

Note on a Viscoelastic Convection Flow

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Abstract. A mathematical analysis of the free convection flow of an incompressible viscoelastic (Rivlin-Ericksen) fluid from an infinite flat plate under variable suction acted upon by a transverse magnetic field is presented. It is found that the effect of viscoelasticity is to appreciably alter the skin-friction.

1. Introduction

Lighthill¹ discussed the unsteady laminar fluid flow in boundary layers. Later Messiha² considered the free convection flow problem for the laminar boundary layers pertaining to a vertical flat plate with constant suction and an extension to the hydro-magnetic case was taken up by Pop³. Then Krishna Lal⁴ extended the above problem to variable suction for the oscillatory-type motion. Further, Krishna Lal⁵ mathematically analysed the magnetic boundary layer equations, using linear and spiral group transforms after obtaining similarity solutions.

Harinath & Ramesan⁶ considered an unsteady motion of an electrically conducting incompressible viscoelastic fluid through a porous medium bounded by two infinite parallel plates in the presence of a transverse magnetic field and extended certain results on the oscillatory motion of a homogeneous isotropic Rivlin-Ericksen (viscoelastic) fluid between parallel plates considered earlier by Siddappa & Shanker Hegde⁷. The problem of an unsteady viscoelastic flow through a circular pipe was solved by Harinath, Madegowda & Ramesan⁸ and various extensions are currently under progress.

For some other viscoelastic flows we refer to Verma, Choudhry & Rajvanshi⁹ and Dubey & Sharma¹⁰ and for a convection flow past a porous wall, we refer to Siddappa & Bujurke¹¹. A comprehensive account of several theories of applied mechanics may be found in the treatise by Duvaut and Lions¹². The basis for the theory outlined here is the classic paper by Rivlin & Ericksen¹³.

The aim of this note is to undertake the mathematical analysis of a free convection flow of an incompressible Rivlin-Ericksen fluid from an infinite plate under variable suction acted upon by a transverse magnetic field. The problem considered has immense applications in various branches of engineering and sciences, such as, soil

mechanics, hydraulics, ceramics, powder metallurgy, etc. and to geophysics, mainly in the study of geopressurized regions in the earth's crust.

2. Basic Equations

Suppose we consider an infinite vertical flat plate in an incompressible viscoelastic fluid. We set up a rectangular cartesian coordinate system (x, y, z) in such a manner that the z -axis is along the vertical, the x -axis lies along the plate and the y -axis is perpendicular to the plate. Initially, the entire media is maintained at a constant reference temperature T_0 and a uniform magnetic field of strength B_0/p , where p is the magnetic permeability of the fluid and B_0 the initial magnetic induction is applied perpendicular to the plate. We use the constitutive relations due to Rivlin and Ericksen & assume that the cross-viscosity coefficient is small. In addition the electrical conductivity σ is assumed to be quite small, so that the governing equations are further simplified.

If U denotes the component of velocity along the plate and V is the component of velocity perpendicular to the plate, the governing uni-dimensional differential equations describing the free convection flow of a viscoelastic incompressible fluid with magnetic field past an infinite vertical flat plate are:

$$U_{,x} + V_{,y} = 0 \quad (1)$$

$$T_{,t} + vT_{,y} = kT_{,yy} \quad (2)$$

$$\rho U_{,t} + \rho v U_{,y} = \rho \eta g T - \sigma B_0^2 U + \alpha U_{,yy} + \beta U_{,yyt} \quad (3)$$

where T is the temperature deviation in the boundary layer from T_0 and v denotes the suction velocity. $\rho, k, \eta, \alpha, \beta$ respectively represent the density, the coefficient of thermal conductivity the coefficient of volume expansion, the coefficient of viscosity, the coefficient of viscoelasticity; g denotes the acceleration due to gravity, t the time variable and commas indicate partial differentiation. It may be noted that if gravity effects are ignored, then there would be no coupling between the temperature and velocity fields. In case we assume that the normal component of velocity V is independent of y , then $V = -\gamma v(t)$ is a plausible solution, where $\gamma > 0$ is a measure of the suction velocity. We assume that V is independent of y , so that the equation of continuity (1) now has the form: $U_{,x} = 0$.

3. Solutions

The unknown functions in this problem are the coupled functions—the temperature $T(y, t)$ and the velocity $U(y, t)$ along the plate. The velocity of suction $v(t)$ is assumed to be a variable quantity given as a finite sum

$$v(t) = 1 + A\epsilon e^{i\omega t} + \dots + A^n \epsilon^n e^{ni\omega t} \quad (4)$$

where A is a constant, ω is a frequency parameter, $i = (-1)^{1/2}$ and ϵ is the perturbation parameter, very small compared to unity.

