Free Convection Flow with Constant Heat Sources in a Porous Channel

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Abstract. The effect of constant heat sources on fully developed free convection flow of a viscous fluid in a porous channel oriented in the direction of the body force has been studied when the walls are maintained at constant temperatures. It has been found that both the velocity and temperature depend on the heat source parameter α and the dimensionless group Q representing the free convection effects.

Introduction

The free or natural convection heat transfer has received considerable attention in recent years in view of its wide applications in the design of nuclear reactor, cooling of electronic equipments, aircraft cabin insulation and thermal storage system. Ostrach[1,2] has studied the laminar natural convection flow between vertical heated plates when they are kept either at a constant temperature or the temperature varies linearly along the plate. Rao[3] analysed the corresponding problem with porous walls. Nanda and Sharma[4] have extended this to the case of flow in a circular pipe. The present work deals with fully developed free convection flow in a porous channel including the effect of frictional heating, in presence of constant heat sources or sinks distributed uniformly in the fluid. As is usual in free convection flows, a linear density temperature variations has been adopted to express the body force term as buoyancy term. The case when both the walls are maintained at the same temperature has been discussed in detail.

Formulation of the Problem

Consider the fully developed steady laminar free convection flow of a viscous incompressible fluid between two infinite parallel porous flat walls of equal permeability, oriented in the direction of the body force. The x-axis is taken along one of the walls and y axis normal to it. Since the boundaries are of infinite dimentions in x-direction, we can in general, assume that the velocity and temperature both depend only upon y. The velocity field is taken to be $(u, V, 0)$, where $V > 0$ represents injection at $y = 0$ and suction at $y = h$, h being the distance between the walls, while $V < 0$ represents the vice versa.

The equation of continuity is identically satisfied while that of motion and energy take the form

$$\rho V \frac{du}{dy} = \rho f + \mu \frac{d^2u}{dy^2} - \frac{\partial p}{\partial x} \tag{1}$$

$$o = \frac{\partial p}{\partial y} \tag{2}$$

$$\rho C_p V \frac{dT}{dy} = k \frac{d^2T}{dy^2} + \mu \left(\frac{du}{dy}\right)^2 + D \tag{3}$$

where D denotes the heat added due to constant heat sources, f the generating body force per unit mass, C_p, the specific heat at constant pressure, k the coefficient of thermal conductivity, p the pressure, μ the coefficient of viscosity and ρ, the density of the fluid.

The boundary conditions to be satisfied, are

$$\left.\begin{aligned} &u = 0,\ v = V,\ T = T_1 \quad \text{at } y = 0 \\ \text{and} \quad &u = 0,\ v = V,\ T = T_2 \quad \text{at } y = h \end{aligned}\right\} \tag{4}$$

Following Ostrach[1], the body force term can be expressed as a buoyancy term. In hydrostatic condition

$$\rho_0 f - \frac{\partial p_0}{\partial x} = 0, \tag{5}$$

Using this, the body force and pressure gradient terms in Eqn. (1) are

$$\rho f - \frac{\partial p}{\partial x} = (\rho - \rho_0) f - \frac{\partial p_1}{\partial x} \tag{6}$$

where $p_1 = p - p_0$. Further assuming a linear density temperature variation, we write, for small temperature differences,

$$\rho - \rho_0 = - \beta\rho(T - T_0) \tag{7}$$

Thus Eqn. (6) is simplified as

$$\rho f - \frac{\partial p}{\partial x} = \beta\rho(T - T_0) f_x - \frac{\partial p_1}{\partial x}, \tag{8}$$

where $f_x = - f$ and β is the coefficient of volumetric expansion. Also, a suffix 0 denotes the respective quantities in hydrostatic state.

