STRUCTURE OF THE CONTACT REGION AND ITS APPLICATION TO THE REFLEXION OF A PLANE SHOCK WAVE FROM A HEAT-CONDUCTING WALL

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A study of the structure of the contact region has been made taking into account the effects of viscosity, heat conduction and radiative heat transfer. Analytical solutions for the temperature, velocity and pressure distributions in a uniformly moving contact region have been obtained under the optically thick-gas approximation when the thermal conductivity and absorption coefficients are given by power laws. Applying the analysis of the contact region to the situation when a plane shock is reflected from a plane heat-conducting wall it has been shown that the reflected shock is attenuated due to the combined effects of molecular heat conduction and radiative heat conduction.

A contact region can occur in compressible fluid flow in different physical situations. In ideal gas theory this is regarded as a discontinuous front across which density, temperature and entropy undergo jumps but the pressure is the same on both sides of the front so that it moves with the fluid. But in fact it is, like a shock layer, a region of small thickness where dissipative mechanisms are effective. Goldsworthy[1] studied the structure of the contact region on the basis of the concepts associated with the Prandtl's boundary layer theory. He showed that the pressure across it remains constant approximately. Hall[2] in his study of a uniformly moving contact region, experimentally verified that very little pressure change occurs across it.

Goldsworthy[1] applied his analysis of the contact region to the situation when a plane shock is reflected from a plane heat-conducting wall. He showed that the shock is attenuated due to the presence of the contact region adjacent to the wall, in which the effects of viscosity and heat conduction were considered.

Sturtevant & Slachmuylders[3] have experimentally measured the position and the velocity of the reflected shock and found that Goldsworthy's approach to the problem of shock reflexion from a heat-conducting wall is in a quite reasonably agreement with their observations. Further they concluded that temperature jump effects at the interface are not significant when the reflected shock is more than ten shock thicknesses distant from the wall. Baganoff's[4] measurements of the pressure rise at the wall are, as he states, not conclusively in favour of Goldsworthy's results, neither are they seriously at variance with it. The cause of the uncertainties is attributed to the inherent difficulties of pressure measurement on sub-microsecond scale. Clarke[5] has included the temperature jump effects in his study of the problem using the technique of matched asymptotic expansions and his results agree reasonably well with experimental observations. He concludes that the temperature jump effects may account for some observations of reflected shock trajectories which, according to Sturtevant & Slachmuylders[3] is attributed to experimental inaccuracies.

Goldsworthy[1] considered the effects of viscosity and heat conduction in his analysis of the structure of the contact region. I, following him, has studied this problem including radiative heat transfer under the optically thick-gas approximation. To

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illustrate the modified theory, which incorporates the general problem of an accelerating contact region, I have obtained analytical solutions for the temperature, velocity and pressure distributions in a uniformly moving contact region when the thermal conductivity and absorption coefficients are given by power laws. Under the caption "Structure of a Uniformly Moving Contact Region", I have obtained analytical solution to the problem for any ratio T_2/T_1 of temperatures. A series solution has been developed for the case when the ratio $(T_1-T_2)/T_1$ is much less than unity. It is found that the effect of radiative heat transfer is, under the thick-gas approximation, similar to that of molecular heat conduction as expected. Under the caption "Normal Reflexion of a Shock from a Heat-Conducting Wall", I have applied the theory of the contact region to determine the flow set up when a plane shock is reflected from a heat-conducting wall. I have found that the combined effects of radiative heat conduction and molecular heat conduction decrease the strength of the reflected shock.

EQUATION GOVERNING THE FLOW

I have considered a uniform gas of infinite extent which is initially at rest and in the region $y < 0$ it is heated at a rate depending only on the distance y from the fixed plane $y = 0$ and the time t. This situation involves a shock wave which is propagated into the non-heated gas. The shock wave is followed by a contact surface which separates the heated gas from the non-heated gas. If viscous and heat conduction effects are neglected, the motion and conditions at the two sides of the contact discontinuity can be determined given the rate of generation of energy in the fluid. It is assumed that this 'ideal-gas' solution is known. The one-dimensional gas-dynamic equations including radiant heat transfer but disregarding the radiation energy and pressure are

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial y} (\rho u) = 0, \tag{1}$$

