Def Sci J, Vol 32, No. 2, April 1982, pp 99-104

Unsteady Flow Between Two Oscillating Plates

EVELYN CHANDRA SEKHARAN & G. RAMANAIAH
Faculty of Science, Madras Institute of Technology, Madras-600 044

Received 29 October 1980

Abstract. The flow of an incompressible viscous fluid between two infinite harmonically oscillating plates is considered, when an oscillatory body force (having the same frequency as that of the plates) is applied in the direction of motion of the plates. Using the technique adopted in the classical Stokes second problem, detailed analytical expressions for the velocity field of the fluid, volume flow rate and stress at the plates have been obtained. It is found that the fluid velocity oscillates in time with the same frequency as that of the plates or the body force, but with a phase difference.

1. Introduction

Oscillatory flows are of great importance in a variety of engineering and biological situations and hence such flows have been considered and reconsidered by many researchers from time to time. Stokes[1] has discussed the motion of a fluid in contact with a harmonically vibrating plate. Since then many authors have studied the flow of a fluid between two oscillating parallel flat plates and through oscillating channels of various geometries. Khan[2] has discussed the motion of a fluid between two oscillating plates when a constant body force is applied in the direction of motion of the plates. In this paper the flow of an incompressible viscous fluid between two infinite harmonically oscillating plates is considered when an oscillatory body force (having the same frequency as that of the plates) is applied in the direction of motion of the plates. The technique adopted is that used in the classical Stokes' second problem.

2. Governing Equations

Consider a cartesian coordinate system centred between the two plates, where the direction of flow is taken as the x axis and the y axis is perpendicular to the two plates. The region $-h \leqslant y \leqslant h$ between the two plates is filled with viscous incompressible fluid. The two plates are oscillating in time with frequency w in a direction parallel to themselves and an oscillatory body force f (having the same frequency as that of the plates) is applied in the direction of motion of the plates. The fluid flow is characterized by the velocity field $\bar{u}\,[u(y, t), 0, 0]$.

The equation of continuity is automatically satisfied and the equation of motion reduces to

$$\frac{\partial u}{\partial t} = f + \nu \frac{\partial^2 u}{\partial y^2} \tag{1}$$

where ν, the kinematic viscosity of the fluid, is μ/ρ, μ being the viscosity and ρ the density of the fluid.

Introducing the dimensionless quantities $Y = \frac{y}{h}$, $T = wt$, $U = \frac{u}{hw}$, $Re = \frac{h^2 w}{}$

$F = \frac{f}{hw^2}$

Eqn. (1) becomes

$$\frac{\partial U}{\partial T} = F + \frac{1}{Re}\frac{\partial^2 U}{\partial Y^2} \tag{2}$$

Eqn. (2) is to be solved under the boundary conditions,

$$U \quad U_0 \sin T \text{ at } Y \quad \pm \tag{3}$$

3. Solution of the Problem

To solve the problem assume $U = U(Y)e^{iT}$, $F = F_0 e^{iT}$. Substituting these in Eqn. (2), we have

$$\frac{d^2 U}{dY^2} - Re\, i\, U = -\,Re\, F_0 \tag{4}$$

where U in Eqn. (4) is clearly a function of only Y.

Solving Eqn. (4) through the boundary conditions that $U(Y) \quad U_0$ at $Y \quad \pm 1$, the velocity of the fluid in real terms is given by

$$U = \frac{U_0}{(E^2 \quad G^2)}\left[(EE_Y + GG_Y)\sin T + (EG_Y \quad GE_Y)\cos T\right]$$

$$+ \frac{F_0}{(E^2 + G^2)}\left[(GE_Y - EG_Y)\sin T + (EE_Y \quad GG_Y \quad E^2 - G^2)\cos T\right] \tag{5}$$

where

$$E = \cosh\left(\sqrt{\frac{Re}{2}}\right) \cos\left(\sqrt{\frac{Re}{2}}\right),\ G \quad \sinh\left(\sqrt{\frac{Re}{2}}\right) \sin\left(\sqrt{\frac{Re}{2}}\right)$$

(equation continued)

$$E_Y = \cosh\left(Y\sqrt{\frac{Re}{2}}\right)\cos\left(Y\sqrt{\frac{Re}{2}}\right),\ G_Y = \sinh\left(Y\sqrt{\frac{Re}{2}}\right) \times \sin\left(Y\sqrt{\frac{Re}{2}}\right) \tag{6}$$

The fluid velocity at the mid-region i.e. at $Y \quad 0$ is

$$U \quad \frac{1}{(E^2 + G^2)}[U_0\{E \sin T - G \cos T\} \quad F_0\{G \sin T \quad (E^2 + G^2 - E) \times \cos T\}] \tag{7}$$

