Ekman Layer on a Porous Plate in Slip Flow Regime

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Received 18 August 1981; revised 18 February 1983

Abstract. The rotating flow of a viscous liquid on a porous plate with velocity slip at the wall is studied. Expressions for the velocity field and the skin friction have been obtained analytically. The effects of rotation and the rarefaction parameters on the flow and on the skin friction are discussed.

1. Introduction

The study of the motion of rarefied gases has stimulated considerable interest in recent years due to its important applications in high altitude flights and in space technology. In hypersonic flights, the cooling of body surface of the spacecraft is generally controlled by injection of air or a suitable gas through the porous surface. Moreover, the large scale and moderate motion of the gas medium in space are greatly affected by the vorticity of the rotation of the earth and other planets.

The problem, although idealized, retains the essential features. It is known that for rarefied gases, the ordinary continuum approach fails to yield satisfactory results. However, when the gas is only slightly rarefied, results agreeing with the observed physical phenomena can be obtained only solving the usual Navier-Stokes equations for incompressible flow together with modified boundary conditions allowing for a velocity slip at the surface. The present theoretical investigation is particularly suitable for studying the effects of rotation and gas rarefaction in the slip flow regime associated with any classical viscous flow problem.

One important class of steady two-dimensional viscous flow problems, on a rotating frame of reference, initiated by Batchelor[1] deals with the effect of coriolis force on the flow of an incompressible fluid. Gupta[2] has found some interesting features for such a rotating flow over an infinite flat plate with suction (or blowing) at the surface. It is of interest to find how Gupta's results get modified when his no-slip boundary conditions are replaced by the velocity slip conditions. This indeed is the motivation of the present investigation.

In the present paper, we consider a uniform flow with velocity U past an infinite porous flat plate coinciding with the plane $z = 0$, which is at rest relative to a liquid rotating with constant angular velocity Ω about the z-axis, the uniform flow being along the x-axis. In such a flow the pressure gradient far away from the plate balances the coriolis force $2\Omega U$, while near the plate viscous and coriolis forces are of same order of magnitude in a layer known as Ekman Layer[1]. The slip boundary conditions pertinent to rarefied gases have been introduced to underline the effects of fluid rarefaction on the Ekman Layer flow. The results have been compared with Gupta's results. We further mention that in the above description of flow, the fluid is assumed incompressible to a fairly valid approximation at least for low Mach number flows. However, for a more accurate description of the high Mach number flow of rarefied gases, the compressibility effect may have to be taken into account.

2. Velocity Field

Assuming u, v, w to be the velocity components in respective directions the governing equations relevant to the present steady state problem in a rotating frame of reference are

$$w = -w_0 \tag{1}$$

$$-w_0 \frac{du}{dz} = \nu \frac{d^2u}{dz^2} + 2\Omega v \tag{2}$$

$$-w_0 \frac{dv}{dz} = \nu \frac{d^2v}{dz^2} - 2\Omega (u - U) \tag{3}$$

$$0 = -\frac{\partial P}{\partial z} \tag{4}$$

where $w_0(>0)$ is a constant representing suction velocity and u, v are functions of z alone.

Introducing dimensionless quantities

$$F(\eta) = \frac{u}{U} - 1 + \frac{iv}{U}, \; \eta = \frac{zw_0}{\nu}, \; E = \frac{2\Omega\nu}{w_0} \tag{5}$$

Equations (2) and (3) may be combined as

$$\frac{d^2F}{d\eta^2} + \frac{dF}{d\eta} - iEF(\eta) = 0 \tag{6}$$

First order velocity slip boundary conditions are given[3] by

$$\left.\begin{aligned} u &= \frac{2-f_1}{f_1} L \frac{du}{dz} = L_1 \frac{du}{dz} \\ v &= \frac{2-f_1}{f_1} L \frac{dv}{dz} = L_1 \frac{dv}{dz} \end{aligned}\right\} \text{at } z = 0 \tag{7}$$

where

f_1 = Maxwell's reflexion coefficient.