In terms of $\epsilon \ll 1$, we express the unknown functions $T(y, t)$ and $U(y, t)$ in the boundary layer as the finite sums

$$T(y, t) = T_0(y) + T_1(y) \epsilon e^{i\omega t} + \dots + T_n(y) \epsilon^n e^{n i \omega t} \quad (5)$$

$$U(y, t) = U_0(y) + U_1(y) \epsilon e^{i\omega t} + \dots + U_n(y) \epsilon^n e^{n i \omega t} \quad (6)$$

using self-explanatory notations.

The various $2n + 2$ unknowns $T_0(y), \dots, T_n(y)$ and $U_0(y), \dots, U_n(y)$ occurring in (5) and (6) are subject to the following boundary conditions

$$(i) \text{ whenever } y = 0, U = 0 \text{ and } T = 1 + \epsilon e^{i\omega t} + \dots + \epsilon^n e^{n i \omega t}$$

$$(ii) \text{ as } y \rightarrow \infty, T \rightarrow 0 \text{ and } U \rightarrow 0$$

in terms of the temperature and the velocity.

Substitution of Eqn. (5) in the differential equation (2) and equating the various powers of ϵ leads to $n + 1$ second order ordinary differential equations which have to be solved successively using the boundary conditions. After routine calculations we arrive at the following results

$$T_0(y) = e^{-s_0(y)}$$

where

$$s_0 = \gamma/K > 0 \quad (7)$$

$$T_1(y) = \left(\frac{iA\gamma^2}{\omega K} e^{-s_0 y} - \frac{iA\gamma^2 - \omega K}{\omega K} e^{-s_1 y} \right)$$

where

$$s_1 = \frac{1}{2K} [\gamma + \sqrt{\gamma^2 + 4i\omega K}] \quad (8)$$

$$T_2(y) = (C_0 e^{-s_0 y} - C_1 e^{-s_1 y} + C_2 e^{-s_2 y}) \quad (9)$$

where

$$s_2 = \frac{1}{2K} [\gamma + \sqrt{\gamma^2 + 8i\omega K}]$$

$$C_0 = \frac{i\gamma^2 A^2}{2\omega^2 K^2} (\omega K + i\gamma^2)$$

$$C_1 = \frac{i\gamma A}{2\omega^2 K^2} (i\gamma^2 A - \omega K) [\gamma + \sqrt{\gamma^2 + 4i\omega K}]$$

$$C_2 = \frac{1}{2\omega^2 K^2} [2\omega^2 K^2 + i\gamma A (i\gamma^2 A - \omega K) [\gamma + \sqrt{\gamma^2 + 4i\omega K}] - i\gamma^2 A^2 (\omega K + i\gamma^2)]$$

The other functions $T_a(y), \dots$ may be successively calculated. Hence $T(y, t)$ is determined.

Substituting Eqns. (5) and (6) in Eqn. (3) and equating powers of ϵ , we obtain $n + 1$ second order differential equations for $U_0(y)$, ..., $U_n(y)$. Solving these under the indicated boundary conditions and the above obtained solutions (7), (8), (9) we are lead to the solutions

$$U_0(y) = K_0 e^{-s_0 y} - K_0 e^{-s_0^* y} \quad (10)$$

where

$$s_0^* = \frac{\rho\gamma}{\alpha} + \sqrt{\frac{\rho^2\gamma^2}{\alpha^2} + \frac{4\sigma B_0^2}{\alpha}} > 0, \quad K_0 = \frac{\rho\delta\eta K^2}{\sigma K^2 B_0^2 - \gamma^2\alpha}$$

$$U_1(y) = D_0 e^{-s_0 y} + D_0^* e^{-s_0^* y} + D_1 e^{-s_1 y} + D_1^* e^{-s_1^* y} \quad (11)$$

where

$$s_1^* = \frac{\rho\gamma}{\alpha + i\omega\beta} + \sqrt{\frac{\rho^2\gamma^2}{(\alpha + i\omega\beta)^2} + \frac{4(\sigma B_0^2 + i\omega\rho)}{\alpha + i\omega\beta}}$$

$$D_0 = \frac{\rho\gamma^2 A}{\omega K f(\gamma/K)} (\omega K - i\gamma\eta) \quad D_0^* = -\frac{1}{f(s_0^*)} \cdot \frac{\rho^2 g \eta \gamma A K^2 s_0^*}{(\sigma K^2 B_0^2 - \gamma^2\alpha)}$$

$$D_1 = \frac{\rho g \eta}{\omega K f(s_1)} (iA\gamma^2 - \omega K) \quad D_1^* = -D_0 - D_0^* - D_1$$

$$f(s) = s^2 - \frac{\rho\gamma}{\alpha + i\omega\beta} s - \frac{\sigma B_0^2 + i\omega\rho}{\alpha + i\omega\beta}$$

Proceeding in the same manner we can calculate $U_2(y)$, ... and finally the velocity $U(y, t)$. This therefore completes the theoretical calculations pertaining to the solutions of the temperature and the velocity.