Volume Flow Rate

The volume flow rate for the fluid is given by

$$Q = \int_{-1}^{+1} U dY$$

$$\frac{U_0}{(E^2 + G^2)\left(\sqrt{\frac{Re}{2}}\right)}[\{E(K + H) + G(K - H)\} \sin T \quad \{G(K + H) - E(K - H)\} \cos T] + \frac{F_0}{(E^2 + G^2)\left(\sqrt{\frac{Re}{2}}\right)}[\{G(K + H) \quad E(K - H)\} \sin T \quad \{E(K + H) + G(K \quad H)\} \cos T] \quad 2F_0 \cos T \tag{8}$$

where

$$K = \cosh\left(\sqrt{\frac{Re}{2}}\right)\sin\left(\sqrt{\frac{Re}{2}}\right),\ H = \sinh\left(\sqrt{\frac{Re}{2}}\right)\cos\left(\sqrt{\frac{Re}{2}}\right) \tag{9}$$

Stress at the Bounding Plates

The dimensionless stress at the plate $Y =$ is given by

$$T_{XY} = \left(\frac{\partial U}{\partial Y}\right)_{Y=1}$$

$$\frac{U_0\left(\sqrt{\frac{Re}{2}}\right)}{(E^2 + G^2)}[\{G(K + H) \quad E(K - H)\} \sin T + \{E(K + H) + G(K \quad H)\} \cos T] + \frac{F_0\left(\sqrt{\frac{Re}{2}}\right)}{(E^2 + G^2)}[\{G(K + H) \quad E(K - H)\} \cos T - \{E(K + H) + G(K - H)\} \sin T] \tag{10}$$

Since U is an even function of Y, $\frac{\partial U}{\partial Y}$ is an odd function of Y, hence the stress at the two plates are equal in magnitude but opposite in sign.

4. Discussion and Conclusions

Equation (5) describes the fluid velocity which is oscillating in time with the same frequency as that of the plates or the oscillatory body force. The amplitude of oscillation viz. r is given by

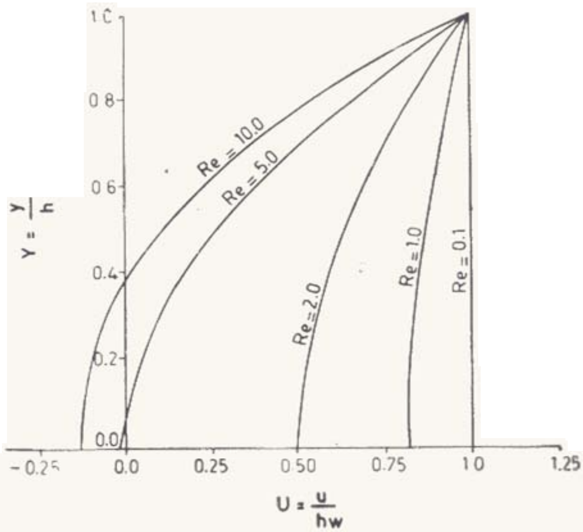

Figure 1. Velocity profiles ($Y \geqslant 0$) for different Reynolds numbers at $T = \pi/2$ when $U_0 = 1$ & $F_0 = 0$.

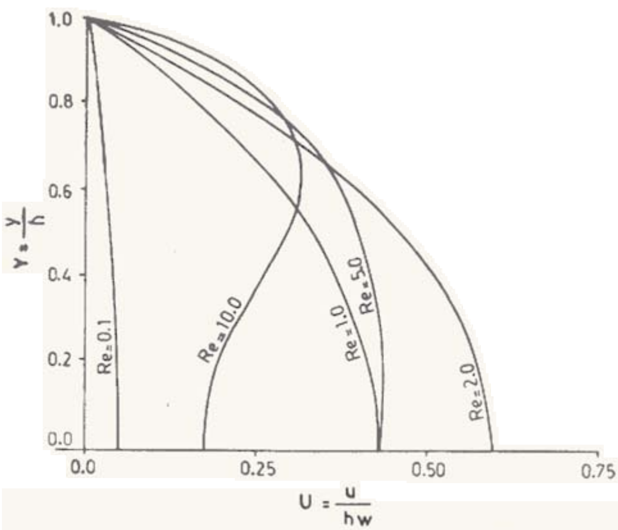

Figure 2. Velocity profiles ($Y \geqslant 0$) for different Reynolds numbers at $T = \pi/2$ when $U_0 = 0$ & $F_0 = 1$.