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial y} = - \frac{\partial p}{\partial y} + \frac{4}{3} \frac{\partial}{\partial y} \left(\mu \frac{\partial u}{\partial y} \right), \tag{2}$$

$$c_p \left(\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial y} \right) - \frac{1}{\rho} \left(\frac{\partial p}{\partial t} + u \frac{\partial p}{\partial y} \right) = Q (y, t) \, . \, H (y_0 - y)$$
$$+ \frac{4}{3} \frac{\mu}{\rho} \left(\frac{\partial u}{y} \right)^2 + \frac{1}{\rho} \frac{\partial}{\partial y} \left(k_g \frac{\partial T}{\partial y} \right) - \frac{1}{\rho} \frac{\partial F_r}{\partial y}, \tag{3}$$

$$p = R \rho T, \tag{4}$$

where u is the fluid velocity, p the pressure, ρ the density, T the temperature, F_r the radiative flux in the positive y direction, y the distance measured from the fixed plane $y = 0$, $y_0 (t)$ the position at time t of the particle initially at the origin, $H (y_0-y)$ the Heaviside unit function which is zero for $y > y_0 (t)$, $Q (y, t)$ the rate at which energy is generated per unit mass of the fluid, μ the coefficient of viscosity, k_g the coefficient of thermal conductivity, c_p the specific heat at constant pressure and R the gas constant.

Since the contact surface moves with the fluid equations (1) to (3) are rewritten in the Lagrangian frame of reference for the sake of convenience

$$\psi = \bar{\rho} x = \int_{y_0 (t)}^{y} \rho dy, \tag{5}$$

$$\frac{\partial u}{\partial t} = - \frac{\partial p}{\partial \psi} + \frac{4}{3} \frac{\partial}{\partial \psi} \left(\mu \rho \frac{\partial u}{\partial \psi} \right), \tag{6}$$

$$c_p \frac{\partial T}{\partial t} - \frac{1}{\rho} \frac{\partial p}{\partial t} = Q(\psi, t) H(-\psi) + \frac{4}{3} \mu\rho \left(\frac{\partial u}{\partial \psi}\right)^2 + \frac{\partial}{\partial \psi}\left(k_g \rho \frac{\partial T}{\partial \psi}\right) - \frac{\partial F_r}{\partial \psi}, \tag{7}$$

where x is the initial position of a particle which is at position y at time t, $\bar{\rho}$ the initial constant density of the gas and ψ and t are the new independent variables.

Writing equation (5) in the form

$$y - y_0(t) = \int_0^{\psi} \frac{1}{\rho} d\psi \tag{8}$$

and differentiating (8) with respect to t keeping ψ constant, the velocity u of a fluid particle is found

$$u = u_0(t) + \int_0^{\psi} \frac{\partial}{\partial t}\left(\frac{1}{\rho}\right) d\psi, \tag{9}$$

where $u = \left(\frac{\partial y}{\partial t}\right)_{\psi}$ and $u_0\ (t)$ is the velocity at time t of the particle initially at the origin.

Writing the actual pressure and gas velocity in the contact region in the form

$$p = P + p', \qquad u = U + u', \tag{10}$$

and substituting (10) in (9) and (6), we obtain

$$u' = u_0'(t) + \int_0^{\psi} \frac{\partial}{\partial t}\left(\frac{1}{\rho} - \frac{1}{\omega}\right) d\psi \tag{11}$$

$$\frac{\partial u'}{\partial t} = -\frac{\partial p'}{\partial \psi} + \frac{4}{3} \frac{\partial}{\partial \psi}\left[\mu\rho\left(\frac{\partial U}{\partial \psi} + \frac{\partial u'}{\partial \psi}\right)\right] \tag{12}$$

where P, ω and U denote the 'ideal-gas' solutions for the pressure, density and velocity.

