# CERTAIN GENERALISATIONS OF BARTHOLOMEW'S PROBLEM IN LIFE TESTING

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Certain generalisations of the Bartholomew's problem in Life Testing have been discussed namely when a number of items are placed on test at different times, when the life of items follows normal distribution and when items with two components are placed on test.

## BARTHOLOMEW'S PROBLEM

Bartholomew[1] has discussed a life test experiment in which all the items are placed on test at different times depending on their availability. Thus when the experiment is terminated at a particular time, either we know the life of an item if it has failed by that time or we know the time since it is on test if it has not failed. He has discussed the estimation of the average life of the items assuming that the life of items follows exponential distribution.

## ITEMS PLACED ON TEST AT DIFFERENT TIMES

First we discuss the generalisation when instead of placing one item at one time, we place a number of items at one time. In such an experiment $N_1$ items are placed initially and samples of $N_2, N_3, \ldots, N_k$ (say) items are placed on test at different times so that when the experiment is finally terminated at a certain time we know the times $T_i$ $(i = 1, 2, \ldots k)$ since these samples of $N_i$ items are on test.

Let $n$ be the number of items that fail during the experiment and $r$ the number of items that survived. Among the $r$ items that survived let $r_i$ be from the $i$th sample of $N_i$ items. Then if $f(t)$ be the probability density function of the variable representing the life of the items and $F(t)$ the distribution function, the likelihood function of the sample arising as a result of the experiment is

$$P(s) = \prod_{j=1}^{n} f(t_j) \prod_{i=1}^{k} [1 - F(T_i)]^{r_i} \tag{1}$$

where $t_1, t_2, \ldots, t_n$ are the failure times of the $n$ items.

Now we shall discuss two cases—

(a) when the life of items follows exponential distribution

(b) when the life of items follows normal distribution

### (a) *Exponential Distribution*

Let $$f(t) = \alpha \exp(-\alpha t), \; t > 0$$

and thus

$$P(s) = \alpha^n \exp\left(-\alpha \sum_{j=i}^{n} t_j\right) \exp\left(-\alpha \sum_{i=1}^{k} r_i T_i\right) \tag{2}$$

This yields the maximum likelihood estimate of $\alpha$ as

$$n \Big/ \left(\sum_{j=1}^{n} t_j + \sum_{i=1}^{k} r_i T_i\right)$$

with asymptotic variance as $\alpha^2/n$.

*(b) Normal Distribution*

Let

$$f(t) = \frac{1}{\sigma (2\pi)^{\frac{1}{2}}} \exp\left[-1/2 \left(\frac{t-\mu}{\sigma}\right)^2\right]$$

where $\mu$ and $\sigma$ are respectively the mean and standard deviation of the variable representing the life of the items.

Then $$F(t) = \frac{1}{\sigma (2\pi)^{\frac{1}{2}}} \int_{-\infty}^{t} \exp\left[-1/2\left(\frac{t-\mu}{\sigma}\right)^2\right] dt$$

or we can write as

$$F\left(\frac{t-\mu}{\sigma}\right) = F(\xi) = \int_{-\infty}^{\xi} \phi(t)\, dt$$

where

$$\phi(t) = \frac{1}{(2\pi)^{\frac{1}{2}}} \exp(-1/2t^2)$$

and thus

$$L = \log P(s) = -n \log \sigma - 1/2 \sum_{j=1}^{n} \left(\frac{t_j-\mu}{\sigma}\right)^2 + \sum_{i=1}^{k} r_i \log[1-F(\xi_i)] \tag{3}$$