With the help of Eqn. (8), the Eqns. (1) to (3) reduce as

$$\frac{d^2u}{dy^2} + \frac{\rho f_x \beta(T - T_0)}{\mu} - \frac{\rho V}{\mu}\frac{du}{dy} - \frac{1}{\mu}\frac{\partial p_1}{\partial x} = 0 \tag{9}$$

$$\frac{\partial p_1}{\partial y} = 0 \tag{10}$$

and

$$\frac{d^2(T-T_0)}{dy^2}+\frac{\mu}{k}\left(\frac{du}{dy}\right)^2-\frac{\rho C_p V}{k}\frac{d}{dy}(T-T_0)+\frac{D}{k}=0 \tag{11}$$

Dropping the pressure term as in Ostrach[1], and making use of the transformations

$$y=\eta h,\ u=\frac{k}{\rho f_x \beta h^2}U,\ T-T_0=\frac{ku}{\rho^2 f_x^2\beta^2 h^4}T^*, \tag{12}$$

The Eqns. (9) to (11) now transform to

$$\frac{d^2U}{d\eta^2}+T^*-R\frac{dU}{d\eta}=0 \tag{13}$$

and

$$\frac{d^2T^*}{d\eta^2}+\left(\frac{dU}{d\eta}\right)^2-R\sigma\frac{dT^*}{d\eta}+Q\alpha=0 \tag{14}$$

where $R(=Vh/\nu)$ and $\sigma(=\mu C_p/k)$ are respectively the suction Reynolds number and Prandtl number. Also, $Q(=\sigma G\beta f_x h/C_p)$ is a dimensionless group where $G(=\beta f_x h^3(T_1-T_0)/\nu^2)$ is the Grashof's number and $\alpha\left(=\frac{Dh^2}{k(T_1-T_0)}\right)$ is the heat source parameter.

It may be pointed out that as the velocity field depends on the temperature as well, so the frictional heating term in Eqn. (14) must not be neglected, and thus the solution consists in solving the set of simultaneous differential equations (13) and (14). The boundary conditions (4) become

$$\left.\begin{array}{l} U(0)=0,\ U(1)=1, \\ \text{and}\quad T^*(0)=Q,\ T^*(1)=mQ, \end{array}\right\} \tag{15}$$

where $m=\left(\frac{T_2-T_0}{T_1-T_0}\right)$ is another dimensionless group with T_1 and T_2 representing temperature on the lower and upper walls respectively.

Eliminating T^* between (13) and (14), we get

$$U^{iv}-R(1+\sigma)U'''+\sigma R^2U''-(U')^2-Q\alpha=0 \tag{16}$$

The boundary conditions (15) in terms of velocity can be rewritten as

$$\left.\begin{array}{l} U(0)=0,\qquad U(1)=0, \\ -U''(0)+RU'(0)=Q \\ -U''(1)+RU'(1)=mQ \end{array}\right\} \tag{17}$$

Solution

The equation (16) subject to the boundary conditions (17) is solved by the method of successive approximations. In order to develop the method, consider the equation

$$U_p^{iv} - R(1+\sigma)U_p''' + \sigma R^2 U_p'' - (U_{p-1}') - Q\alpha = 0 \tag{18}$$

where U_p denotes the p^{th} order approximation to the value of U, satisfying the boundary conditions (17). The zeroth order approximation corresponds to $p = 0$ and is obtained by neglecting the viscous dissipation term in Eqn. (18). The fourth order ordinary linear differential equation in U_0 thus obtained is solved by the usual method. The corresponding solution is

$$U_0 = \frac{1}{R^2\sigma}\left[\frac{(1+\sigma+\sigma^2)Q\alpha}{R^2\sigma^2} + \frac{(1+\sigma)a_1}{R\sigma} + a_2 + \left\{a_1 + \frac{Q\alpha(1+\sigma)}{R\sigma}\right\}\eta + \frac{Q\alpha}{2}\eta^2\right] + a_3 e^{R\eta} + a_4 e^{R\sigma\eta} \tag{19}$$