We use suffices 1 and 2 to label quantities in the heated and non-heated parts of the gas adjacent to the contact region respectively. We assume that the thickness δ of the contact region is small and neglect $\frac{\partial p'}{\partial \psi}$ $\left(\text{which is } 0\left\{\left(1 - \frac{\omega_1}{\omega_2}\right)\delta\right\}\right)$ and the viscous term in equation (7) following Goldsworthy[1]. Then the pressure in equation (7) can be replaced by the ideal gas pressure $P_0\ (t)$ evaluated at $\psi = 0$. He[1] also replace $Q\ (\psi, t)$ by $Q_0\ (t)$ on the assumption that the rate of generation of heat in the fluid is not too strongly dependent upon the temperature. Now the equation (7) becomes

$$\frac{\partial T}{\partial t} - \frac{(\gamma - 1)}{\gamma P_0} \cdot \frac{dP_0}{dt} T = \frac{Q_0(t) H(-\psi)}{c_p} + \frac{P_0}{c_p R} \cdot \frac{\partial}{\partial \psi}\left(\frac{k_g}{T} \frac{\partial T}{\partial \psi}\right) - \frac{1}{c_p} \frac{\partial F_r}{\partial \psi}. \tag{13}$$

Under the optically thick-gas approximation

$$F_r = -\frac{16\sigma T^3}{3k_r}\frac{\partial T}{\partial y} \tag{14}$$

which, on using (5), takes the form

$$F_r = -\frac{16\sigma\rho T^3}{3k_r}\frac{\partial T}{\partial \psi}, \tag{15}$$

where k_r is the absorption coefficient per unit volume and σ the Stefan-Boltzmann constant.

We solve (13) subject to the boundary conditions $T = T_1$ at $\psi = -\infty$, $T = T_2$ at $\psi = \infty$, where T_1 and T_2 are the temperatures of the gas on either side of the contact discontinuity and they, using the relation (15), satisfy the equations

$$\frac{dT_1}{dt} - \frac{(\gamma-1)}{\gamma P_0}\frac{dP_0}{dt}T_1 = \frac{Q_0(t)}{c_p} \tag{16}$$

$$\frac{dT_2}{dt} - \frac{(\gamma-1)}{\gamma P_0}\frac{dP_0}{dt}T_2 = 0\,. \tag{17}$$

Putting $T = T_1\theta(\psi, t)$ or $T = T_1\theta + T_2$, the equation (13) using (16) and (17) reduces to

$$\frac{\partial}{\partial t}\left[\frac{T_1}{T_2}\{\theta - H(-\psi)\}\right] = \frac{P_0T_1}{c_pRT_2}\frac{\partial}{\partial\psi}\left(\frac{k_g}{T}\frac{\partial\theta}{\partial\psi}\right) - \frac{1}{T_2c_p}\frac{\partial F_r}{\partial\psi}\,. \tag{18}$$

Making use of (15) in (18) we have

$$\frac{\partial}{\partial t}\left[\frac{T_1}{T_2}\{\theta - H(-\psi)\}\right] = \frac{P_0T_1}{c_pRT_2}\frac{\partial}{\partial\psi}\left(\frac{k_g}{T}\frac{\partial\theta}{\partial\psi}\right) + \frac{16\sigma T_1}{3T_2c_p}\frac{\partial}{\partial\psi}\left(\frac{\rho T^3}{k_r}\frac{\partial\theta}{\partial\psi}\right). \tag{19}$$

Equation (19), in general, can be solved numerically once the laws governing the variations of k_g and k_r are prescribed. In what follows we have solved (19) analytically when k_g and k_r are given by power laws. Once the temperature is known as a function of ψ and t, the position y of a particle is determined from equation (8) which can be written

$$y - y_0(t) = \left(\frac{R}{P_0}\right)\int_0^{\psi} T\,d\psi. \tag{20}$$

By using equations (11), (12), (15), (18) and (20) the expressions for the velocity and pressure are obtained.

$$u - U = -\frac{1}{\gamma P_0}\frac{dP_0}{dt}(y - Y) + \frac{k_g}{c_pT}\frac{\partial T}{\partial\psi} + \frac{16\sigma R\rho T^3}{3c_pP_0k_r}\frac{\partial T}{\partial\psi} + f(t), \tag{21}$$

$$p - P = \left[\frac{d}{dt} \left(\frac{1}{\gamma P_0} \frac{dP_0}{dt} \right) - \left(\frac{1}{\gamma P_0} \frac{dP_0}{dt} \right)^2 \right] \int_0^{\psi} (y - Y)\, d\psi - f'(t)\, \psi$$

$$+ \frac{1}{\gamma P_0} \frac{dP_0}{dt} f(t)\, \psi + \left(\frac{4}{3} P_r - 1 \right) \frac{k_g}{c_p T} \frac{\partial T}{\partial t}$$

