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Optimization Problem of Mortar Barrel and Bomb Clearances

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ABSTRACT

Optimum mortar windage to achieve maximum accuracy and required velocity for impacting the firing stud under two conditions of constraint is considered. These control constraints are considered to be bounded and the extremals have been studied.

1. INTRODUCTION

Mortars are smooth bore muzzle loading infantry weapons. Bombs are dropped from the muzzle end. The velocity of the bomb hitting a striker/stud located in the breech piece must, therefore, be above a certain critical value to initiate the cap composition. This amongst other factors depends on the barrel to bomb clearance, known as windage. If windage is large, it will increase the velocity of hit. It may be undesirable from the point of view of ballistics and accuracy of fire. An optimum value will be one which gives a velocity just above the critical value decided by the sensitivity of cap compositions.

The problem of optimization[1] of mortar windage is treated as a variational problem of Bolza-Mayer type with limitations imposed on controls. The methods of the variational calculus are applied to solve the general problem of extremising a given 'windage' for mortars on loading the bomb. The necessary conditions for the existence of a minimum barrel-to-bomb clearance and the nature of extremals subjected to the various control variables have been studied.

2. MOTION OF THE BOMB

If the parallel portion of the bomb is small, then forces acting on the bomb at any instant time *t*, when the bomb has dropped inside the barrel, are as shown in Fig. 1.

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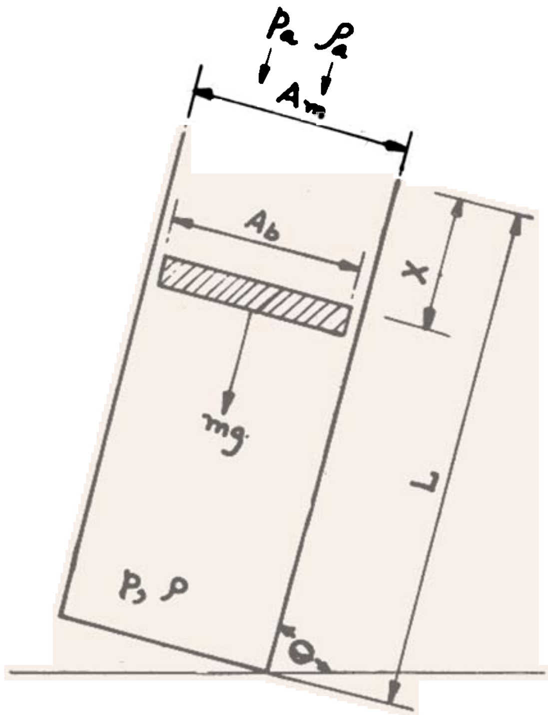

Figure 1. Motion of bomb inside mortar.

Neglecting the effect of buoyancy which is less than 0.1 per cent of the weight the equation of motion of the bomb inside the barrel, inclined at 45° to the horizontal can be written as

$$\ddot{x} + \frac{p A_b}{m} = g \left(\frac{1-\mu}{\sqrt{2}} + \frac{p_a A_b}{mg} \right) \tag{1}$$

where μ is the coefficient of friction between the bomb and the barrel.

From conservation of mass we have

$$\frac{d}{dt}\left[A_m (L-x) \rho \right] = -(A_m - A_b) \rho_a V_l \tag{2}$$

where V_l is the velocity of escaping air past the bomb and can be readily obtained for a compressible flow.

Assuming that the compression of air inside the barrel due to the fall of the bomb follows adiabatic law, the system of Eqns. (1) and (2) of motion of the bomb can be reduced to the form

$$\dot{x} = u \tag{3}$$

$$\dot{u} = g\left[\frac{1-\mu}{\sqrt{2}} - \frac{p_a A_m K}{mg} \left\{ \left(\frac{\rho}{\rho_a} \right) \right\} \right] \tag{4}$$

$$\dot{\rho} = \frac{\rho}{(L-x)} \left\{ u - \beta V_l \left(\frac{\rho_a}{\rho} \right) \right\} \tag{5}$$

where

β is the ratio of windage area to the mortar barrel sectional area

$$\text{where } \beta = \frac{A_m - A_b}{A_m} = 1 - K \tag{6}$$

$$V_l = \sqrt{\left[\frac{2\gamma}{\gamma-1} RT_a \left\{ \left(\frac{\rho}{\rho_a} \right)^{\gamma-1} - 1 \right\} \right]} \tag{7}$$

$$K = \frac{A_b}{A_m} \tag{8}$$

3. FORMULATION OF THE PROBLEM

The system of differential Eqns (3) to (5) contains five physical variables x, u, ρ, K and m for any particular motion. Two variables, K and m are termed as controlled variables[1]. The other remaining quantities x, u, and ρ being termed as state variables.

The constraints to make the 'optimal' bounded are defined as

$$K^* \leqslant K \leqslant K^{**}$$
$$m^* \leqslant m \leqslant m^{**}$$

We can replace it by the equality constraint

$$(K - K^*) \; (K^{**} - K \; - a^2 = 0 \tag{9}$$

$$(m - m^*)(m^{**} - m) - \eta = 0 \tag{10}$$

where a and η are real variables.

