SOME CONTOUR INTEGRALS INVOLVING G-FUNCTION OF TWO VARIABLES

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The object of this paper is to evaluate contour integrals for G-function of two variables. Some results for Meijer's G-function have been obtained as particular cases.

Some contour integrals involving G-function of two variables have been evaluated. On specialising the parameters the results for Meijer's G-function as particular cases are obtained.

For the sake of brevity we have used the symbol $\triangle(\delta,\alpha)$ for the set of parameters α/δ, $(\alpha+1)/\delta,\ldots,(\alpha+\delta-1)/\delta$ and (a_p) stands for $a_1, a_2,\ldots,a_p$ throughout.

Agarwal[1] and Sharma[2] defined the G-function of two variables in the form of Mellin-Barnes type integral which has been represented by Bajpai[3] as

$$G\,{}^{(m_1,\,m_2);\,(n_1,\,n_2),\,n_3}_{(p_1,\,p_2),\,p_3;\,(q_1,\,q_2),\,q_3}\left[\begin{array}{c|c}\eta & \begin{array}{l}(a_{p_1});\ (c_{p_2})\\ (e_{p_3})\end{array}\\ \zeta & \begin{array}{l}(b_{q_1});\ (d_{q_2})\\ (f_{q_3})\end{array}\end{array}\right]$$

$$=\frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2}\frac{\prod_{j=1}^{m_1}\Gamma(b_j-s)\prod_{j=1}^{n_1}\Gamma(1-a_j+s)\prod_{j=1}^{m_2}\Gamma(d_j-t)\prod_{j=1}^{n_2}\Gamma(1-c_j+t)}{\prod_{j=m_1+1}^{q_1}\Gamma(1-b_j+s)\prod_{j=n_1+1}^{p_1}\Gamma(a_j-s)\prod_{j=m_2+1}^{q_2}\Gamma(1-d_j+t)\prod_{j=n_2+1}^{p_2}\Gamma(c_j-t)}\times$$

$$\times\frac{\prod_{j=1}^{n_3}\Gamma(1-e_j+s+t)\quad\eta^s\ \zeta^t}{\prod_{j=n_3+1}^{p_3}\Gamma(e_j-s-t)\prod_{j=1}^{q_3}\Gamma(1-f_j+s+t)}\,ds\,dt. \qquad (1)$$

The contour L_1 is in the s-plane and runs from $-i\infty$ to $+i\infty$ with loops if necessary, to ensure that the poles of $\Gamma(b_j-s)$, $j=1,2,\ldots,m_1$ lie on the right and the poles of $\Gamma(1-a_j+s)$, $j=1,2,\ldots,n_1$ and $\Gamma(1-e_j+s+t)$, $j=1,2,\ldots,n_3$ to the left of the contour. Similarly the contour L_2 is in the t-plane and runs from $-i\infty$ to $+i\infty$ with loops, if necessary, to ensure that the poles of $\Gamma(d_j-t)$, $j=1,2,\ldots,m_2$ lie on the right and the poles of $\Gamma(1-c_j+t)$, $j=1,2,\ldots,n_2$ and $\Gamma(1-e_j+s+t)$, $j=1,2,\ldots,n_3$ lie to the left of the contour. Provided that

$$0\leqslant m_1\leqslant q_1;\ 0\leqslant m_2\leqslant q_2;\ 0\leqslant n_1\leqslant p_1;\ 0\leqslant n_2\leqslant p_2;\ 0\leqslant n_3\leqslant p_3;$$

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the integral converges if

$$\left.\begin{aligned}&(p_3+q_3+p_1+q_1)<2(m_1+n_1+n_3);\ (p_3+q_3+p_2+q_2)<2(m_2+n_2+n_3);\\&|\arg\eta|<\left[m_1+n_1+n_3-\frac{1}{2}(p_3+q_3+p_1+q_1)\right]\pi\\&|\arg\zeta|<\left[m_2+n_2+n_3-\frac{1}{2}(p_3+q_3+p_2+q_2)\right]\pi\end{aligned}\right\}\quad(2)$$