This yields the maximum likelihood equations for estimating $\mu$ and $\sigma$ as,

$$\left.\begin{aligned} \frac{\partial L}{\partial \mu} &= \frac{n}{\sigma}\left[\frac{\bar{t}-\mu}{\sigma} + \frac{1}{n}\sum_{i=1}^{k} r_i z_i\right] = 0 \\ \frac{\partial L}{\partial \sigma} &= \frac{1}{\sigma}\left[\sum_{j=1}^{n}\left(\frac{t_j-\mu}{\sigma}\right)^2 - n + \sum_{i=1}^{k} r_i \xi_i z_i\right] = 0 \end{aligned}\right\} \tag{4}$$

where $z_i$ is the failure rate defined as

$$z_i = z(\xi_i) = \frac{\phi(\xi_i)}{1 - F(\xi_i)} \tag{5}$$

To solve (4), Newton's method may be used. It is an iterative method based on the Taylor's series expansion of (4) in the neighbourhood of their simultaneous solution. Assuming $\mu_0$, $\sigma_0$ to be the approximate solution of (4), let

$$\hat{\mu} = \mu_0 + u \text{ and } \hat{\sigma} = \sigma_0 + v$$

where $u$ and $v$ are corrections to be determined by the iterative process. Using Taylor's theorem and neglecting second and higher powers of $u$ and $v$, we have

$$\left.\begin{aligned} u \frac{\partial^2 L}{\partial \mu^2_0} + v \frac{\partial^2 L}{\partial \mu_0 \, \partial \sigma_0} &= -\frac{\partial L}{\partial \mu_0} \\ u \frac{\partial^2 L}{\partial \mu_0 \, \partial \sigma_0} + v \frac{\partial^2 L}{\partial \sigma^2_0} &= -\frac{\partial L}{\partial \sigma_0} \end{aligned}\right\} \tag{6}$$

To solve these equations for $u$ and $v$, we differentiate (4) once more to get,

$$\left.\begin{aligned} \frac{\partial^2 L}{\mu^2} &= -\frac{n}{\sigma^2}\left[1 + 1/n \sum_{i=1}^{k} r_i A_i\right] \\ \frac{\partial^2 L}{\mu \, \partial \sigma} &= -\frac{n}{\sigma^2}\left[\frac{2(\bar{t} - \mu)}{\sigma} + 1/n \sum_{i=1}^{k} r_i B_i\right] \\ \frac{\partial^2 L}{\partial \sigma^2} &= -\frac{1}{\sigma^2}\left[\frac{3 \sum\limits_{j=1}^{n} (t_j - \mu)^2}{\sigma^2} - n + \sum_{i=1}^{k} r_i C_i\right] \end{aligned}\right\} \tag{7}$$

where

$$A_i = Z_i (Z_i - \xi_i)$$
$$B_i = Z_i + \xi_i A_i$$
$$C_i = \xi_i (Z_i + B_i)$$

Now $u$ and $v$ can be obtained by substituting $\mu = \mu_0$ and $\sigma = \sigma_0$ in (7) and then solving (6). The number of iterations will depend on the approximate solutions.

## ITEMS WITH TWO COMPONENTS

Now we discuss the extension of the above generalisation of Bartholomew's problem to a case where items have got two components each. Let the two components be $A$ and $B$.

We shall discuss the estimation of parameters of two models for the joint probability density function of the random variables $x$ and $y$ representing lives of the two components. The models relate to the situations :

*(i) When Failure Rate of Each Component does not Change with Time*

Let

$n_1 =$ the number of items in which $A$ failed first,

$n_2 =$ the number of items in which $B$ failed first,

$r =$ the number of items in which only $A$ failed,

$s =$ the number of items in which only $B$ failed, and

$n' =$ the number of items in which none failed.

Further out of the sample of $N_i$ items placed on test let $n_{1i}$ be the number of items in which $A$ failed first, $n_{2i}$ be the number of items in which $B$ failed first, $r_i$ be the number of items in which only $A$ failed, $s_i$ be the number of items in which only $B$ failed and $n'_i$ be the number of items in which none failed.