$$+ \frac{1}{c_p P_0} \frac{dP_0}{dt} \int^{T} \left(\frac{k_g}{\gamma T} - \frac{4P_r}{3} \frac{dk_g}{dT} \right) dT$$

$$- \frac{\gamma + 1}{\gamma} \frac{R}{P_0^2 c_p} \frac{dP_0}{dt} \int_0^{\psi} F_r\, d\psi + \frac{R}{P_0 c_p} \int_0^{\psi} \frac{\partial F_r}{\partial t}\, d\psi + g(t), \tag{22}$$

where P_r is the Prandtl number, $f(t)$ and $g(t)$ are arbitrary functions of time, Y is the position of a particle at time t given by 'ideal-gas' flow theory and y the corresponding position of the particle when the effects of viscosity, heat conduction and radiative heat conduction are considered. The functions $f(t)$ and $g(t)$ are determined by considering the effect of the contact region on the external ideal-gas flow. For a uniformly moving contact surface which we have considered, $\dfrac{dP_0}{dt} = 0$, $\dfrac{\partial T}{\partial t} \to 0$ at the edges of the contact region. As such, in this case, it follows from (21) and (22) that $u \to U$ and $p \to P$ at both edges of the contact region if $f(t) \equiv 0$, $g(t) \equiv 0$.

STRUCTURE OF A UNIFORMLY MOVING CONTACT REGION

Since the contact region is assumed to move with constant velocity, the ratio T_2/T_1 of temperatures across it remains constant. It is assumed that k_g and k_r vary according to the power laws.

$$\left. \begin{aligned} k_g &= c_1 T, \\ k_r &= c_2 \rho T^3 \end{aligned} \right] \tag{23}$$

where c_1 and c_2 are constants. By making use of (23) and $T = T_1\,\theta$ equation (19) reduces to

$$\frac{\partial \theta}{\partial t} = \left(\frac{P_0 c_1}{c_p R} + \frac{16\sigma}{3 c_p c_2} \right) \frac{\partial^2 \theta}{\partial \psi^2}, \tag{24}$$

which in terms of the similarity variable defined by

$$\eta = \psi / (t\beta)^{\frac{1}{2}} \tag{25}$$

where

$$\beta = \frac{P_0 c_1}{c_p R} + \frac{16\sigma}{3 c_p c_2} \tag{26}$$

transforms to

$$\frac{d^2\theta}{d\eta^2} + \frac{\eta}{2}\;\frac{d\theta}{d\eta} = 0. \tag{27}$$

Equation (27) under the boundary conditions

$$\theta(\infty) = \frac{T_2}{T_1}\;,\; \theta(-\infty) = 1,$$

admits the solution

$$\theta = \tfrac{1}{2}\left(1 + \frac{T_2}{T_1}\right) - \tfrac{1}{2}\left(1 - \frac{T_2}{T_1}\right) \operatorname{erf}\left(\frac{\eta}{2}\right). \tag{28}$$

Equations (21) and (22) on using (28), $\dfrac{dP_0}{dt} = 0,$

$$f(t) = 0 \text{ and } g(t) = 0$$

give the velocity and pressure distributions in a uniformly moving contact region respectively.

Series Solutions

It is assumed that k_g and k_r vary according to power laws given by

$$k_g = k_1 T,\; k_r = k_2 \rho T^2, \tag{29}$$

and in order to obtain analytical solution, the ratio of temperatures across the contact region has to be restricted.

It is assumed that

$$\frac{T_1 - T_2}{T_1} << 1.$$

By making use of (29) and $T = T_1\theta + T_2$ the equation (19) reduces to

$$\frac{\partial\theta}{dt} = \left(\frac{P_0 k_1}{c_p R} + \frac{16\sigma T_2}{3k_2 c_p}\right)\frac{\partial^2\theta}{\partial\psi^2} + \frac{16\sigma T_1}{3k_2 c_p}\left[\left(\frac{\partial\theta}{\partial\psi}\right)^2 + \theta\,\frac{\partial^2\theta}{\partial\psi^2}\right]. \tag{30}$$

