]$$

$$X_{43,4} = -\, 2\mu\, \frac{(1 + \nu_2\, ip)}{(1 + \nu_1\, ip)}\, [\lambda_1 b J,\ Y_{\mu_1-1}(\lambda_1 b) - (\mu_1 + 1)\, J, Y_{\mu_1}(\lambda_1 b)]$$

Giving numerical values to $a, b, c, a_{66}, a_{55}, H, P, \rho_0, \mu, \nu_1, \nu_2, \rho_0'$ in a particular problem we can find the corresponding frequency equation in a simple form.

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References

1. Olszak, W., 'Non Homogeneity in Elasticity and Plasticity' (Pergamon Press, London), 1959, p. 171.
2. Gibson, R. E., *Geotechnique*, **17**, (1967), 58–67.
3. Huston, R. L., *Zamm.*, **44**, (1964), 573–578.
4. Chakravorty, J. G., *Proc. Nat. Inst. Sc.*, **A224**, (1956), 220.
5. Asthana, K. M., *Acta Ciencia Indica*, **1**, (1975), 219.
6. Narain, S., *Proc. Ind. Acad. Sc.*, **87**, (1976), 137–145.
7. Pipes, A. & Harvill, L. R. 'Applied Mathematics for Engineers and Physicists' (McGraw Hill', Kogakusha Ltd.), 1970, p. 790.