Thus

$$\sum_{i=1}^{k} r_i = r, \quad \sum_{i=1}^{k} s_i = s, \quad \sum_{i=1}^{k} n_{1i} = n_1, \quad \sum_{i=1}^{k} n_i' = n' \quad \text{and} \quad \sum_{i=1}^{k} n_{2i} = n_2$$

Then assuming the following model which is due to Freund[2],

$$f(x, y) = \begin{cases} \alpha\beta' \exp\left[-(\alpha+\beta-\beta')\, x - \beta' y\right], & 0 < x < y \\ \alpha'\beta \exp\left[-(\alpha+\beta-\alpha')\, y - \alpha' x\right], & 0 < y < x \end{cases}$$

the likelihood function of the sample arising as a result of the experiment can be written as

$$\begin{aligned} P(s) = {} & (\alpha\,\beta')^{n_1} \exp\left[-(\alpha+\beta-\beta') \sum^{n_1} x - \beta' \sum^{n_1} y\right] \cdot \\ & \cdot\, (\alpha'\,\beta)^{n_2} \exp\left[-(\alpha+\beta-\alpha') \sum^{n_2} y - \alpha' \sum^{n} x\right] \cdot \\ & \cdot\, \alpha^{r} \exp\left[-(\alpha+\beta-\beta') \sum^{r} x - \beta' \sum^{k} r_i\, T_i\right] \cdot \\ & \cdot\, \beta^{s} \exp\left[-(\alpha+\beta-\alpha') \sum^{s} y - \alpha' \sum^{k} s_i\, T_i\right] \cdot \\ & \cdot\, \exp\left[-(\alpha-\beta) \sum^{k} n_i'\, T_i\right] \end{aligned}$$

where $\alpha$ is the failure rate of $A$ which changes to $\alpha'$ on $B$'s failure, $\beta$ is the failure rate of $B$ which change to $\beta'$ on $A$'s failure. $\Sigma\, x$ and $\Sigma\, y$ represent the sum of the lives of $A$ and $B$ respectively.

Differentiating the logarithm of the likelihood function partially with respect to the parameters $\alpha$, $\alpha'$, $\beta$ and $\beta'$ and equating the partial derivatives to zero we get the following maximum likelihood estimates for the parameters.

$$\hat{\alpha} = \frac{n_1 + r}{\overset{n_1+r}{\Sigma} x + \overset{n_2+s}{\Sigma} y + \overset{k}{\Sigma} n_i' T_i}$$

$$\hat{\alpha}' = \frac{n_2}{\overset{n_2}{\Sigma} (x - y) - \overset{s}{\Sigma} y + \overset{k}{\Sigma} s_i T_i}$$

$$\hat{\beta} = \frac{n_2 + s}{\overset{n_1+r}{\Sigma} x + \overset{n_2+s}{\Sigma} y + \overset{k}{\Sigma} n_i' T_i}$$

$$\hat{\beta}' = \frac{n_1}{\overset{n_1}{\Sigma} (y - x) - \overset{r}{\Sigma} x + \overset{k}{\Sigma} r_i T_i}$$

The asymptotic variances of these maximum likelihood estimates are as follows.

$$\text{var } (\hat{\alpha}) = \frac{\alpha (\alpha + \beta)}{\sum_{i=1}^{k} N_i [1 - \exp \{-(\alpha + \beta) T_i\}]}$$

$$\text{var } (\hat{\alpha}') = \alpha'^2 \Big/ \sum_{i=1}^{k} N_i \alpha [\{1/(\alpha + \beta)\} \{1 - \exp(-\overline{\alpha + \beta}\, T_i)\} -$$

$$- \exp(-\alpha' T_i) \{1/(\alpha + \beta' - \alpha')\} \{1 - \exp(-\overline{\alpha + \beta - \alpha'}\, T_i)\}]$$

The expressions for variances of $\hat{\beta}$ and $\hat{\beta}'$ can be found easily by interchanging $\alpha$ with $\beta$ and $\alpha'$ with $\beta'$ in the corresponding expressions for variances of $\hat{\alpha}$ and $\hat{\alpha}'$.