4. Discussion of Results

The rate of heat transfer Q from the plate to the fluid and the skin-friction coefficient calculated from the shearing stress τ , have the following expressions

$$\dot{Q} = -(KT, y)|_{y=0} \quad \tau = (\alpha U, y)|_{y=0} \quad (12)$$

These are calculated by using the above solutions as

$$\begin{aligned} \dot{Q} = & \gamma + \frac{1}{2\omega K} [2iA\gamma^3 - (iA\gamma^2 - \omega K) (\gamma + \sqrt{\gamma^2 + 4i\omega K})] \epsilon e^{t\omega t} \\ & + K [s_0 C_0 - s_1 C_1 + s_2 C_2] \epsilon^2 e^{2i\omega t} + O(\epsilon^3) \end{aligned} \quad (13)$$

$$\begin{aligned} \tau &= \alpha K(\gamma/K - s_0^*) \\ &+ \alpha \left[AK \left(\frac{\gamma}{K} - s_0^* \right) + \frac{\gamma}{K} D_0 + s_0^* D_0^* + s_1 D_1 + s_1^* D_1^* \right] \epsilon e^{i\omega t} \\ &+ O(\epsilon^2) \end{aligned} \tag{14}$$

For $|\omega| \gg \gamma$ Eqns. (13) and 14 yield the following approximations in which the contributions due to viscosity, viscoelasticity, magnetic field, gravity are clearly pronounced

$$\begin{aligned} \dot{Q} &\approx \gamma + \left[\frac{iA\gamma^3}{\omega K} + (1+i) \sqrt{\frac{\omega K}{2}} \right] \epsilon e^{i\omega t} \\ &+ \left[\frac{i\gamma^3 A^2}{2\omega K} - \sqrt{2} \cdot \gamma A + (1+i) \sqrt{\omega K} - i\sqrt{2} + \frac{(1-i)\gamma^2 A^2}{2\sqrt{\omega K}} \right] \\ &\times \epsilon^2 e^{2i\omega t} \end{aligned} \tag{15}$$

$$\begin{aligned} \tau &\approx \frac{Ag\eta\rho K^2\alpha}{(\sigma K^2 B_0^2 - \gamma^2\alpha)} \left(\frac{\gamma}{K} - \frac{\rho\gamma}{\alpha} - \sqrt{\frac{\rho^2\gamma^2}{\alpha^2} + \frac{4\sigma B_0^2}{\alpha}} \right) \\ &+ \left[\frac{\rho K^2 g \alpha \eta A}{(\sigma K^2 B_0^2 - \gamma^2\alpha)} \left(\rho + \frac{\gamma}{K} - \frac{\rho\gamma}{\alpha} - \sqrt{\frac{\rho^2\gamma^2}{\alpha^2} + \frac{4\sigma B_0^2}{\alpha}} \right) + \rho g \eta \alpha (1-i) \sqrt{\frac{K}{2\omega}} \right. \\ &\quad \left. - \frac{\rho^2 g \eta \gamma \alpha^2 K^2 A}{\sigma K^2 B_0^2 - \gamma^2\alpha} \frac{1}{[\rho\gamma + \sqrt{\rho^2\gamma^2 + 4\sigma B_0^2\alpha}]} \right. \\ &\quad \left. + \frac{2i\rho^{3/2}g\alpha\eta}{\sqrt{\beta}} \left\{ (1-i) \sqrt{\frac{K}{2\omega}} - \frac{\rho K}{(\sigma K^2 B_0^2 - \gamma^2\alpha)} \right. \right. \\ &\quad \left. \left. + \frac{1}{(\sigma K^2 B_0^2 - \gamma^2\alpha) (\rho\gamma + \sqrt{\rho^2\gamma^2 + 4\sigma B_0^2\alpha})} \right\} \right] \\ &\times \epsilon e^{i\omega t} \end{aligned} \tag{16}$$

In equations (15) and (16), we note that the viscosity occurs as a factor in most of the terms. Since electrical conductivity is assumed to be small, the effects of viscosity and viscoelasticity are primary while the magnetic effects are secondary.

In the case of constant (magnetic) suction, some of the above expressions take neater forms, but still the heat-transfer and the skin-friction depend on the various parameters.

5. Conclusions

From equation (15), we observe that as $|\omega|$ increases, \dot{Q} decreases. i.e. the rate of heat transfer diminishes. Also from (15) we note that γ and \dot{Q} increase or decrease together. It is clear from Eqn. (16) that the effect of viscosity as well as viscoelasticity is to appreciably alter the shearing frictional force in the boundary layer. The expressions in (16) also reveal the fact that for highly viscous fluids in which the viscoelastic coefficients

are significant, the skin-friction coefficient is almost negligible for large $|\omega|$ and that the magnetic effects are secondary when compared to the viscoelastic effects, even though there is coupling of electrical conductivity, magnetic induction and viscoelasticity. Moreover, the expressions (15) and (16) are easily amenable for numerical computations.

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