The boundary conditions are

$$\theta(\infty) = 0,\; \theta(-\infty) = \epsilon, \tag{31}$$

where

$$\epsilon = \frac{T_1 - T_2}{T_1}\;.$$

We seek the solution of (30) in the form

$$\theta = \epsilon\theta_0 + \epsilon^2\theta_1 + \ldots . \tag{32}$$

On substituting (32) in (30) and on equating coefficients of equal powers of ϵ we obtain

$$\frac{\partial\theta_0}{dt} = \left(\frac{P_0 k_1}{c_p R} + \frac{16\sigma T_2}{3k_2 c_p}\right)\frac{\partial^2\theta_0}{\partial\psi^2}, \tag{33}$$

$$\frac{\partial \theta_1}{\partial t} = \left(\frac{P_0 k_1}{c_p R} + \frac{16\sigma T_2}{3k_2 c_p} \right) \frac{\partial^2 \theta_1}{\partial \psi^2} + \frac{16\sigma T_1}{3k_2 c_p} \left[\left(\frac{\partial \theta_0}{\partial \psi} \right)^2 + \theta_0 \frac{\partial^2 \theta_0}{\partial \psi^2} \right] \tag{34}$$

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The boundary conditions (31) with the help of (32) take the form

$$\left. \begin{array}{l} (i)\ \theta_0(\infty) = \theta_1(\infty) = 0, \\ (ii)\ \theta_0(-\infty) = 1, \theta_1(-\infty) = 0 \end{array} \right] \tag{35}$$

Zeroth Order Approximation

The zeroth order equation (33) in terms of the similarity variable defined by

$$\eta = \psi/(t\beta_1)^{\frac{1}{2}} \tag{36}$$

where

$$\beta_1 = \frac{P_0 k_1}{c_p R} + \frac{16\sigma T_2}{3k_2 c_p}, \tag{37}$$

reduces to

$$\frac{d^2\theta_0}{d\eta^2} + \frac{\eta}{2} \frac{d\theta_0}{d\eta} = 0, \tag{38}$$

which under the boundary conditions

$$\theta_0(\infty) = 0,\ \theta_0(-\infty) = 1,$$

admits the following solution

$$\theta_0 = (1/2) \operatorname{erfc} \left(\frac{\eta}{2} \right). \tag{39}$$

First Order Approximation

The first order equation (34) by using (36) reduces to

$$\frac{d^2\theta_1}{d\eta^2} + \frac{\eta}{2} \frac{d\theta_1}{d\eta} = -\frac{16\sigma T_1}{3k_2 c_p \beta_1} \left[\left(\frac{d\theta_0}{d\eta} \right)^2 + \theta_0 \frac{d^2\theta_0}{d\eta^2} \right], \tag{40}$$

which by using (39) and the boundary conditions

$$\theta_1(\infty) = \theta_1(-\infty) = 0,$$

admits the following solution

$$\theta_1(\eta) = \frac{\beta_2}{8} \left[1 - \left\{ \operatorname{erf} \left(\frac{\eta}{2} \right) \right\}^2 + \int_{-\infty} \eta \operatorname{erfc} \left(\frac{\eta}{2} \right) \cdot \operatorname{erf} \left(\frac{\eta}{2} \right) d\eta \right.$$
$$\left. + \operatorname{erf} \left(\frac{\eta}{2} \right) \int_{-\infty} \eta \operatorname{erfc} \left(\frac{\eta}{2} \right) d\eta, \right. \tag{41}$$

where

$$\beta_2 = \frac{16\sigma T_1}{3k_2 \beta_1 c_p}.$$

NORMAL REFLEXION OF SHOCK FROM A HEAT-CONDUCTING WALL

It is assumed that at time $t = 0$, a plane shock of given strength is reflected from the face, $y = 0$, of a wall which occupies the region $y > 0$. If viscosity, heat conduction and radiation heat conduction are neglected, the velocity U_s of the reflected shock can be determined. A contact region near the face of the wall exists which influences the external ideal-gas flow. In what follows we determine the distribution of temperature, pressure and velocity in the contact region and thereby show how the shock is affected by the conducting wall. We use the subscripts 2 and 3 to denote the flow variables ahead of and behind the reflected shock respectively.