***(ii) When Failure Rate of Each Component Changes with Time***

In this case we assume the following model :

$$f(x, y) = \begin{cases} \alpha_1 \beta'_1 \exp [-(\alpha_1 + \beta_1 - \beta'_1) x - \beta'_1 y], & 0 < x < y \leqslant T_0 \\ \alpha_1 \beta'_{11} \exp [-(\beta'_1 - \beta'_{11}) T_0 - (\alpha_1 + \beta_1 - \beta'_1) x - \beta'_{11} y], & 0 < x \leqslant T_0 < y \\ \alpha_2 \beta'_2 \exp [-(\alpha_1 + \beta_1 - \alpha_2 - \beta_2) T_0 - (\alpha_2 + \beta_2 - \beta'_2) x - \beta'_2 y)], & 0 < T_0 < x < y \\ \alpha'_1 \beta_1 \exp [-(\alpha_1 + \beta_1 - \alpha'_1) y - \alpha'_1 x], & 0 < y < x \leqslant T_0 \\ \alpha'_{11} \beta_1 \exp [-(\alpha'_1 - \alpha'_{11}) T_0 - (\alpha_1 + \beta_1 - \alpha'_1) y - \alpha'_{11} x], & 0 < y \leqslant T_0 < x \\ \alpha'_2 \beta_2 \exp [-(\alpha_1 + \beta_1 - \alpha_2 - \beta_2) T_0 - (\alpha_2 + \beta_2 - \alpha'_2) y - \alpha'_2 x], & 0 < T_0 < y < x \end{cases}$$

It should be noted that in the above bivariate model the random variables $x$ and $y$ are dependent on each other such that the failure of $B$ changes the failure rate of $A$ from $\alpha_1$ to $\alpha'_1$ or from $\alpha_2$ to $\alpha'_2$ depending upon whether $B$ fails before or after time $T_0$ has elapsed since the placing of item on test. Similarly the failure of $A$ changes the failure rate of $B$ from $\beta_1$ to $\beta'_1$ or from $\beta_2$ to $\beta'_2$ depending upon whether $A$ fails before or after time $T_0$.

Moreover the failure rates of $A$ and $B$ have been assumed to change, after time $T_0$, from $\alpha_1$ and $\beta_1$ to $\alpha_2$ and $\beta_2$ or from $\alpha'_1$ and $\beta'_1$ to $\alpha'_{11}$ and $\beta'_{11}$ respectively. Further we assume that

$$\operatorname*{Min}_{i} \{ T_i \} > T_0$$

Let,

$n_1$ = the number of items in which $A$ failed first while $B$ also failed by time $T_0$,

$n_2$ = the number of items in which $A$ failed by time $T_0$ while $B$ failed after time $T_0$,

$n_3$ = the number of items in which $A$ failed first after time $T_0$,

$n_4$ = the number of items in which $B$ failed first while $A$ also failed by time $T_0$,

$n_5$ = the number of items in which $B$ failed by time $T_0$ while $A$ failed after time $T_0$.

$n_6$ = the number of items in which $B$ failed first but after time $T_0$,

$p$ = the number of items in which only $A$ failed but by time $T_0$,

$q$ = the number of items in which only $A$ failed but after time $T_0$,

$r$ = the number of items in which only $B$ failed but by time $T_0$,

$s$ = the number of items in which only $B$ failed after time $T_0$, and

$n'$ = the number of items in which none failed.

Further, let $n_{1i}, n_{2i}, n_{3i}, n_{4i}, n_{5i}, n_{6i}, p_i, q_i, r_i, s_i$ and $n'_i$ be the number of items out of $n_1, n_2, n_3, n_4, n_5, n_6, p, q, r, s$ and $n'$ respectively, which are from $i$th sample of $N_i$ items. Then the likelihood function of the sample arising as a result of the experiment can be written as