The right hand side of (1) shall, henceforth be denoted by $G\begin{bmatrix}\eta\\\zeta\end{bmatrix}$

We establish the following integrals :

$$\frac{1}{(2\pi i)}\int_{c-i\infty}^{c+i\infty} y^{\frac{1}{2}-\rho}\, I_\nu(xy)\, G_{(p_1+p_2),p_3;(q_1,q_2),q_3}^{(m_1,m_2);(n_1,n_2),n_3}\left[\begin{matrix}\eta y^{2h}\\ \\ \zeta\end{matrix}\left|\begin{matrix}(a_{p_1});\ (c_{p_2})\\(e_{p_3})\\(b_{q_1});\ (d_{q_2})\\(f_{q_3})\end{matrix}\right.\right]dy.$$

$$=\left(\frac{x}{2}\right)^{\rho-\frac{3}{2}}(h)^{\frac{1}{2}-\rho}(2\pi)^{h-1}\,G_{(p_1+2h,p_2),p_3;(q_1,q_2),q_3}^{(m_1,m_2);(n_1,n_2),n_3}$$

$$\left[\begin{matrix}\eta\left(\frac{2h}{x}\right)^{2h}\\ \\ \zeta\end{matrix}\left|\begin{matrix}(a_{p_1}),\ \triangle\left(h,\frac{\rho+\nu+\frac{1}{2}}{2}\right),\triangle\left(h,\frac{\rho-\nu+\frac{1}{2}}{2}\right);\ (c_{p_2})\\(e_{p_3})\\(b_{q_1});\ (d_{q_2})\\(f_{q_3})\end{matrix}\right.\right]\quad(3)$$

where h is a positive number and

$$Re\,[\rho+2h\,(1-a_j)]>|\,Re\,\nu\,|-\tfrac{1}{2},\qquad j=1,\ 2,\ldots n_1.$$

Similar results hold for

$$G\begin{bmatrix}\eta\\\zeta\,(y)^{2h}\end{bmatrix}\text{ and }G\begin{bmatrix}\eta\,(y)^{2h}\\\zeta\,(y)^{2h}\end{bmatrix}$$

$$\frac{1}{(2\pi i)}\int_{c-i\infty}^{c+i\infty} e^{x\mu}\,(x+\alpha)^{-\nu}\,G_{(p_1,p_2),p_3;(q_1,q_2),q_3}^{(m_1,m_2);(n_1,n_2),n_3}\left[\begin{matrix}\eta\,(x+\alpha)^{\delta}\\ \\ \zeta\end{matrix}\left|\begin{matrix}(a_{p_1});(c)_{p_2}\\(e_{p_3})\\(b_{q_1});(d_{q_2})\\(f_{q_3})\end{matrix}\right.\right]dx$$

$$=\frac{\mu^{\nu-1}\,e^{-\alpha\mu}}{(2\pi)^{\frac{1}{2}-\frac{\delta}{2}}\,(\delta)^{\nu-\frac{1}{2}}}\,G_{(p_1+\delta,p_2),p_3;(q_1,q_2),q_3}^{(m_1,m_2);(n_1,n_2),n_3}\left[\begin{matrix}\eta\left(\frac{\delta}{\mu}\right)^{\delta}\\ \\ \zeta\end{matrix}\left|\begin{matrix}(a_{p_1}),\ \triangle(\delta,\nu);\ (c_{p_2})\\(e_{p_3})\\(b_{q_1});\ (d_{q_2})\\(f_{q_3})\end{matrix}\right.\right]\quad(4)$$

where δ is a positive number and

$$Re\,[\,\nu + \delta\,(1 - a_j)\,] > 0,\ j = 1, 2, \ldots, n_1$$

Similar results hold for

$$G\begin{bmatrix}\eta\,(x+\alpha)^{\delta} \\ \zeta\,(x+\alpha)^{\delta}\end{bmatrix} \text{ and } G\begin{bmatrix}\eta \\ \zeta\,(x+\alpha)^{\delta}\end{bmatrix}$$

In all the above integrals, the conditions of validity are same as (2).

