STABILITY OF SELF-SIMILAR FLOWS OF SECOND KIND IN THE NEIGHBOURHOOD OF A CRITICAL POINT

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Following the method developed by Bhatnagar & Prasad, based on the investigations of Kulikovskii & Slobodkina, we study the stability of self-similar flows generated by the propagation of shock-wave in an inhomogeneous medium with density varying either exponentially or as a power of distance. Also we consider the shocks produced by impulsive load. We find that all these flows are stable in the neighbourhood of critical point, which is a saddle point of the system of differential equations governing the flow in its neighbourhood.

The stability of the self-similar flows of the second kind as defined by Zeldovich & Raizer[1], in the neighbourhood of a critical point by reducing the system of equations to a quasi-linear equation has been investigated. Bhatnagar & Prasad[2] who have shown that the investigations of Kulikovskii and Slobodkina[3] for the propagations of disturbances in a steady flow can be extended to self-similar flows, have been followed. It has been shown that a self-similar flow of second kind due to imploding shock-wave in the neighbourhood of a critical point is stable with respect to radially symmetric disturbances. Applying this method to the problem of the flow into a cavity, discussed by Hunter[4], it has been shown[5] that the flow is unstable, when the boundary of the cavity is accelerating. It is found that the critical point is a node[5].

The expressions for the discussion of stability, for a very general unsteady one-dimensional flow with variable density have been obtained. As a particular case, for spherically symmetric flow with constant initial density, the result of Bhatnagar & Prasad[2] are obtained. For spherically symmetric isentropic flow with constant initial density, results of Prasad & Tagare[5] are obtained.

Next the stability of the self-similar flow behind a shock-wave propagating towards the edge of a gas is discussed. This problem was first studied by Sakurai[6] and has been given as one of the examples of self-similar flows of second kind by Zeldovich and Raizer[1]. Further the stability of the self-similar flow due to propagation of a shock-wave due to an impulsive load, a problem discussed in Zeldovich & Raizer[1] have been considered. Next the stability of the self-similar flow due to propagation of a shockwave in an exponential medium, a problem considered by Hayes[7] and discussed in Zeldovich & Raizer[1] has been considered.

FORMULATION OF THE PROBLEM AND BASIC EQUATIONS

The equations of one dimensional unsteady motion of a polytropic gas are

$$\rho_t + u\rho_r + \rho u_r + \frac{(\nu - 1)}{r} \rho u = 0 \tag{1}$$

$$\rho \, (u_t + u u_r) + p_r = 0 \tag{2}$$

and

$$\left(p_t + u p_r \right) - \frac{\gamma p}{\rho} \left(\rho_t + u \rho_r \right) = 0 \tag{3}$$

$\nu = 1, 2$ and 3 correspond respectively to plane flow, cylindrically symmetric flow and spherically symmetric flow.

In terms of the new dependent variables G, V and Z and new independent variables η and τ defined by

$$\left.\begin{aligned} \rho &= \rho_c G(\eta, \tau), \; u = \frac{\dot{R}\xi}{\delta} V(\eta, \tau) \\ p &= \rho_0 \dot{R}^2 \frac{\xi^2}{\gamma\delta^2} Z(\eta, \tau)\, G(\eta, \tau) \\ \xi &= \frac{r}{R(t)} \end{aligned}\right\} \tag{4}$$

and

$$\eta = ln \left\{ \frac{r}{A \mid t \mid^{\delta}} \right\}, \; \tau = \frac{1}{\delta} ln \left\{ A \mid t \mid^{\delta} \right\} \tag{5}$$

where $\rho_0(t) = \rho^* R^{\omega}$ and $R(t) = A \mid t \mid^{\delta}$ are functions of time and ρ^*, ω, A and δ are constants, the eqns. (1) to (3) reduce to

$$\frac{G\tau}{G} + \frac{(V - \delta)}{G} G\eta + V\eta + \nu V + \delta\omega = 0 \tag{6}$$

$$V\tau + (V - \delta) V\eta + \frac{Z}{\gamma G} G\eta + \frac{Z\eta}{\gamma} + \frac{2Z}{\gamma} + V(V - 1) = 0 \tag{7}$$

and

$$\frac{(\gamma - 1) Z}{(V - \delta)} G\tau - \frac{Z\tau}{(V - \delta)} + \frac{(\gamma - 1) Z}{G} G\eta - Z\eta - 2 \left[\frac{(V - 1)}{(V - \delta)} - \frac{\delta\omega (\gamma - 1)}{2 (V - \delta)} \right] Z = 0 \tag{8}$$