L = Mean free path $[\mu(\pi/2\rho p)^{1/2}]$ which is constant for incompressible fluids.

$$L_1 = \frac{2-f_1}{f_1} L = \text{constant for liquids.}$$

and

$$\frac{2-f_1}{f_1} = 0(1) \text{ in general.}$$

The boundary conditions at infinity are

$$u \to U, \; v \to 0 \text{ as } z \to \infty \tag{8}$$

Hence for Eqn. (6), the modified boundary conditions become

$$\begin{aligned} &F(0) + \quad = h_1\left(\frac{dF}{d\eta}\right)_{\eta=0} \\ &F(\infty) = 0 \end{aligned} \tag{9}$$

where

$$\left.\frac{L_1 w_0}{\nu} = \frac{L_1}{l_1}\,\frac{l_1 w_0}{\nu} = K_n Re\right) \text{ is known as rarefaction parameter,}$$

l_1 = characteristic dimension of the flow,

$\frac{L_1}{l_1}$, the Knudsen number,

$\frac{l_1 w_0}{\nu}$, the suction Reynolds number,

and constant factor $\frac{2-f_1}{f_1}$ involved in L_1 does not effect K_n since l_1 itself is rather arbitrary.

The solution of the Eqn. (6) subject to conditions (9) are given by

$$F(\eta) = -\frac{1}{1 + h_1(\alpha + i\beta)} e^{-(\alpha+i\beta)\eta} \tag{10}$$

with

$$\alpha = \frac{1}{2}\left\{\frac{+(1+16E^2)^{1/2}}{2}\right\}^{1/2} \quad \frac{1}{2}(1+q)$$

$$\beta = \frac{1}{2}\left\{\frac{(1+16E)^{1/2}-1}{2}\right\}^{1/2} = \frac{1}{2}(q^2-1)^{1/2} \tag{11}$$

$$q = \left\{\frac{+(1+16E^2)^{1/2}}{2}\right\}^{1/2}$$

Expanding Eqn. (10) into real and imaginary parts, we obtain

$$\frac{u}{U} = \frac{e^{-\alpha\eta}}{\{(1+\alpha h_1)^2+\beta^2 h_1^2\}^{1/2}}\cos(\beta\eta+\theta) \tag{12}$$

$$\frac{v}{U} = \frac{e^{-\alpha\eta}}{\{(1+\alpha h_1)^2+\beta^2 h_1^2\}^{1/2}}\sin(\beta\eta+\theta) \tag{13}$$

where

$$\tan\theta = \frac{\beta h_1}{1+\alpha h_1} \tag{14}$$

The results (12) and (13) corresponding to no-slip ($h_1 = 0$) situation become

$$\frac{u}{U} = - e^{-\alpha\eta}\cos(\beta\eta) \tag{15}$$

$$\frac{v}{U} = e^{-\alpha\eta}\sin(\beta\eta) \tag{16}$$

which appeared in Gupta[2].

It is obvious from the results (12), (13) and (14) that the presence of velocity slip ($h_1 \neq 0$) at the boundary modifies the flow field both in respect of magnitude and phase. It is further observed that the thickness of the boundary layer remains independent of h_1, but there occurs a phase shift of the velocity vector at all heights due to this. The speed of flow however yields.

$$\left|\frac{u+iv}{U}\right| = \left[1 - \frac{2e^{-\alpha\eta}}{M}\left\{(1+\alpha h_1)\cos(\beta\eta) - \beta h_1\sin(\beta\eta)\right\} + \frac{e^{-2\alpha\eta}}{M}\right]^{1/2} \tag{17}$$

where $\quad M = (1+\alpha h_1)^2 + \beta^2 h_1^2.$

The appearance of cos ($\beta\eta$) and sin ($\beta\eta$) in Eqn. (17) clearly indicate offshoots in the speed of flow for various values of the product $\beta\eta$. Since β characterizes speed of rotation of the system and η, the distance from the plate, the offshoots in the velocity field are entirely due to rotation of the system. The occurence of such a situation in rotating fluids even in the absence of wall slip has not been pointed out by Gupta[2]. Furthermore, it is evident from Eqn. (17) that the offshoots in the velocity field occurs at a distance far away from the plate when the speed of rotation is small and appears very near the plate for any moderate speed of rotation of the system. It is also found that these offshoots diminish with the increase of the rarefaction parameter h_1. So the large offshoots in the velocity field due to rotation could be kept within required limits by introducing velocity slip at the boundary. This phenomenon appears to be a deviation from that occurs in an inertial system and Figs. 1 and 2 clearly demonstrate this fact.