Proof :—To prove (3), expressing the G-function on the left as in (1) changing the order of integration and evaluating the inner integral with the help of the integral

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^{\frac{1}{2}-\rho}\, I_\nu\,(xy)\,dy = \frac{\left(\frac{x}{2}\right)^{\rho-\frac{3}{2}}}{\Gamma\left(\frac{\rho+\nu+\frac{1}{2}}{2}\right)\Gamma\left(\frac{\rho-\nu+\frac{1}{2}}{2}\right)},\ Re\,(\rho) > |\,Re\,\nu\,| - \frac{1}{2}$$

which follows from reference 4. We get that left hand side of (3) equals

$$\frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2}\frac{\prod_{j=1}^{m_1}\Gamma(b_j-s)\prod_{j=1}^{n_1}\Gamma(1-a_j+s)\prod_{j=1}^{m_2}\Gamma(d_j-t)}{\prod_{j=m_1+1}^{q_1}\Gamma(1-b_j+s)\prod_{j=n_1+1}^{p_1}\Gamma(a_j-s)\prod_{j=m_2+1}^{q_2}\Gamma(1-d_j+t)\prod_{j=n_2+1}^{p_2}\Gamma(c_j-t)}\times$$

$$\times\frac{\prod_{j=1}^{n_2}\Gamma(1-e_j+t)\prod_{j=1}^{n_3}\Gamma(1-e_j+s+t)\left(\frac{x}{2}\right)^{\rho-2hs-\frac{3}{2}}\eta^{s}\,\xi^{t}}{\prod_{j=n_3+1}^{p_3}\Gamma(e_j-s-t)\prod_{j=1}^{q_3}\Gamma(1-f_j+s+t)\,\Gamma\left(\frac{\rho+\nu+\frac{1}{2}}{2}-hs\right)\Gamma\left(\frac{\rho-\nu+\frac{1}{2}}{2}-hs\right)}\,ds\,dt.$$

Now using (1) and multiplication formula for Gamma functions[5], the integral (3) is proved.

The integral (4) is established by adopting the same method as above and using the formula[6] viz.

$$\frac{1}{(2\pi i)}\int_{c-i\infty}^{c+i\infty} e^{x\mu}\,(x+\alpha)^{-\nu}\,dx = \frac{\mu^{\nu-1}\,e^{-a\mu}}{\Gamma\nu},\ Re\,(\nu) > 0$$

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Putting $m_2 = q_2 = 1$, $n_2 = n_3 = p_2 = p_3 = q_3 = 0$, and making use of the formula given by Bajpai[3] viz.

$$G^{(m,\,1);\,(n,\,0),\,0}_{(p,\,0),\,0;\,(q,\,1),\,0}\left[\begin{array}{c|cc} x & (a_p)\,; & \text{—} \\ y & (b_q)\,; & 0\end{array}\right] = e^{-y}\,G^{m,\,n}_{p,\,q}\left[\,x\,\middle|\begin{array}{c}(a_p)\\(b_q)\end{array}\right] \quad (5)$$

we get from (4),

$$\frac{1}{(2\pi i)} \int_{c-i\infty}^{c+i\infty} e^{x\mu} (x+\alpha)^{-\nu} G^{m_1, n_1}_{p_1, q_1}\left[\eta (x+\alpha)^{\delta} \middle| \begin{matrix} (a_{p_1}) \\ (b_{q_1}) \end{matrix} \right] dx$$

$$= \frac{\mu^{\nu-1} \; e^{-a\mu}}{(2\pi)^{\frac{1}{2}-\frac{\delta}{2}} \; (\delta)^{\nu-\frac{1}{2}}} \quad G^{m_1, n_1}_{p_1+\delta, q_1}\left[\eta \left(\frac{\delta}{\mu}\right)^{\delta} \middle| \begin{matrix} (a_{p_1}), \triangle(\delta, \nu) \\ (b_{q_1}) \end{matrix} \right] \qquad (6)$$

Specialising the parameters as above and making use of (5), we get an integral[7] as a particular case of (3).

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