The next higher approximation which includes the effect of frictional heating also, is given by the differential equation

$$U_1^{iv} - R(1+\sigma)U_1''' + \sigma R^2 U_1'' = (U_0')^2 + Q\alpha.$$

Accordingly the solution is

$$\begin{aligned}U_1 = {} & b_3 e^{R\eta} + b_4 e^{R\sigma\eta} + \frac{a_3^2 e^{2R\eta}}{4R^2(2-\sigma)} + \frac{a_4^2 e^{2R\sigma\eta}}{4R^2\sigma(2\sigma-1)} + \frac{2a_3a_4 e^{R(1+\sigma)\eta}}{R^2(1+\sigma)^2} \\ & + \frac{2a_3}{R^5\sigma(1-\sigma)}\left\{R\left(\frac{Q\alpha\eta^2}{2} + K_{1\eta}\right) - 2Q\alpha\eta - \frac{Q\alpha\eta}{1-\sigma}\right\}e^{R\eta} \\ & - \frac{2a_4 e^{R\sigma\eta}}{R^5\sigma^3(1-\sigma)}\left\{R\sigma\left(\frac{Q\alpha\eta^2}{2} + K_1\eta\right) - 2Q\alpha\eta + \frac{Q\sigma\alpha\eta}{1-\sigma}\right\} \\ & + \frac{1}{\sigma R^2}\left[b_2 + \frac{b_1(1+\sigma)}{R\sigma} + d + \left\{b_1 + \frac{Q\alpha(1+\sigma)}{R\sigma}\right\}\eta + \frac{Q\alpha\eta^2}{2}\right. \\ & + \frac{2Q\alpha}{R^7\sigma^5}(1+\sigma+\sigma^2+\sigma^3).(Q\alpha\eta + K_1) + \\ & \frac{(1+\sigma+\sigma^2)}{R^6\sigma^4}(Q\alpha\eta + K_1)^2 + \frac{(1+\sigma)\left(\frac{Q^2\alpha^2}{3}\eta^3 + Q\alpha K_1\eta^2 + K_1^2\eta\right)}{R^5\sigma^3} \\ & \left. + \frac{\left(\frac{Q_2\alpha^2\eta^4}{12} + \frac{Q\alpha K_1\eta^3}{3} + \frac{K_1^2\eta^2}{2}\right)}{R^4\sigma^2}\right]\end{aligned} \tag{20}$$

where, $K_1 = a_1 + \dfrac{Q\alpha(1+\sigma)}{R\sigma}$; $d = \dfrac{Q\alpha}{R^2\sigma^2}(1+\sigma+\sigma^2) + \dfrac{2Q^2\alpha^2}{R^8\sigma^6}$

$$\times (1+\sigma+\sigma^2+\sigma^3+\sigma^4)$$

and $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$, are constants determined under the boundary conditions (17).

The first order approximate to the temperature is

$$T_1^* = - U_1'' + RU_1'$$

$$= R^2\sigma(1-\sigma)\, b_4 e^{R\sigma\eta} - \frac{a_3^2 e^{2R\eta}}{2(2-\sigma)} - \frac{a_4^2}{2} e^{2R\sigma\eta} - \frac{2a_3a_4\sigma\, e^{R(1+\sigma)\eta}}{(1+\sigma)}$$

$$+ \frac{2a_3 e^{R\eta}}{R^4\sigma(1-\sigma)}\left\{- R(Q\alpha\eta + K_1) + Q\alpha + \frac{Q\alpha}{1-\sigma}\right\}$$

$$+ \frac{2a_4 e^{R\sigma\eta}}{R^4\sigma^3(\sigma-1)}\Bigg[R\sigma\left\{R\sigma\left(\frac{Q\alpha\eta^2}{2} + K_1\eta\right) - 2Q\alpha\eta + \frac{Q\alpha\sigma\eta}{1-\sigma}\right\}(1-\sigma)$$

$$+ \frac{Q\alpha\sigma(1-2\sigma)}{1-\sigma} + R\sigma(Q\alpha\eta + K_1)(1-2\sigma) - Q\alpha(2-3\sigma)\Bigg]$$

$$+ \frac{1}{R^2\sigma}\Bigg[Rb_1 + \frac{Q\alpha}{\sigma} + RQ\alpha\eta + \frac{2Q^2\alpha^2}{R^6\sigma^5} + \frac{2Q\alpha}{R^5\sigma^4}(Q\alpha\eta + K_1)$$

$$+ \frac{(Q\alpha\eta + K_1)^2}{R^4\sigma^3} + \frac{\left(\frac{Q^2\alpha^2}{3}\eta^3 + Q\alpha K_1\eta^2 + K_1^2\eta\right)}{R^3\sigma^2}\Bigg] \tag{21}$$