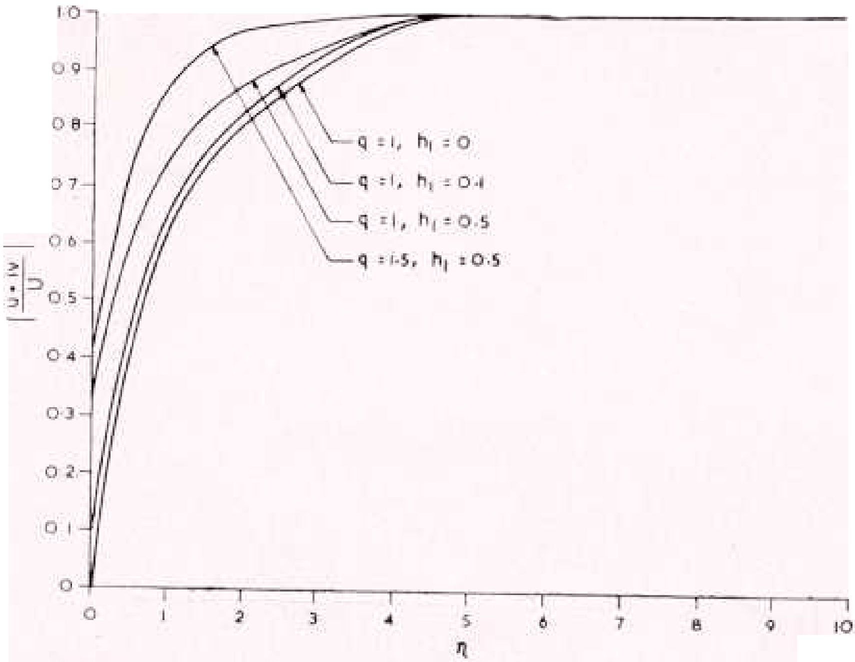

Figure 1. Effects of rotation and rarefaction parameters on the speed of flow.

As regard phase shift of velocity field it is found from Eqn. (14) that

$$\left.\begin{array}{ll} q = 1,\ h_1 \neq 0, & \theta = 0 \\ q \neq 1,\ h_1 = 0, & \theta = 0 \\ q \to \infty,\ h_1 \neq 0, & \theta = \pi/4 \\ h_1 \to \infty,\ q \neq 1, & \theta = \pi/4 \end{array}\right\} \tag{18}$$

So whatever the values of q (rotation parameter) and h_1 (rarefaction parameter) may be, the phase shift of the velocity field in the present case, compared to its classical situation ($q = 1,\ h_1 = 0$) is $\pi/4$ and hence remains bounded for all q and h_1.

3. Skin Friction

The skin-friction, in dimensionless form, is given by

$$\tau^* = -\frac{\tau}{\rho U w_0} = -\left.\frac{\partial F}{\partial \eta}\right|_{\eta=0}$$

