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Maximum-Entropy Principle in Flexible Manufacturing Systems

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Abstract. It is shown that the entropy of the joint probability distribution of the queue lengths of the M machine-groups in a closed queuing network model of a flexible manufacturing system is maximum when the loads on the different machine-groups are equal for both single-machine machine-groups and for those multiple-machine machine-groups of which group sizes are equal. It is also shown that for unequal machine-group-sizes, the entropy is not maximum when the workload is balanced. The simultaneous variations of the entropy function of the load distribution, the entropy function of joint probability distribution lengths of queues and the expected production function are studied in order to investigate the relationship between the information content and productive capacity of manufacturing systems. Four measures of load balance in a flexible manufacturing system are given.

1. Introduction

We consider the closed queuing network model for a flexible manufacturing system suggested by Solberg[1,2]. Let s_i be the number of machines in the ith machine-group and let x_i be the scaled workload in it ($i = 1, 2, \ldots, M$) so that (Stecke[3])

$$x_1 + x_2 + \ldots + x_M = s_1 + s_2 + \ldots + s_M = m, \tag{1}$$

where m is the total number of machines in the system.

The probability that there are $n_1, n_2, \ldots, n_M$ parts in the M machine-groups, either being processed or waiting to be processed is (Buzen[4], Gordon & Newell[5,6])

$$p(n_1, n_2, \ldots, n_M) = \frac{g_1(n_1)\, g_2(n_2) \ldots g_M(n_M)}{\sum\limits_{S(M,N)} g_1(n_1)\, g_2(n_2) \ldots g_M(n_M)}, \tag{2}$$

where $S(M, N)$ is the set of all non-negative integers whose sum is N (the number of parts in the closed system) and

$$g_i(n_i) = x_i^{n_i} \text{ for single-machine machine-groups}$$

$$\left.\begin{aligned} &= \frac{x_i^{n_i}}{n_i!}, && n_i \leqslant s_i \\ &\quad \frac{x_i^{n_i}}{s_i!\, s_i^{n_i - s_i}}, && n_i > s_i \end{aligned}\right\} \text{ for multiple-machine machine-groups} \tag{3}$$

The expected production function (EPF) is

$$\phi(x_1, x_2, \ldots, x_M) = \frac{\sum\limits_{S(M, N-1)} g_1(n_1)\, g_2(n_2) \ldots g_M(n_M)}{\sum\limits_{S(M,N)} g_1(n_1)\, g_2(n_2) \ldots g_M(n_M)} \tag{4}$$

In the special case of single-machine machine-groups, this gives

$$\phi(x_1, x_2, \ldots, x_M) = \frac{\sum\limits_{S(M, N-1)} x_1^{n_1}\, x_2^{x_2} \ldots x_M^{n_M}}{\sum\limits_{S(M,N)} x_1^{n_1}\, x_2^{n_2} \ldots x_M^{n_M}} \tag{5}$$

Stecke[3] has shown partially analytically, but mostly numerically and graphically that the EPF (5) is maximum when

$$x_1 = x_2 = \ldots = x_M = 1, \tag{6}$$

so that

$$(EPF)_{max} = \frac{N}{M + N - 1} \tag{7}$$

Thus the EPF is maximum when the load is distributed uniformly over the M machines, i.e., when the entropy of the load distribution is maximum[7].

This suggests a possible relationship between maximization of production and entropy functions.

If $s_1 = s_2 = \ldots = s_M = s$, i.e, if all machine-groups are of same size, Stecke's discussion suggests that for maximization of production function, we should set $x_1 = x_2 = \ldots = x_M$ which again suggests maximization of entropy.

If the machine-group sizes are not equal, balancing of workloads is still suggested i.e. we are asked to choose

$$\frac{x_1}{s_1} = \frac{x_2}{s_2} = \quad = \frac{x_M}{s_M} \tag{8}$$

This, however, does not maximize the *EPF.* We shall see that in this case, entropy is also not maximized.

The aim of the present paper is to investigate the relationship between maximization of entropy and of expected production function for closed queuing network models of flexible manufacturing systems.