In the gas (in the region $y < 0$) adjacent to the wall, the temperature satisfies equation (19) which we rewrite in the form

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial \psi} \left(\frac{P_3}{c_p R} \frac{k_g}{T} \frac{\partial T}{\partial \psi} \right) + \frac{16\sigma}{3c_p} \frac{\partial}{\partial \psi} \left(\frac{\rho}{k_r} \cdot T^3 \frac{\partial T}{\partial \psi} \right), \tag{42}$$

which for $k_g = c_1 T$ and $k_r = c_2 \rho T^3$, reduces to

$$\frac{\partial T}{\partial t} = \left(\frac{P_3 c_1}{c_p R} + \frac{16\sigma}{3c_2 c_p} \right) \frac{\partial^2 T}{\partial \psi^2}. \tag{43}$$

In the region $y > 0$, the temperature of the solid satisfies the diffusion equation

$$\frac{\partial T}{\partial t} = \frac{k_w}{\rho_w c_w} \frac{\partial^2 T}{\partial y^2}, \tag{44}$$

where k_w is the thermal conductivity, ρ_w the density and c_w the specific heat of the wall. The initial temperature of the wall is denoted by T_1. Equation (43) under the boundary condition

$$T \to T_3 \text{ as } \psi \to -\infty,$$

admits the solution

$$T - T_3 = B \left[1 + \operatorname{erf} \left(\frac{\psi}{2 (\beta_4 t)^{\frac{1}{2}}} \right) \right] (\text{in } y < 0), \tag{45}$$

where

$$\beta_4 = \left(\frac{P_3 c_1}{c_p R} + \frac{16\sigma}{3c_2 c_p} \right)$$

and B is the constant of integration.

Equation (44) under the boundary condition

$$T \to T_1 \text{ as } y \to +\infty,$$

admits the solution

$$T - T_1 = A \left[1 - \operatorname{erf} \left(\frac{y}{2 (\beta_5 t)^{\frac{1}{2}}} \right) \right] (\text{in } y > 0), \tag{46}$$

where

$$\beta_5 = \frac{k_w}{\rho_w c_w}$$

and A is the constant of integration.

Further, by using the condition

$$\left[k_g \frac{\partial T}{\partial y} + F_r \right]_{y=-0} = \left[k_w \frac{\partial T}{\partial y} \right]_{y=+0} \tag{47}$$

the constants A and B are determined

$$A = \frac{m (T_3 - T_1)}{(m+1)}, \; B = \frac{-(T_3 - T_1)}{(m+1)}, \tag{48}$$

where

$$m = \left\{ \frac{\beta_4}{\rho_w c_w k_w} \right\}^{\frac{1}{2}} c_p . \tag{49}$$

By substituting (45) in (20) we determine the Eulerian distance y given by

$$(1 + m) \frac{\omega_3 y}{2 (\beta_4 t)^{\frac{1}{2}}} = \left(m + \frac{T_1}{T_3}\right) \frac{\psi}{2 (\beta_4 t)^{\frac{1}{2}}} - \left(1 - \frac{T_1}{T_3}\right) \left\{ \frac{\psi}{2 (\beta_4 t)^{\frac{1}{2}}} \operatorname{erf}\left(\frac{\psi}{2 (\beta_4 t)^{\frac{1}{2}}}\right) + \frac{1}{\pi^{\frac{1}{2}}} \left(e^{-(\psi^2/4\beta_4 t)} - 1\right) \right\}. \tag{50}$$

The velocity distribution in the gas is determined from (21) by using the condition that the particle velocity must be zero at the wall, hence

$$u = \left(\frac{c_1}{c_p} + \frac{16\sigma R}{3P_3 c_2 c_p}\right) \frac{\partial T}{\partial \psi} - \left\{ \left(\frac{c_1}{c_p} + \frac{16\sigma R}{3P_3 c_2 c_p}\right) \frac{\partial T}{\partial \psi} \right\}_{\psi = 0}, \tag{51}$$

where T is given by (45).

Outside the contact region $\frac{\partial T}{\partial \psi} \to 0$, and therefore, the velocity there is

$$u_\infty = -\left\{ \left(\frac{c_1}{c_p} + \frac{16\sigma R}{3P_3 c_2 c_p}\right) \frac{\partial T}{\partial \psi} \right\}_{\psi = 0}$$

$$= - \frac{(T_3 - T_1)}{(\beta_4 \pi)^{\frac{1}{2}} (m + 1) (t)^{\frac{1}{2}}} . \tag{52}$$

Expression (22) for pressure contains the unknown function $g(t)$ which is determined on the assumption that the effect of the contact region on the external ideal-gas flow is small.