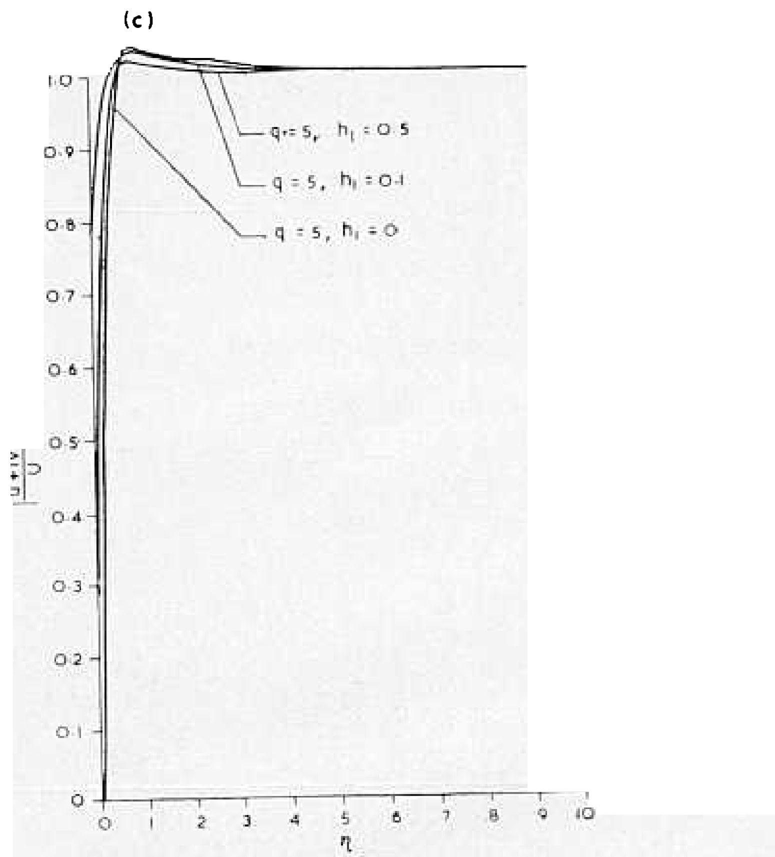

Figure 2. Effects of rotation and rarefaction parameters on the speed of flow

substituting Eqn. (10) in Eqn. (19), we have

$$\tau^* \qquad \frac{\alpha + i\beta}{+ h_1(\alpha + i\beta)} \qquad |\tau^*| e^{i\theta} \tag{20}$$

with

$$|\tau^*| \qquad \frac{\alpha + i\beta}{h_1(\alpha + i\beta)} \qquad \frac{[\beta^2 + \{\alpha + h_1(\beta^2 + \alpha^2)\}^2]^{1/2}}{[(1 + \alpha h_1)^2 + h_1^2\beta^2]}$$

$$\frac{\sqrt{(\alpha^2 + \beta^2)}}{\sqrt{\{(1 + h_1\alpha)^2 + h_1^2\beta^2\}}} \tag{21}$$

and

$$\theta^* \quad \tan^{-1}\left\{\frac{\beta}{\alpha + h_1(\beta^2 + \alpha^2)}\right\} = \tan^{-1}\left(\frac{\beta}{\alpha}\right) \quad \tan^{-1}\left(\frac{h_1\beta}{+ h_1\alpha}\right) \tag{22}$$

where α, β and q are defined in Eqn. (11).

Equation (21) shows that in the no-slip case ($h_1 = 0$), large rotation of the system results in large magnitude of the skin friction. But when a slip velocity is allowed at the wall, the skin friction depends on the slip velocity which, however, can never exceed some finite value. This fact is evident from Fig. 3 where $\log_{10} | \tau^* |$ is plotted against $\log_{10} q$, $| \tau^* |$ being the magnitude of the skin friction and q, a parameter connected with angular velocity of the solid body rotation. In fact the maximum value of $| \tau^* |$ is $1/h_1$ which attains in the limit of infinitely large speed of rotation. Fig. 3 shows that as h_1 increases moderately, the value of $| \tau^* |$ decreases whether or not rotation of the system is large. This also is a consequence of the slip boundary condition.

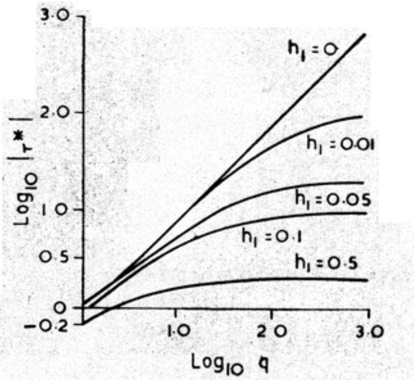

Figure 3. Magnitude of the skin friction

Since α and β are always positive, Eqn. (22) shows that θ^* (principal value) is also positive, implying that the skin friction acts with a phase lead, θ^*, over the main stream. With increasing rotation E of the system the phase lead θ^* increases sharply initially, attains a maximum and then falls to zero ultimately as $E \to \infty$. This result is in contrast with the no-slip case where the phase lead is a monotonically increasing function of E and tends to a constant value $\pi/4$ as $E \to \infty$. Fig. 4 shows

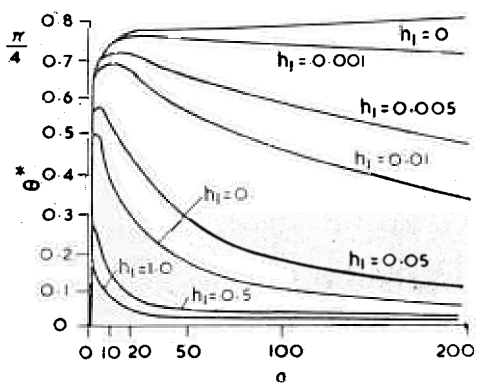

Figure 4. Phase of the skin friction.

the graphs of the phase lead θ^* against q for different values of h_1. The graph for $h_1 = 0$ corresponds to the Gupta's no-slip case. The figure also shows that if h_1 is moderately high the phase lead becomes insignificant irrespective of E and the skin friction continues in phase with main stream.

References

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