The characteristics of the eqns. (6) to (8) in (η, τ)-plane are

$$\frac{d\eta}{d\tau} = (V - \delta) \text{ and } \frac{d\eta}{d\tau} = (V - \delta) \pm \sqrt{Z} \tag{9}$$

Let the suffix zero represent the values of the flow variables in a self-similar flow, so that G_0, V_0, and Z_0 are functions of η only, and satisfy the system of ordinary differential equations

$$\frac{dV_0}{d\eta} + \frac{(V_0 - \delta)}{G_0} \frac{dG_0}{d\eta} + \nu V_0 + \delta\omega = 0, \tag{10}$$

$$(V_0 - \delta) \frac{dV_0}{d\eta} + \frac{Z_0}{\gamma G_0} \frac{dG_0}{d\eta} + \frac{1}{\gamma} \frac{dZ_0}{d\eta} + \frac{2Z_0}{\gamma} + V_0 (V_0 - 1) = 0 \tag{11}$$

and

$$\frac{(\gamma - 1) Z_0}{G_0} \frac{dG_0}{d\eta} - \frac{dZ_0}{d\eta} - 2 \left[\frac{(V_0 - 1)}{(V_0 - \delta)} - \frac{\omega\delta (\gamma - 1)}{2 (V_0 - \delta)} \right] Z_0 = 0 \tag{12}$$

In order that the solution of the eqns. (10) to (12) satisfy the correct boundary conditions in a self-similar flow of second kind, it is necessary that the integral curve in (Z_0, V_0)-plane must pass through the singular point (Z_0^*, V_0^*) determined by the equations

$$f(V_0^*) \equiv V_0^{*2}(\nu-1) + V_0^* \left\{ \frac{2(\delta-1)}{\gamma} + \frac{\delta\omega}{\gamma} - \delta\nu + 1 \right\} - \delta \left\{ \frac{2(\delta-1)}{\gamma} + \frac{\delta\omega}{\gamma} \right\} = 0 \tag{13}$$

and

$$Z_0^* = (\delta - V_0^*)^2 \tag{14}$$

For planar motion with $\nu=1$, eqns. (13) and (14) determine a unique singular point (Z_0^*, V_0^*). The condition that the integral curve passes through this singular point (Z_0^*, V_0^*) determines the value of the similarity exponent δ. In the case of non-planar motion $\nu \neq 1$ it is found in all cases we have considered that only the larger root V_0^* of eqn. (13) corresponds to physically realisable flows. Consequently, the similarity exponent is determined through that integral curve which passes through the singular point corresponding to this larger root.

The characteristic velocity

$$C_0 = (V_0 - \delta) + \sqrt{Z_0}$$

which vanishes at (V_0^*, Z_0^*) satisfies

$$\frac{dC_0}{d\eta} = -\nu V_0 - \frac{(\delta-1)}{\gamma} - \frac{\delta\omega}{\gamma} \cdot \frac{(\gamma+1)}{\gamma} - \sqrt{Z_0} + \frac{\left\{ (V_0-\delta) - \frac{(\gamma-1)}{\gamma}\sqrt{Z_0} \right\} f(V_0)}{(V_0-\delta)^2 - Z_0} \tag{15}$$

We can also show that at the point (V_0^*, Z_0^*)

$$\left(\frac{dV_0}{d\eta}\right)^* = \frac{2}{(\gamma+1)} \left[\left(\frac{dC_0}{d\eta}\right)^* + 1 - \left\{ \frac{\nu(\gamma-1)+2}{2} \right\} V_0^* \right] \tag{16}$$