Substituting

$$u = u', \ \rho = \omega_3 + \rho', \ p = P_3 + p', \tag{53}$$

in the ideal-gas flow equations and on neglecting the squares and higher powers of the perturbed quantities the wave-equation for u' which admits the solution

$$u' = F(A_3 t + y) + G(A_3 t - y), \tag{54}$$

where A_3 is the isentropic sound speed behind the reflected shock.

The perturbed pressure p' is given by

$$p' = - \omega_3 A_3 [F(A_3 t + y) - G(A_3 t - y)]. \tag{55}$$

Making use of the boundary condition (52) at the edge of the contact region (at the wall $y = 0$), we obtain

$$F(A_3 t) + G(A_3 t) = - \frac{L}{t^{\frac{1}{2}}}, \tag{56}$$

where

$$L = \frac{(T_3 - T_1)}{(\beta_4 \pi)^{\frac{1}{2}} (m + 1)} .$$

At the unperturbed shock position given by $y = - U_s t$ ($U_s \equiv$ shock speed), the velocity and pressure perturbations are related by the equation

$$p' = \omega_3 A_3 \phi (M_s) u', \tag{57}$$

where

$$M_s = \frac{U_s}{A_3}$$

and

$$\phi (M_s) = \frac{2 M_s [(\gamma - 1) M_s^2 + 2]}{[(3\gamma - 1) M_s^2 + (3 - \gamma)]} \text{ [See Appendix, eqn. (10)]}$$

Substituting expressions (54) and (55) in equation (57) we obtain

$$F \left\{ (A_3 - U_s) t \right\} = - \left\{ \frac{1 + \phi (M_s)}{1 - \phi (M_s)} \right\} \times G \left\{ (A_3 + U_s) t \right\}, \tag{58}$$

which can be put in the form

$$F (\xi) = - NG (\lambda \xi), \tag{59}$$

where ξ is a variable,

$$N = \frac{1 + \phi (M_s)}{1 - \phi (M_s)}$$

and

$$\lambda = \frac{1 + M_s}{1 - M_s} .$$

We put $\xi = A_3 t$ in (56) and on substituting for $F(\xi)$ from (59), we obtain the equation for $G(\xi)$

$$- NG (\lambda \xi) + G (\xi) = - L (A_3 / \xi)^{\frac{1}{2}} , \tag{60}$$

which has the solution

$$G (\xi) = \frac{L}{\{ N / (\lambda)^{\frac{1}{2}} \} - 1} (A_3 / \xi)^{\frac{1}{2}} \tag{61}$$

Making use of (59) we obtain

$$F (\xi) = - \frac{LN}{\lambda^{\frac{1}{2}} \{ (N / \lambda^{\frac{1}{2}}) - 1 \}} (A_3 / \xi)^{\frac{1}{2}} \tag{62}$$

Expressions (54) and (55) on using (61) and (62) are

$$u' = - \frac{L}{N / (\lambda)^{\frac{1}{2}} - 1} \left[\frac{N / (\lambda)^{\frac{1}{2}}}{\{ t + (y / A_3) \}^{\frac{1}{2}}} - \frac{1}{\{ t - (y / A_3) \}^{\frac{1}{2}}} \right], \tag{63}$$

$$p' = \frac{w_3 A_3 L}{\{ N/(\lambda)^{\frac{1}{2}} \} - 1} \left[\frac{N/(\lambda)^{\frac{1}{2}}}{\{t + (y/A_3)\}^{\frac{1}{2}}} + \frac{1}{\{ t - (y/A_3) \}^{\frac{1}{2}}} \right]. \tag{64}$$

It is noted that the function $g(t)$ in (22) is equal to the perturbed pressure at the edge of the contact region, so that from (64) and (22) we have

$$g(t) = (p')_{y=0} = \frac{\omega_3 A_3 L}{(N/\lambda^{\frac{1}{2}}) - 1} \cdot \frac{(N/\lambda^{\frac{1}{2}}) + 1}{(N/\lambda^{\frac{1}{2}}) + 1} \cdot \frac{1}{t^{\frac{1}{2}}} \tag{65}$$