Skin Friction and Nusselt Number

The skin friction at the walls can be calculated from

$$S_f(0,1) = \frac{\mu K}{\rho f_x \beta h^3} = \left(\frac{dU}{d\eta}\right)_{(0,1)} \tag{22}$$

And the Nusselt numbers on the walls are given by

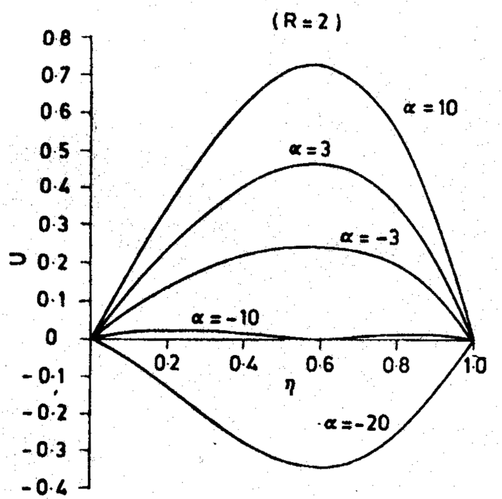

Figure 1. Variation of U with η for different α ($Q = 3.0$).

$$Nu_{(0,1)} = \frac{1}{(m-1)Q}\left(\frac{dT^*}{d\eta}\right)_{(0,1)}, \; m \neq 1$$

$$= \frac{1}{Q}\left[\frac{dT^*}{d\eta}\right]_{(0,1)}, \; m = 1$$

Results and Discussion

The non-dimensional velocity and temperature have been calculated for $\alpha = 10, 3, -10, -20$, $Q = 1, 2, 3, 4, 5$ and $R = \pm 1, \pm 2$, for $\sigma = 0.75$ and $m = 1$. But to

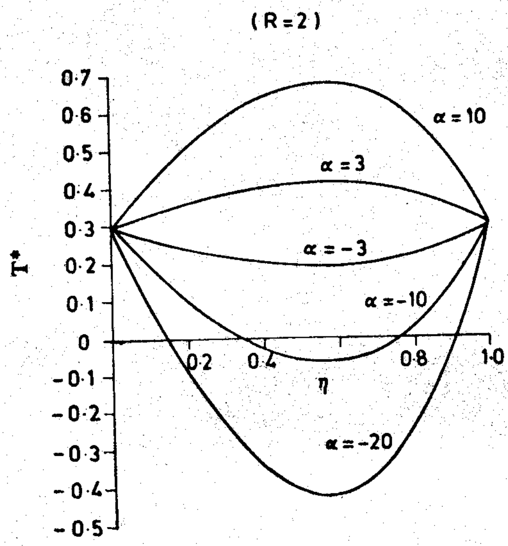

Figure 2. Variation of T^* with η for different α ($Q = 3.0$).

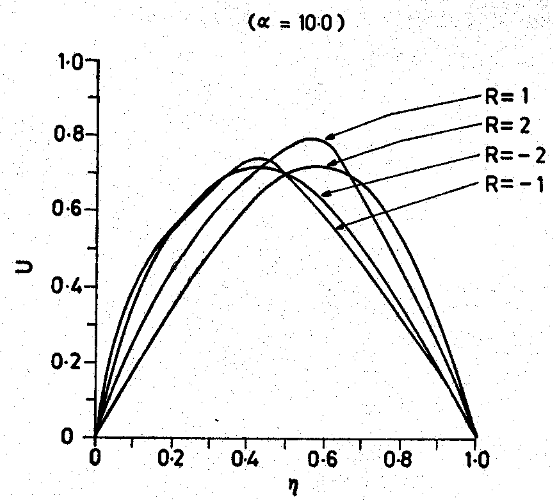

Figure 3. Variation of U with η for different R ($Q = 3.0$).