$$\begin{aligned}
P(s) = {} & (\alpha_1 \beta'_1)^{n_1} \exp [-(\alpha_1 + \beta_1 - \beta'_1) \sum^{n_1} x - \beta'_1 \sum^{n_1} y] \,. \\
& \cdot (\alpha_1 \beta'_{11})^{n_2} \exp [-n_2 (\beta'_1 - \beta'_{11}) T_0 - (\alpha_1 + \beta_1 - \beta'_1) \sum^{n_2} x - \beta'_{11} \sum^{n_2} y] . \\
& \cdot (\alpha_2 \beta'_2)^{n_3} \exp [-n_3 (\alpha_1 + \beta_1 - \alpha_2 - \beta_2) T_0 - (\alpha_2 + \beta_2 - \beta'_2) \sum^{n_3} x - \beta'_2 \sum^{n_3} y] . \\
& \cdot (\alpha'_1 \beta_1)^{n_4} \exp [-(\alpha_1 + \beta_1 - \alpha'_1) \sum^{n_4} y - \alpha'_1 \sum^{n_4} x] \,. \\
& \cdot (\alpha'_{11} \beta_1)^{n_5} \exp [-n_5 (\alpha'_1 - \alpha'_{11}) T_0 - (\alpha_1 + \beta_1 - \alpha'_1) \sum^{n_5} y - \alpha'_{11} \sum^{n_5} x] . \\
& \cdot (\alpha'_2 \beta_2)^{n_6} \exp [-n_6 (\alpha_1 + \beta_1 - \alpha_2 - \beta_2) T_0 - (\alpha_2 + \beta_2 - \beta'_2) \sum^{n_6} y - \alpha'_2 \sum^{n_6} x] . \\
& \cdot \alpha^{p}_1 \exp [-p (\beta'_1 - \beta'_{11}) T_0 - (\alpha_1 + \beta_1 - \beta'_1) \sum^{p} x - \beta'_{11} \sum^{k} p_i T_i] \,. \\
& \cdot \alpha^{q}_2 \exp [-q (\alpha_1 + \beta_1 - \alpha_2 - \beta_2) T_0 - (\alpha_2 + \beta_2 - \beta'_2) \sum^{q} x - \beta'_2 \sum_{i=1}^{k} q_i T_i] . \\
& \cdot \beta^{r}_i \exp [-r (\alpha'_1 - \alpha'_{11}) T_0 - (\alpha_1 + \beta_1 - \alpha'_1) \sum^{r} y - \alpha'_{11} \sum_{i=1}^{k} r_i T_i] . \\
& \cdot \beta^{s}_2 \exp [-s (\alpha_1 + \beta_1 - \alpha_2 - \beta_2) T_0 - (\alpha_2 + \beta_2 - \alpha'_2) \sum^{s} y - \alpha'_2 \sum_{i=1}^{k} s_i T_i]] . \\
& \cdot \exp [-n' (\alpha_1 + \beta_1 - \alpha_2 - \beta_2) T_0 - (\alpha_2 + \beta_2) \sum_{i=1}^{k} n'_i T_i ] \,.
\end{aligned}$$

This likelihood function yields the following maximum likelihood estimates for the various parameters involved in the model

$$\hat{\alpha}_1 = \frac{n_1 + n_2 + p}{\sum^{n_1+n_2+p} (x - T_0) + \sum^{n_4+n_5+r} (y - T_0) + T_0 \sum^{k} N_i}$$

$$\hat{\alpha}'_1 = \frac{n_4}{\sum^{n_4} (x - y) + \sum^{n_5+q} (T_0 - y)}, \qquad \hat{\alpha}'_{11} = \frac{n_5}{\sum^{n_5}(x - T_0) + \sum^{k} r_i (T_i - T_0)}$$

$$\hat{\alpha}_2 = \frac{n_3 + q}{\sum^{n_3+q} (x - T_0) + \sum^{n_6+s} (y - T_0) + \sum^{k} n'_i (T_i - T_0)}$$

$$\hat{\alpha}'_2 = \frac{n_6}{\sum^{n_6} (x - y) + \sum^{k} s_i T_i - \sum^{s} y}, \qquad \hat{\beta}_1 = \frac{n_4 + n_5 + r}{n_1 + n_2 + p} \hat{\alpha}_1$$

