ROCKET LOCATIONS FOR MINIMUM TIME INTERCEPTION

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Interception of target rocket moving along a trajectory in a gravitational field by an interceptor rocket has been analysed to achieve interception in minimum time duration. Numerical values of the parameters of the intercepting trajectory and locations of the target rocket for launch and target orbits in Earth's gravitational field have been calculated for different impulsive velocity changes applied to the interceptor rocket.

In reference (1) has been studied the interception of moving rockets at preassigned destinations in gravitational field. In that the emphasis has been laid on the fuel minimization in the interception operation. However, conditions may arise when interception in minimum flight time of interceptor rocket may become necessary. The problem of interception of a moving target rocket by an interceptor rocket in minimum time with specified fuel expenditure in the interception operation is analysed in this paper. The paths of both the target and interceptor rockets are supposed to be known. The location of the target rocket at the instant the interception rocket is fired so as to hit the former in minimum time with given impulsive velocity change imparted to the interceptor rocket has been found out. The numerical results have been calculated for different impulsive velocity changes.

INTERCEPTION TRAJECTORY PARAMETERS

Let the equations of the paths of the interceptor and target rockets be

$$\frac{l_1}{r} = 1 + e_1 \cos \theta \tag{1}$$

$$\frac{l_2}{r} = 1 + e_2 \cos (\theta - \xi) \tag{2}$$

where l, e and ξ denote latus rectum, eccentricity, angle between the major axes of the two orbits respectively and suffixes 1 and 2 correspond to the paths of interceptor and target rockets (hereafter called launch and destination orbits). r and θ represent polar coordinates with force centre as pole and line drawn from the force centre to the periapsis of the launch orbit as initial line. Let (r_1, θ_1) be the given launch point (Fig 1) at which the interceptor rocket is fired and (r_2, θ_2) the destination point where the target is hit by the interceptor rocket. Suppose $\theta_2 = \theta_1 + \phi$ where ϕ is the transfer angle. Now if $\triangle V$ is the impulsive velocity change applied to the interceptor rocket, we have

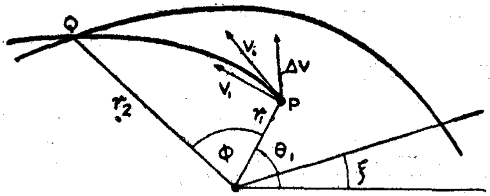

Fig. 1—Geometry of interception.

$$V_i^2 = V_1^2 + (\triangle V)^2 + 2 \triangle V . \; V_1 \cos \alpha \tag{3}$$

where V represents velocity and suffix i denotes values at the launch point just after application of the impulse and α is the angle between V_1 and $\triangle V$. If γ is the angle between velocity vector and local horizontal, α can be expressed as

$$\alpha = (\gamma_i - \gamma_1) + \sin^{-1}\left[\frac{V_1}{\triangle V}\sin(\gamma_i - \gamma_1)\right] \tag{4}$$

where γ_1 and V_1 are given by

$$\tan \gamma_1 = \frac{e_1 \sin \theta_1}{1 + e_1 \cos \theta_1} \tag{5}$$

$$V_1 = \left[\frac{K}{l_1}(1 + e_1^2 + 2e_1 \cos \theta_1)\right]^{\frac{1}{2}} \tag{6}$$

K being the gravitational parameter. With the help of (4) to (6), (3) shows that for specified $\triangle V$; V_i is a function of γ_i.

If E and a are the eccentricity and semi-major axis of the elliptic interception trajectory, we have

$$E = \left[\left(\frac{r_1 V_i^2}{K} - 1\right)^2 \cos^2 \gamma_i + \sin^2 \gamma_i\right]^{\frac{1}{2}} \tag{7}$$

$$a = \frac{r_1}{\left(2 - \frac{r_1 V_i^2}{K}\right)} \tag{8}$$

Then flight time T of the interceptor rocket will be given[2] by

$$T = \left(\frac{a^3}{K}\right)^{\frac{1}{2}}\left[(\psi_2 - \psi_1) - E(\sin \psi_2 - \sin \psi_1)\right]$$

where ψ_1 and ψ_2 are the eccentric anomalies of the launch and destination points corresponding to the interception trajectory and