$$r^2 = \frac{1}{(E^2 + G^2)} [U_0(E_Y^2 + G_Y^2) + F_0 \quad (E^2 + G^2 + E_Y^2 + G_Y^2 \qquad 2EE_Y$$

$$- 2GG_Y) + 2U_0F_0(GE_Y - EG_Y)]$$

and the phase lag is given by

$$\theta = \text{arc tan } [\{U_0(EG_Y - GE_Y) + F_0(EE_Y + GG_Y - E^2 - G^2)\}$$

$$\div \{U_0\,(EE_Y + GG_Y) + F_0(GE_Y - EG_Y)\}]$$

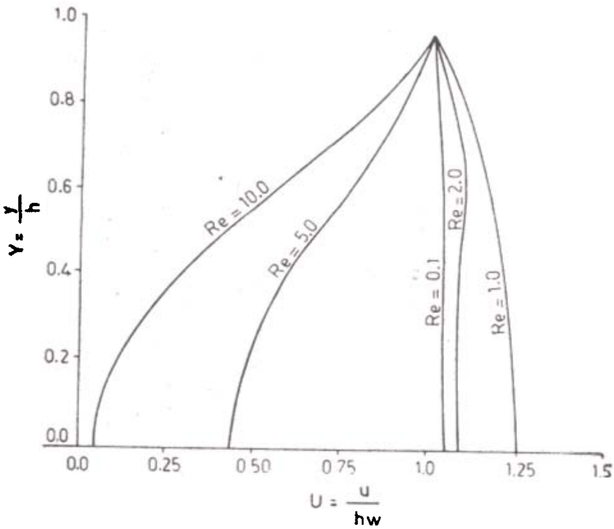

Figure 3. Velocity profiles ($Y \geqslant 0$) for different Reynolds numbers at $T = \pi/2$ when $U_0 = 1$ & $F_0 = 1$.

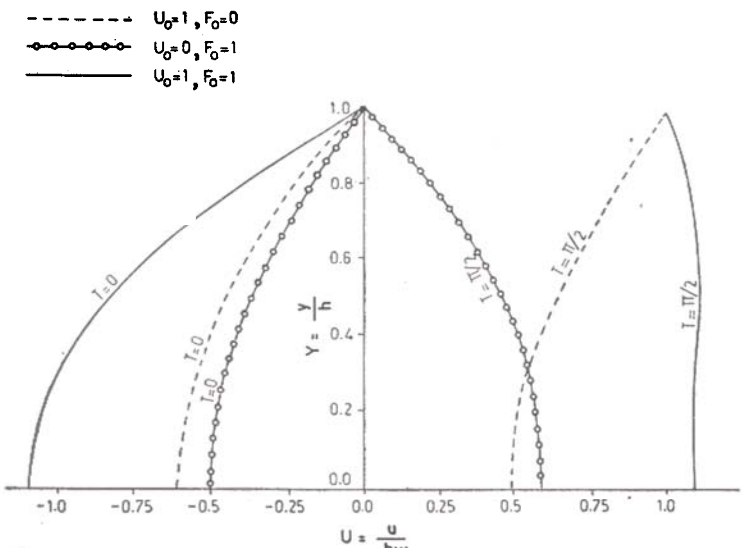

Figure 4. Velocity profiles ($Y \geqslant 0$) when $Re = 2$, for different times.

Since the flow is symmetrical about the plane $Y = 0$ velocity profile curves depicting the variation of U with Y are drawn for the upper half region i.e. $0 \leqslant Y \leqslant 1$. Fig. 1 to 3 show the velocity profile curves for the three cases viz. $U_0 = 1, F_0 = 0$, $U_0 = 0, F_0 = 1$ and $U_0 = 1, F_0 = 1$ at time $T = \pi/2$ for different Reynolds numbers. Fig. 4 shows the velocity profile curves for $Re = 2$ at times $T = 0; \pi/2$ for the three cases considered. The situations at $T = \pi$ and $3\pi/2$ have not been depicted since $U(Y, \pi) = -U(Y, 0)$ and $U(Y, 3\pi/2) = -U(Y, \pi/2)$. At times $T = 0$, $3\pi/2$ the fluid has a backward motion and at times $T = \pi/2, \pi$ the motion is forward i.e. along the positive x-direction. From Fig. 4, it is clear that the effect of the oscillatory body force on the fluid between two oscillating plates is to increase the velocity of the fluid and this effect is greatest at $Y = 0$.

When $U_0 = 1$, $F_0 = 1$, the fluid particles in the central region $Y = 0$ are momentarily at rest at times

$$t = \arc\tan \{(E^2 + G^2 + G - E)/(E + G)\} \pm n\pi, \text{ where } n \text{ is an integer.}$$

For example, when $Re = 2$, this time is $0.25\pi \pm n\pi$.

When $U_0 = 1$, $F_0 = 1$, the volume flow rate $Q = 0$ at times given by

$$t = \arc\tan \left[\left\{ GH - EK + (E^2 + G^2) \left(\sqrt{\frac{Re}{2}} \right) \right\} \Big/ (EH + GK) \right] \pm n\pi.$$

When $Re = 2$, this time is $0.19\,\pi \pm n\pi$.

When $U_0 = 1$, $F_0 = 1$, the stress at the plates vanish, when $t = \arc\tan [(EH + GK)/(EK - GH)] \pm n\pi$. For $Re = 2$, this time is $0.42\pi \pm n\pi$.

References

1. Schlichting, H., 'Boundary-Layer Theory' (McGraw-Hill Book Company, New York), 1968, p. 85.
2. Khan, M. A. A., *Def. Sci. J.*, **19** (1969), 139.