2. Two-machines, Two Parts Case

Here

$$M = 2,\ N = 2,\ s_1 = 1,\ s_2 = 1,\ x_1 = x,\ x_2 = 2-x,$$

so that

$$p(n_1, n_2) = \frac{x^{n_1}(2-x)^{n_2}}{\sum_{n_1+n_1=2} x^{n_1}(2-x)^{n_2}},\quad 0 \leqslant x \leqslant 2$$

or

$$p(2,0) = \frac{x^2}{x^2-2x+4},\ p(1,1) = \frac{x(2-x)}{x^2-2x+4},\ p(0,2) = \frac{(2-x)^2}{x^2-2x+4} \tag{11}$$

$$\bar{n}_1 = \frac{\sum_{n_1+n_1=2} n_1\, p(n_1, n_2)}{\sum_{n_1+n_1=2} p(n_1, n_2)} = \frac{x^2+2x}{x^2-2x+4}$$

$$\bar{n}_2 = \frac{\sum_{n_1+n_1=2} n_2\, p(n_1, n_2)}{\sum_{n_1+n_1=2} p(n_1, n_2)} = \frac{x^2-6x+8}{x^2-2x+4} = 2-\bar{n}_1$$

$$\phi(x) = \frac{2}{x^2-2x+4} = \frac{2}{(x-1)^2+3}$$

The production function is maximum when $x = 1$ Also the entropy is given by

$$S(x) = -\sum_{n_1+n_2=2} p(n_1, n_2)\ln p(n_1, n_2)$$

$$\sum_{n_1+n_2=2} p(n_1, n_2)\,[n_1 \ln x + n_2 \ln(2-x) - \ln(x^2-2x+4)]$$

$$= \ln(x^2-2x+4) - \bar{n}_1 \ln x - \bar{n}_2 \ln(2-x)$$

$$= \ln(x^2-2x+4) - \frac{x^2+2x}{x^2-2x+4}\ln x - \frac{x^2-6x+8}{x^2-2x+4}\ln(2-x)$$

$$S(0) = 0,\quad S(1) \quad \ln 3,\quad S(2) = 0$$

$$S(x) = S(2-x),\ S'(x) = -S'(2-x),\ S''(x) = S''(2-x)$$

$$S'(1) = 0$$

When $x = 1$, the three probabilities in Eqn. (11) are each equal to 1/3 and then this gives the globally maximum value of $S(x)$, viz. $\ln 3$. Also

$$\frac{dS}{dx} = \ln x\,\frac{d}{dx}\left(\frac{x^2+2x}{x^2-2x+4}\right) - \ln(2-x)\,\frac{d}{dx}\left(\frac{x^2-6x+8}{x^2-2x+4}\right) \tag{20}$$

When $x < 1$, $\ln x < 0$, $\ln(2-x) > 0$, the first derivative in Eqn. (20) is positive and the second derivative there is negative so that

$$dS/dx > 0 \tag{21}$$

Thus the entropy is a continuously increasing function of x as x increases from 0 to 1 and then the entropy decreases as x increases from 1 to 2 as shown in Fig. 1.

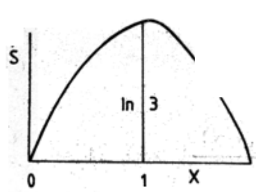

Figure 1.

3. Two-Machines N Parts Case

Here

$$M = 2,\ N > 2,\ s_1 = \quad,\ s_2 \quad,\ x_1 = x,\ x_2 = 2-x$$

$$p(n, N-n) = \frac{x^n(2-x)^{N-n}}{\sum_{n=0}^{N} x^n(2-x)^{N-n}},\qquad n = 0, 1, 2, \ldots, N \tag{23}$$

All these $N+1$ probabilities will be equal to $1/(N+1)$ if $x =$ and in this case the entropy will have the maximum possible value, viz., $\ln(N+1)$

In fact

$$S(x) = -\sum_{n=0}^{N} p(n, N-n)\left[n\ln x + (N-n)\ln(2-x) - \ln\sum_{n=0}^{N} x^n(2-x)^{N-n}\right] \tag{24}$$

Equations (18) and (19) continue to hold and instead of Eqn. (17), we have

$$S(0) = 0,\ S(1) = ln\,(N + 1),\ S(2) = 0$$

It can be shown that $S(x)$ is an increasing function of x in the interval [0, 1] and also that $S(x)$ is a concave function of x.