In the neighbourhood of the singular point (V_0^*, Z_0^*) we have,

$$\frac{f(V_0)}{Z_0 - (\delta - V_0)^2} = \frac{f'(V_0^*) \left(\frac{dV_0}{d\eta}\right)^*}{\left\{ \sqrt{Z_0^*} - (\delta - V_0^*) \right\} \left(\frac{dC_0}{d\eta}\right)^*} + 0\left[(\eta - \eta^*) \right] \tag{17}$$

where η^* is the value of when $V_0 = V_0^*$. Therefore, it follows from (15) that $\left(\frac{dC_0}{d\eta}\right)^*$ is given by

$$\left\{ \left(\frac{dC_0}{d\eta}\right)^* \right\}^2 - \alpha \left(\frac{dC_0}{d\eta}\right)^* - \beta = 0 \tag{18}$$

where

$$\alpha = -\frac{[\delta(\nu+2) - 1 + \delta\omega]}{2} \tag{19}$$

and

$$\beta = -\frac{[\{\nu(\gamma-1)+2\}V_0^*-2]}{4} \times \left\{2(\nu-1)V_0^* + \frac{2(\delta-1)}{\gamma} + \frac{\delta\omega}{\gamma} - \delta\nu + 1\right\} \quad (20)$$

Following Kulikovskii & Slobodkina[3], the propagation of the self-similar flow and the self-similar flow in the neighbourhood of the critical point are governed by

$$\frac{\partial C}{\partial \tau} + C\frac{\partial C}{\partial \eta} = \alpha C + \beta(\eta - \eta^*) \quad (21)$$

and

$$\frac{d\eta}{d\tau} = C_0, \quad \frac{dC_0}{d\tau} = \alpha C_0 + \beta(\eta - \eta^*). \quad (22)$$

STABILITY OF SELF-SIMILAR FLOW DUE TO PROPAGATION OF SHOCK-WAVE TOWARDS THE EDGE OF THE GAS

Consider the stability of a plane flow with $\nu = 1$, when a shock-wave propagates through a non-uniform medium of decreasing density ($\omega \neq 0$) and reaches the boundary where the density vanishes. This problem was first discussed by Sakurai[6]. Then

$$\alpha = -\frac{\{(3\delta-1)+\delta\omega\}}{2} \quad (23)$$

and

$$\beta = \frac{(\gamma+1)}{4\gamma}\left(V_0^* - \frac{2}{(\gamma+1)}\right)\{\delta(\gamma-2-\omega)-(\gamma-2)\} \quad (24)$$

The same values of the parameters ω, δ and γ are taken as given by Zeldovich & Raizer[1]. We then tabulate α and β (See Tables 1 to 3).

TABLE 1

FOR $\gamma = 5/3$

ω	δ	α	β
3·25	0·590	—1·344	0·2752
2	0·696	—1·240	0·1879
1	0·816	—1·132	0·0349
0·5	0·877	—1·035	0·0279

TABLE 2

FOR $\gamma = 7/5$

ω	δ	α	β
1	0·718	—1·295	0·1842
1	0·831	—1·162	0·8500
0·5	0·906	—1·085	0·0387

TABLE 3

FOR $\gamma = 6/5$

ω	δ	α	β
2	0·752	—1·380	0·1096
1	0·855	—1·210	0·0865
0·5	0·920	—1·110	0·3850

From Tables 1 to 3 it can be seen that for all these cases, $\alpha < 0$ and $\beta > 0$ and hence the singular point is a saddle point.

In this problem the time is measured when the shock reaches the edge of the gas so that $\tau = ln\,(-t), R\,(t) = A\,(-t)^{\delta}$ and t increases from $-\infty$ to 0. $\tau = ln\,|\,t\,|$ decreases from $+\infty$ to $-\infty$ as t increases from $-\infty$ to 0. Hence as t increases τ decreases. But $\alpha < 0$. Hence as t increases $\alpha\tau$ also increases and hence the area of the perturbation $S = S_0 e^{\alpha\tau}$ in $(C\,,\eta)$-plane increases. Following Kulikovskii & Slobodkina[3] it is concluded that only one of the four steady flows passing through the saddle point is stable. Here actual flow is represented by l of in Fig. 1 and is stable in the neighbourhood of the singular point.