$$\cos \psi = \frac{a - r}{aE}$$

Therefore T can be written as

$$T = \left(\frac{a^3}{K}\right)^{\frac{1}{2}}\left[\cos^{-1}\left(\frac{a - r_2}{aE}\right) - \cos^{-1}\left(\frac{a - r_1}{aE}\right) - \right.$$
$$\left. - E\left(\sin\left\{\cos^{-1}\left(\frac{a - r_2}{aE}\right)\right\} - \sin\left\{\cos^{-1}\left(\frac{a - r_1}{aE}\right)\right\}\right)\right] \tag{9}$$

where r_2 is given by (2) as

$$r_2 = \frac{l_2}{1 + e_2 \cos(\theta_1 + \phi - \xi)}$$

MINIMUM INTERCEPTION TIME

Since interception trajectory passes through the points (r_1, θ_1) and (r_2, θ_2) we have[1]

$$\frac{r_1 V_i^2}{K} = \frac{\sec^2 \gamma_i (1 - \cos \phi)}{\frac{r_1}{r_2} - \cos \phi + \sin \phi \tan \gamma_i} \tag{10}$$

Substitution of the value of r_2 in (10), rearrangement of the terms and simplification gives

$$(1 + C_1C_3)\cos\phi + C_1C_4\sin\phi + C_1C_2 - 1 = 0 \tag{11}$$

where $$C_1 = \frac{r_1 V_i^2 \cos^2\gamma_i}{K}, \quad C_2 = \frac{r_1}{l_2}, \quad C_3 = \frac{r_1 e_2 \cos(\theta_1 - \xi)}{l_2} - 1$$

and $$C_4 = \tan\gamma_i - \frac{r_1 e_2 \sin(\theta_1 - \xi)}{l_2}$$

From equation (11) we obtain

$$\tan\frac{\phi}{2} = \frac{-C_1C_4 \pm [(C_1C_4)^2 - C_1(C_2 + C_3)\{C_1(C_2 - C_3) - 2\}]^{\frac{1}{2}}}{\{C_1(C_2 - C_3) - 2\}} \tag{12}$$

Positive sign is taken before the radical because negative sign will correspond to greater value of ϕ for the interception trajectory and therefore greater interception time. With (3) to (8) and (10), interception time T in (9) is a function of single variable γ_i and therefore for minimum time interception

$$\frac{\partial T}{\partial \gamma_i} = \frac{\partial T}{\partial a}\cdot\frac{\partial a}{\partial \gamma_i} + \frac{\partial T}{\partial \phi}\cdot\frac{\partial \phi}{\partial \gamma_i} + \frac{\partial T}{\partial E}\cdot\frac{\partial E}{\partial \gamma_i} = 0 \tag{13}$$

where

$$\frac{\partial T}{\partial a} = \frac{3}{2}\,\frac{T}{a} + \frac{1}{(aK)^{\frac{1}{2}}}\left[\frac{r_1^2}{\{(aE)^2 - (a - r_1)^2\}^{\frac{1}{2}}} - \frac{r_2^2}{\{(aE)^2 - (a - r_2)^2\}^{\frac{1}{2}}}\right]$$

$$\frac{\partial T}{\partial \phi} = \left(\frac{a}{K}\right)^{\frac{1}{2}} \frac{e_2 \sin(\phi + \theta_1 - \xi)\, r_2^3}{l_2\,\{(aE)^2 - (a - r_2)^2\}^{\frac{1}{2}}}$$

$$\frac{\partial T}{\partial E} = \left(\frac{a^3}{K}\right)^{\frac{1}{2}}\left[\frac{1}{aE}\left\{\frac{r_2(a - r_2)}{\{(aE)^2 - (a - r_2)^2\}^{\frac{1}{2}}} - \frac{r_1(a - r_1)}{\{(aE)^2 - (a - r_1)^2\}^{\frac{1}{2}}}\right\} - \left\{\sin\left(\cos^{-1}\frac{a - r_2}{aE}\right) - \sin\left(\cos^{-1}\frac{a - r_1}{aE}\right)\right\}\right]$$