4. General Case

For general values of M and N,

$$S(x_1, x_2, \quad, x_M) = - \sum_{S(M;\,N)} p(x_1, x_2, \ldots, x_M)\, ln\, p(x_1, x_2, \ldots, x_M),$$

where $p(x_1, x_2, \ldots, x_M)$ is given by Eqn. (5). It is easily seen that $S(x_1, x_2, \ldots, x_M)$ is a symmetric function of $x_1, x_2, \ldots, x_M$. This has to be maximized subject to

$$x_1 + x_2 + \quad + x_M = M \tag{27}$$

Using Lagrange's method, this gives

$$\frac{\partial S}{\quad} \quad \frac{\partial S}{\quad} = \quad = \frac{\frac{\partial S}{\partial x_M}}{1}$$

whose solution is

$$x_1 = x_2 = \quad x_M = \tag{29}$$

so that the entropy is maximum when the machines are equally loaded. For completing the proof of this result, we have to show that S is a concave or a quasiconcave function of $x_1, x_2, \ldots, x_M$. S is, however, a concave function of $p's$ and $p's$ can be shown to be positive quasiconcave functions of $x_1, x_2, \ldots, x_M$ by the same arguments as used by Stecke[3].

It is easily shown that the maximum entropy is $ln \left(\frac{N+M-}{M-1} \right)$ and this function increases both with M and N.

5. Two Groups of Two Machines Each

Here $M = 2$, $m = 4$, $s_1 = 2$, $s_2 = 2$, $N = 4$, so that

$$p(0, 4) \propto \frac{x_1^0}{\quad} \frac{x_2^4}{\quad} \quad \frac{x_2^4}{8},\ p(1, 3) \propto \frac{x_1^1}{1} \frac{x_2^3}{2.2^1} = \frac{x_1 x_2^3}{4}$$

$$\frac{\quad}{\quad} = \frac{x_1^2 x_2^2}{4},\ p(3, 1) \propto \frac{x_2^1}{1} \frac{x_1^3}{2{\cdot}2} = \frac{x_2 x_1^3}{4}$$

For a balanced system $x_1 = x_2$, so that

$$p\,(0,4) = \frac{1}{8},\; p\,(1,3) = \frac{1}{4},\, p\,(2,2) = \frac{1}{4},\, p\,(3,1) = \frac{1}{4},\, p\,(4,0) = \frac{1}{8}$$

(:

and the entropy of this distribution

$$= \frac{1}{8}\,ln\,8 + \frac{1}{4}\,ln\,4 + \frac{1}{4}\,ln\,4 + \frac{1}{4}\,ln\,\frac{1}{4} + \frac{1}{8}\,ln\,8 = \frac{9}{4}\,ln\,2$$

Now there are five possible states here and the maximum possible entropy for such a system arises, when each state has a probability 1/5 and the entropy for the resulting distribution is *ln* 5 which is more than 9/4 *ln* 2.

We may note that whatever be the loads x_1, x_2 we choose, we cannot make the five probabilities in Eqn. (30) equal so that the entropy *ln* 5 cannot be attained.

In fact the entropy of the distribution in Eqn. (30) is

$$\sum_{n=0}^{4} p\,(n, 4-n)\; ln\; p\,(n, 4-n)$$

and it is easily shown by using Eqn. (30) that this is a symmetric function of x_1, x_2 of which the maximum value arises when $x_1 = x_2 = 1$ so that Eqn. (32) gives the maximum entropy for the case of two groups with two machines in each group.

If the number of parts is N, then in Eqn. (32), the entropy of the resulting distribution when $x_1 = x_2 = 1$, is

$$= \frac{1}{2N}\; ln\; 2N + \frac{N}{N}\quad ln\; N + \frac{1}{2N}\; ln\; 2N \tag{34}$$

$$= ln\; N + \frac{1}{N}\; ln\; 2$$

If all the probabilities are equal, the maximum possible entropy is $ln\,(N+1)$. Moreover,

$$ln\,(N+1) - ln\,N - \frac{1}{N}\,ln\,2 = ln\;\frac{N+1}{N} - \frac{1}{N}\;ln\,2$$

$$\frac{\;}{N}\left[ln\left(\frac{N+1}{N}\right)^{N} - ln\,2\right]$$

Now $\left(1 + \frac{1}{N}\right)^{N}$ is a monotonic increasing function of N and approaches e as $N \to \infty$.