Table 1. Skin Friction on the Walls.

$Q = 3.0$

$R \backslash \alpha$	$(\rho f_x \beta h^3 S_f/\mu k)_{\eta=0}$				$(\rho f_x \beta h^3 S_f/\mu k)_{\eta=1}$			
	10.0	3.0	−10.0	−20.0	10.0	3.0	−10.0	−20.0
− 1.0	3.27000000	2.2120000	−0.060000000	−1.39000000	−2.29100000	−1.57390000	−0.37100000	0.67000000
− 2.0	3.772610	2.521288	0.26056000	−1.42680000	−1.770340	−1.256888	−0.32906200	0.36161000
+ 1.0	2.23300000	1.562100	0.33600000	−0.95000000	−3.40000000	2.2520000	−0.25000000	1.04000000
2.0	1.77016000	1.256883	0.32891200	−0.36218000	−3.77314000	−2.521278	−0.26111000	1.42380000

Table 2. Nusselt Number on the Walls.

$Q = 3.0$

$R \backslash \alpha$	$(QNu)_{\eta=0}$				$(QNu)_{\eta=1}$			
	10.0	3.0	−10.0	−20.0	10.0	3.0	−10.0	−20.0
− 1.0	6.22633330	1.93660000	−5.61500000	−11.07666600	−4.75900000	−1.47810000	4.37900000	8.66666660
− 2.0	6.844460	2.13461600	−6.20416330	−12.25870000	−4.0584200	−1.25470160	3.79242660	7.53329330
1.0	4.76366600	1.47723330	−4.37866660	− 8.66000000	−6.21166660	−1.937100	5.61300000	11.08666600
2.0	4.05851660	1.25470300	−3.79239330	− 7.53346000	−6.84417000	−2.13461030	6.20420660	12.25753300

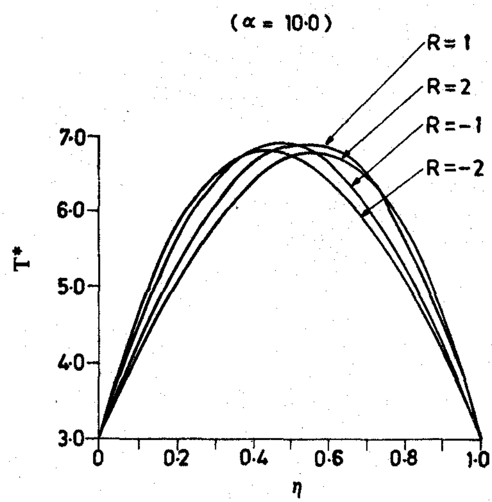

Figure 4. Variation of T^* with η for different R.

conserve space the graphs and tables have been given for $Q = 3$, only. Here, the negative value of α corresponds to the case of sinks.

Figs. 1 and 2 exhibit the behaviour of velocity and temperature for varying α and a fixed R while Figs. 3 and 4 depict the behaviour for varying R and fixed α.

It can be seen that both velocity as well as temperature increase with increase in heat source parameter, reaching a maximum almost in the middle. For large sinks a complete reversal of flow pattern is observed and a cooling takes place between the gap which is more and more pronounced with increase in $-\alpha$. For $\alpha = 0$, i.e., the case without heat sources, our results reduce to that of Rao[3] and thus, completely match with them.

The skin friction and Nusselt number have been calculated for different α and R and given in Tables 1 and 2.

It is found that the Nusselt number at the lower wall increases with increase in α for increasing Q. At the upper wall the behaviour is completely reversed. It is evident from Table 1, that the behaviour of skin friction is quite similar to that of Nusselt number.

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