From the perturbed shock equations we obtain the perturbed shock speed Us'

$$U_s' = \psi(M_s)\, u_3', \tag{66}$$

where

$$\psi(M_s) = \frac{\gamma + 1}{2} \quad \frac{(\gamma - 1) M_s^2 + 2}{(3\gamma - 1) M_s^2 + (3 - \gamma)}, \qquad \text{[See Appendix, eqn. (11)]}$$

Making use of (63) at $y = - U_s t$ (66) takes the form

$$U_s' = - \frac{\psi(M_s)\, L\, (N - 1)}{(1 + M_s)^{\frac{1}{2}} \{(N/\lambda^{\frac{1}{2}}) - 1\}} \cdot \frac{1}{t^{\frac{1}{2}}} \tag{67}$$

which shows that the perturbed shock speed varies inversely as the square root of time so that as $t \to \infty$ the reflected shock speed approaches the value based on ideal gas theory. Hence the reflected shock is attenuated due to the combined effects of molecular heat conduction and radiative heat conduction.

APPENDIX

Taking into account the small perturbations in the flow variables behind the reflected shock due to the presence of the contact region, we write the Ranking-Hugoniot conditions

$$\omega_2 (- U_s - U_s' + u_2) = (\omega_3 + \rho_3') (- U_s - U_s' + u_3'), \tag{1}$$

$$\omega_2 (- U_s - U_s' + u_2)^2 + P_2 = (\omega_3 + \rho_3') (- U_s - U_s' + u_3')^2 + P_3 + p_3', \tag{2}$$

$$\frac{P_3 + p_3'}{P_2} = \frac{\left\{ (\gamma + 1) \dfrac{\omega_3 + \rho_3'}{\omega_2} - (\gamma - 1) \right\}}{\left\{ (\gamma + 1) - (\gamma - 1) \dfrac{\omega_3 + \rho_3'}{\omega_2} \right\}}. \tag{3}$$

Neglecting second and higher powers of perturbed quantities, we deduce from equation (3)

$$p_3' = \frac{4\gamma \rho_3' P_3}{\omega_2 \left\{ (\gamma + 1) \dfrac{\omega_3}{\omega_2} - (\gamma - 1) \right\} \left\{ (\gamma + 1) - (\gamma - 1) \dfrac{\omega_3}{\omega_2} \right\}^2} \tag{4}$$

Similarly, from equation (1) we obtain

$$\frac{\omega_3}{\omega_2} u_3' = \frac{\rho_3'}{\omega_2} U_s + \left(\frac{\omega_3}{\omega_2} - 1 \right) U_s', \tag{5}$$

and from equation (2) we have

$$p_3' = 2\omega_2 U_s u_3' - U_s^2 \rho_3'. \tag{6}$$

Using the relation

$$\frac{\omega_3}{\omega_2} = \frac{(\gamma - 1) M_s^2 + 2}{(\gamma + 1) M_s^2}, \tag{7}$$

in equation (5) we get

$$\frac{\rho_3'}{\omega_2} U_s = \frac{(\gamma - 1) M_s^2 + 2}{(\gamma + 1) M_s^2} u_3' - 2\left\{ \frac{1 - M_s^2}{(\gamma + 1) M_s^2} \right\} U_s'. \tag{8}$$

From equations (4) and (7) we obtain

$$p_3' = \gamma P_3 \frac{\rho_3'}{\omega_2} \left[\frac{(\gamma + 1) M_s^4}{2\gamma M_s^2 - (\gamma - 1)} \right]. \tag{9}$$

From equations (6), (7) and (9) we deduce,

$$p_3' = \omega_3 A_3 \phi (M_s) u'$$

where

$$\phi (M_s) = \frac{2M_s\left\{(\gamma - 1) M_s^2 + 2\right\}}{(3\gamma - 1) M_s^2 + (3 - \gamma)}. \tag{10}$$

Eliminating p_3', ρ_3' and ω_2 between equations (6), (7), (8) and (10), we obtain

$$U_s' = \frac{\gamma + 1}{2} \left\{ \frac{(\gamma - 1) M_s^2 + 2}{(3\gamma - 1) M_s^2 + (3 - \gamma)} \right\} u_3'. \tag{11}$$

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