Thus the entropy when there is one machine in each group is more than the entropy when there are two machines in each group and the difference between the two entropies tends to zero as $N \to \infty$.

Similarly it is expected that the entropy when there are k machines in each group is more than the entropy when there are $(k+1)$ machines in each group.

6. Generalisation to the Case of M Machine-Groups of Equal Sizes

Here

$$s_1 \quad s_2 = \quad = s_M = m/M = s \text{ (say)}$$

The entropy function is a symmetric function of $x_1, x_2, \ldots, x_M$ and its maximum occurs when $x_1 = x_2 = \ldots = x_M = m/M$, i.e., when the system is balanced. Thus for equal sizes of machine-groups, the entropy function and the expected production function are maximum for the same distribution of load.

If $s = 1$, the maximum entropy is $ln \binom{N+M-1}{M-1}$ and is obtained by distributing the load equally among the machines. If $s > 1$, the maximum entropy obtained by distributing the load equally among the machines is less than $ln \binom{M+sM-1}{sM-1}$ and the difference between $ln \binom{N+sM-1}{sM-1}$ and the maximum entropy attainable increases as s increases, but each difference tends to zero as $N \to \infty$.

7. Comparison of Expected Production and Entropy Functions

The expected production function for single-machine machine-groups is

$$\phi(x_1, x_2, \ldots, x_M) = \frac{\sum\limits_{S(M, N-1)} x_1^{n_1} x_2^{n_2} \cdot x_M^{n_M}}{\sum\limits_{S_{M,N}} x_1^{n_1} x_2^{n_2} \cdot x_M^{n_M}}$$

It is a symmetric function of $x_1, x_2, \ldots, x_M$ and its maximum value is

$$\phi(1, 1, \ldots, 1) = \frac{\binom{N+M-2}{M-1}}{\binom{N+M-1}{M-1}} = \frac{N}{\quad}$$

This maximum value is always less than unity and approaches unity as N approaches infinity. The normalised expected production function is

$$\bar{\phi}(x_1, x_2, \quad, x_M) = \frac{\phi(x_1, x_2, \ldots, x_M)}{\phi(1, 1, \ldots, 1)} = \frac{N+M-}{N} \quad \phi(x_1, x_2, \quad, x_M)$$

This is always $\leqslant 1$ and attains its maximum value unity when $x_1 = x_2 =$ $x_M = 1$. Again the entropy function for single-machine machine-groups is

$$S(x_1, x_2, \ldots, x_M) \quad \sum_{S(M,N)} \frac{x_1^{n_1} x_2^{n_2} \ldots x_M^{n_M}}{\sum_{S(M,N)} x_1^{n_1} x_2^{n_2} \ldots x_M^{n_N}} \times ln \frac{x_1^{n_1} x_2^{n_2} x_M^{n_M}}{\sum_{S(M,N)} x_1^{n_1} x_2^{n_2} x_M^{n_M}} \tag{40}$$

This is also a symmetric function of $x_1, x_2, \ldots, x_M$ and attains its maximum value when $x_1 = x_2 = \ldots = x_M = 1$ and the maximum value is

$$S(1, 1, \ldots, 1) = ln \binom{N+M-1}{M-1} \tag{41}$$

The normalised entropy function is

$$= \frac{S(x_1, x_2, \ldots, x_M)}{S(1, 1, \ldots\ldots, 1)} = \frac{S(x_1, x_2, \ldots, x_M)}{ln \binom{N+M-1}{M-1}} \tag{42}$$

8. Conclusion

Performance criterion functions for computer-controlled flexible manufacturing systems include expected production function, proportion of busy machines, the probability of a machine chosen at random being found busy and the probability of all the machines being found busy[3]. In this paper, an information-theoretic performance criterion function is proposed, viz, the entropy of the system. It is shown that this criterion gives the same optimal load distribution as the *EPF* criterion when the number of machines in different groups are equal. Even when the numbers of machines in different groups are different, this criterion still gives almost the same optimal load distribution as the *EPF* criterion. This new criterion is also easier to apply. As such it can be used to improve the performance of flexible manufacturing systems. The discussion also shows that the performance of an FMS is highly correlated with the information-content of the system.

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