$$\frac{\partial a}{\partial \gamma_i} = \frac{Ma^2}{K}$$

$$\frac{\partial \phi}{\partial \gamma_i} = \frac{\frac{r_1}{K}\cos^2\gamma_i\,(M - 2V_i^2\tan\gamma_i)\,(1 - \cos\phi) + \sin\phi\, C_1^2\sec^2\gamma_i}{C_1[\sin\phi - C_1(C_4\cos\phi - C_3\sin\phi)]}$$

$$\frac{\partial E}{\partial \gamma_i} = \frac{\cos\gamma_i}{E}\left[\sin\gamma_i\left\{1 - \left(\frac{r_1 V_i^2}{K} - 1\right)^2\right\} + \frac{r_1}{K}M\cos\gamma_i\left(\frac{r_1 V_i^2}{K} - 1\right)\right]$$

and

$$M = -2\triangle V\cdot V_1\left[1 + \frac{V_1\cos(\gamma_i - \gamma_1)}{[(\triangle V)^2 - \{V_1\sin(\gamma_i - \gamma_1)\}^2]^{\frac{1}{2}}}\right]\sin\theta$$

Equation (13) can be numerically solved which will give γ_i and then E, a and ϕ can be obtained from equations (7), (8) and (12) respectively. Having known E, a and ϕ, interception time for minimum time interception trajectory can be evaluated from (9) and corresponding θ_2 also becomes known.

TARGET ROCKET LOCATIONS

Let $(r_d\,,\ \theta_d)$ be the position of the target rocket on the destination orbit at the instant the interceptor rocket is fired from the launch point so as to intercept the former at the point $(r_2\,,\ \theta_2)$ the destination point corresponding to the minimum time interception trajectory. Then

$$\frac{1}{\sqrt{K}}\left(\frac{l_2}{1-e_2{}^2}\right)^{\frac{3}{2}}\left[2\left\{\tan^{-1}\left(\left\{\frac{1-e_2}{1+e_2}\right\}^{\frac{1}{2}}\tan\frac{\theta_2-\xi}{2}\right)-\tan^{-1}\left(\left\{\frac{1-e_2}{1+e_2}\right\}^{\frac{1}{2}}\tan\frac{\theta_d-\xi}{2}\right)\right\}-e_2\left(1-e_2^2\right)^{\frac{1}{2}}\left\{\frac{\sin(\theta_2-\xi)}{1+e_2\cos(\theta_2-\xi)}-\frac{\sin(\theta_d-\xi)}{1+e_2\cos(\theta_d-\xi)}\right\}\right]=T \quad (14)$$

From (14), the unknown θ_d can be evaluated and then from (2) r_d can be found out. $(r_d\,,\ \theta_d)$ will give the location of the target rocket for minimum time interception. The equations have been numerically solved for two arbitrarily chosen launch and destination orbits in Earth's gravitational field to obtain rocket locations for minimum time interception and parameters of the corresponding interception trajectories for different specified values of $\triangle V$. The results obtained are given in Table 1. The launch orbit is specified by $l_1 = 4243$ miles, $e_1 = 0 \cdot 04$ and the destination orbit by $l_2 = 6006$ miles, $e_2 = 0 \cdot 3$ and $\xi = 30°$. The vectorial $\angle\ \theta_1$ of the launch point is taken equal to 60°. A study of Table 1 shows that as expected with increasing velocity impulse $\triangle V$, minimum interception time decreases.

TABLE 1

ROCKET LOCATIONS AND TRAJECTORY PARAMETERS FOR MINIMUM TIME INTERCEPTION

$\triangle V$ (miles/sec)	γ_i	T (sec)	r_2 (miles)	θ_2	E	a (miles)	θ_d	r_d (miles
1·00	7° 15′	1839·2	7035·73	149° 12′	0·437	7200·54	76° 30′	4978·00
1·25	11° 48′	1039·4	5928·93	117° 31′	0·457	7141·43	67° 2′	4845·56
1·50	15° 23′	706·2	5475·30	101° 9′	0·480	7107·13	63° 54′	4808·65
1·75	18° 44′	534·7	5256·14	91° 36′	0·502	7021·42	62° 12′	4790·02
2·00	21° 26′	431·7	5140·11	85° 50′	0·546	7365·63	61° 35′	4783·51

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