STABILITY OF SELF-SIMILAR FLOW OF A GAS UNDER THE ACTION OF AN IMPULSIVE LOAD

The stability of self-similar flow of a gas under the action of an impulsive load have been discussed. When the gas-surface is subjected to an impulsive load by methods described in

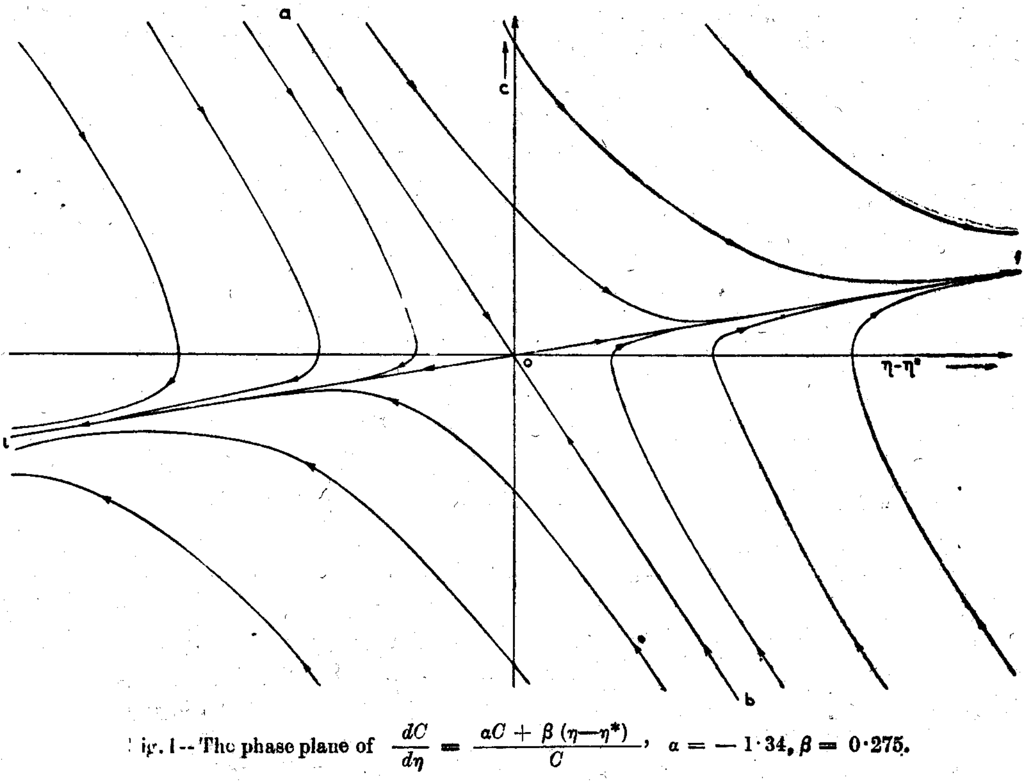

Fig. 1—The phase plane of $\dfrac{dC}{d\eta} = \dfrac{\alpha C + \beta\,(\eta - \eta^*)}{C}$, $\alpha = -1{\cdot}34,\ \beta = 0{\cdot}275.$

Zeldovich & Raizer[1], the motion is self-similar motion of second kind. In this case plane wave ($\nu = 1$) propagates in a gas of uniform density ($\omega = 0$). The time is measured from the instant the impulsive load is applied and the self-similar, motion is realised for $t > 0$.

Thus

$$R(t) = A\ (t)^{\delta},\ \tau = ln\ (t) \tag{25}$$

By putting $\nu = 1$, $\omega = 0$ in equations (13), (19) and (20), we get

$$V_0^* = \frac{2\delta}{2-\gamma} \tag{26}$$

$$\alpha = -\frac{(3\delta - 1)}{2} \tag{27}$$

and

$$\beta = \frac{(\gamma + 1)}{4\gamma}(\gamma - 2)(\delta - 1)\left(V_0^* - \frac{2}{(\gamma + 1)}\right) \tag{28}$$

In Table 4 the values of α, β for values of δ and γ as given in Zeldovich & Raizer[1] are given.