$$\hat{\beta}'_1 = \frac{n_1}{\sum^{n_1} (y - x) + \sum^{n_2+p} (T_0 - x)}, \qquad \hat{\beta}'_{11} = \frac{n_2}{\sum^{n_2} (y - T_0) + \sum^{k} p_i (T_i - T_0)}$$

$$\hat{\beta}_2 = \frac{n_6 + s}{n_3 + q} \hat{\alpha}_2 \quad \text{and} \quad \hat{\beta}'_2 = \frac{n_3}{\sum^{n_3} (y - x) + \sum^{k} q_i T_i - \sum^{q} x}$$

The asymptotic variances of these maximum likelihood estimates are as follows:

$$\text{var}\, (\hat{\alpha}_1 = \frac{\alpha_1 (\alpha_1 + \beta_1)}{N [1 - \exp\{-(\alpha_1 + \beta_1) T_0\}]}$$

$$\text{var}\, (\hat{\alpha}'_1) = \alpha'^2_1 \Big/ N \beta_1 [\{1/(\alpha_1 + \beta_1)\} \{1 - \exp - \overline{\alpha_1 + \beta_1} T_0\} - \exp(-\alpha'_1 T_0) \times$$
$$\times \{1/(\alpha_1 + \beta_1 - \alpha'_1)\} \{1 - \exp \overline{- \alpha_1 + \beta_1 - \alpha'_1} T_0\}]$$

$$\text{var}\, (\hat{\alpha}'_{11}) = \alpha'_{11} (\alpha_1 + \beta_1 - \alpha'_1) \Big/ \beta_1 \exp \{-(\alpha'_1 - \alpha'_{11}) T_0\} [1 - \exp$$
$$\{-(\alpha_1 + \beta_1 - \alpha'_1) T_0\}] \times [\sum_{i=1}^{k} N_i \{\exp(-\alpha'_{11} T_0) - \exp(-\alpha'_{11} T_i)\}]$$

$$\text{var}\, (\hat{\alpha}_2) = \alpha_2 (\alpha_2 + \beta_2) \Big/ \sum_{i=1}^{k} N_i \exp\{-(\alpha_1 + \beta_1 - \alpha_2 - \beta_2) T_0\} [\exp\{-(\alpha_2 + \beta_2) T_0 -$$
$$- \exp\{-(\alpha_2 + \beta_2) T_i\}]$$

$$\text{var}\, (\hat{\alpha}'_2) = \alpha'^2_2 \Big/ \beta_2 \exp\{-(\alpha_1 + \beta_1 - \alpha_2 - \beta_2) T_0\} \sum_{i=1}^{k} N_i [\{1/(\alpha_2 + \beta_2)\} \times$$
$$\times \{\exp(-\overline{\alpha_2 + \beta_2} T_0) - \exp(-\overline{\alpha_2 + \beta_2} T_i)\} - \{\exp(-\alpha'_2 T_i)/(\alpha_2 + \beta_2 - \alpha'_2)\} \times$$
$$\times \{\exp(-\overline{\alpha_2 + \beta_2 - \alpha'_2}) T_0 - \exp(-\overline{\alpha_2 + \beta_2 - \beta'_2}) T_i\}]$$

The expressions for variances of $\hat{\beta}_1$, $\hat{\beta}'_1$, $\hat{\beta}'_{11}$, $\hat{\beta}_2$ and $\hat{\beta}'_2$ can be found easily by interchanging $\alpha_1$ with $\beta_1$, $\alpha'_1$ with $\beta'_1$, $\alpha'_{11}$ with $\beta'_{11}$, $\alpha_2$ with $\beta_2$ and $\alpha'_2$ with $\beta'_2$ in the corresponding expressions for the variances of $\hat{\alpha}_1$, $\hat{\alpha}'_1$, $\hat{\alpha}'_{11}$, $\hat{\alpha}_2$ and $\hat{\alpha}'_2$.

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