It will be seen from Table 4 that the singular point is a saddle point.

Here t increases from 0 to $+\infty$ and hence $\tau = ln\ (t)$ also increases from $-\infty$ to $+\infty$ Thus as t increases τ also increases and $\alpha\tau$ decreases. Thus all the four possible flows near the singular point are stable. Here the actual flow is represented by *aob* in Fig. 1 and is stable.

STABILITY OF SELF-SIMILAR FLOW DUE TO PROPAGATION OF SHOCK-WAVE IN AN EXPONENTIAL MEDIUM

Let a strong shock propagate in a medium in which the density varies exponentially with initial density, producing a self-similar motion of second kind. Such models have been used for atmospheres of stars as well as of earth by many.

The gas is assumed to be polytropic with polytropic exponent γ. The gas is initially at rest at zero temperature and zero pressure under no body force, with a density distribution given by

$$\rho = \rho^* e^r / \triangle \tag{29}$$

TABLE 4

γ	δ	α	β
1·1	0·569	—0·3535	0·0577
1.4	0·600	—0·4000	0·1715
1.6667	0·611	—0·4165	0·1512
2·8	0·627	—0·4405	0·2120

Here ρ^* is the density at surface and $\triangle$ is the characteristic length for the density variation. Following Hayes[7], we suppose that the density increases as r increases downwards the surface.

Let a plane shock-wave be located at $r = R(t)$, and the motion of the shock is assumed to follow the law

$$R = \delta \triangle \ln \mid t \mid \equiv \delta \triangle \tau \text{ (say)} \tag{30}$$

where δ is the similarity parameter. The velocity of the shock is then given by

$$\dot{R} = \delta \triangle t^{-1} \tag{31}$$

The similarity variable ξ is defined by

$$\xi = \frac{(r - R)}{\triangle} \tag{32}$$

When the shock front is propagating towards the surface, the similarity parameter ξ is negative behind the shock and the time t is positive and tends to $+\infty$. The situation is reversed when the shock is propagating outwards from the surface.

In order to study the stability of this flow behind the shock the following non-dimensional variables have been introduced :

$$\pi(\xi, \tau) = \frac{p(r, t)}{\rho_0 \delta^2 \triangle^2 \mid t \mid^{\delta - 2}}, \quad V(\xi, \tau) = \frac{u(r, t)}{\delta \triangle t^{-1}}$$

and

$$g(\xi, \tau) = \frac{\rho(r, t)}{\rho_0 \mid t \mid^{\delta}} \tag{33}$$

Separate analysis is needed according as whether shock is propagating into thinner atmosphere or into denser atmosphere.

Case 1 : Let the shock traverse into thinner atmosphere. The characteristic speed C in (ξ, τ) – plane which vanishes at the singular point satisfies an equation similar to equation (21) and is given by

$$\frac{\partial C}{\partial \tau} + C \frac{\partial C}{\partial \xi} = \frac{(\gamma + \delta - 2)}{2\gamma} (\xi - \xi^*) \tag{34}$$

where ξ^* is the value of ξ when $V_0 = V_0^*$.

The coefficient α is identically equal to zero and the coefficient $\beta = \dfrac{(\gamma + \delta - 2)}{2\gamma} > 0$

since[1] $\gamma > 1$ and $\delta > 1$. Hence the singular point is a saddle point and the area of perturbation $S = S_0 e^{\alpha\tau}$ in (C, ξ) – plane is always a constant for all values of τ. Here τ decreases from ∞ to 0 as t increases from 0 to ∞. The actual flow in this case is *lof* in Fig. 1 and is stable.

Case 2 : When motion of a shock front is in the direction of increasing density, α remains zero and $\beta > 0$ since[1] $\gamma > 1$ and $\delta > 1$. The singular point is a saddle point. The area of perturbation is constant for all values of τ. Here τ increases from $-\infty$ to $+\infty$, as t increase from 0 to $+\infty$. Actual flow in this case is *aob* in Fig. 1 and is stable.

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