The variational problem[1] is formulated as follows :

The functions $x(t)$, $u(t)$, $\rho(t)$, K, m, a and η which are consistent with Eqns. (3) to (5), (9) & (10) and certain prescribed end conditions, it is required to find that particular set which minimizes a certain function of the form,

$$J \equiv J\left[x(t_o), \mathrm{u}(t_o), \rho(t_o), t_o, x(t_f), u(t_f), \rho(t_f), t_f\right] \tag{11}$$

The following development applies only to those problems for which the control variables K and m minimize J of Eqn. (11) are bounded implicitly, without resort to constraints of the Eqn. (9) and (10).

The auxiliary function F is defined as

$$F = \lambda_x(\dot{x}-u)+\lambda_u\left\{\dot{u}-g\left[\frac{1-\mu}{\sqrt{2}}-\frac{p_a A_m}{mg}K\left(\left(\frac{\rho}{\rho_a}\right)^{\gamma}-1\right)\right]\right\}$$

$$+\lambda_\rho\left[\dot{\rho}\quad\frac{\rho}{(L-x)}\left\{u-\beta V_l\frac{\rho_a}{\rho}\right\}\right]$$

$$+\lambda_k\left[\qquad\qquad\right]+\lambda_m\left[(m-m^*)(m^{**}-m)-\eta^2\right] \qquad (12)$$

where $\lambda_x, \lambda_u, \lambda_\rho, \lambda_k$ and λ_m are undetermined Lagrangian multipliers and are function of time. They are also referred as costate variables.

4. EULER-LAGRANGE EQUATIONS

The optimum value of J is governed by the Euler-Lagrange equations[1] for the present problem and may be obtained explicitly as

$$\lambda_x=-\lambda_\rho\frac{\rho}{(L-x)^2}\left\{u-\beta V_l\frac{\rho_a}{\rho}\right\} \qquad 13)$$

$$\lambda_x-\frac{\rho}{(L-x)} \qquad (14)$$

$$=\frac{\lambda_u\gamma\, p_a A_m K}{m\,\rho_a}\left(\frac{\rho}{\rho_a}\right)\qquad\frac{\lambda_\rho}{(L-x)}$$

$$\left\{u-\frac{\beta\gamma R T_a}{V_l}\left(\frac{\rho}{\rho_a}\right)^{\gamma-2}\right\} \qquad 15)$$

$$\frac{\lambda_u\, p_a A_m}{m}\left\{\left(\frac{\rho}{\rho_a}\right)^{\prime}\ 1\right\}-\lambda_\rho\frac{V_l}{(L-x)}\rho_a+\lambda_k(K^{**}+K^*-2K)=0$$

$$-\lambda_u\frac{p_a A_m K}{m^2}\left\{\left(\frac{\rho}{\rho_a}\right)^{\gamma}\quad\right\}+\lambda_m\quad m^{**}+m^*-2m)=$$

$$a = 0$$

$$\lambda_m \quad \eta = 0$$

Since the function F does not contain the independent variable t explicitly, a first integral of Euler equation can readily written as

$$C' + \lambda_x u + \lambda_u g \left\{ \frac{1-\mu}{\sqrt{2}} - \frac{p_a A_m K}{mg} \left[\left(\frac{\rho}{\rho_a}\right) - 1 \right] \right\}$$
$$+ \lambda_\rho \frac{\rho}{(L-x)} \left\{ u - \beta V_l \frac{\rho_a}{\rho} \right\} = 0$$

where C' is an integration constant.

5. TRANSVERSALITY CONDITIONS

The transversality condition which gives changes in boundary conditions as also changes in J must be satisfied by the solution of the optimal control problem and is given by

$$dJ + \left[\lambda_x \, dx + \lambda_u \, du + \lambda_\rho \, d\rho + C' \, dt \right]_{tf}^{to} = 0 \tag{21}$$

6. CORNER CONDITIONS

The boundary conditions on the Lagrange's multipliers at any generic point, where the slope of the problem variables is discontinuous, and derived from the corner conditions as

$$\lambda_{x_i}(t_i) = \lambda_{x_{i+1}}(t_i)$$

$$\lambda_{u_i}(t_i) = \lambda_{u_{i+1}}(t_i)$$

$$\lambda_{\rho_i}(t_i) = \lambda_{\rho_{i+1}}(t_i)$$

$$\left[\lambda_{xi} u + \lambda_{ui} g \left\{ \frac{1-\mu}{} - \frac{p_a A_m K}{} \left(\left(\frac{\rho}{\rho_a}\right)' - \right) \right\} \right.$$
$$\left. + \lambda_{\rho i} \frac{\rho}{(L-x)} \left\{ u - \beta V_l \frac{\rho_a}{\rho} \right\} \right]_{ti} = \left[\lambda_{x_{i+1}} u + \lambda_{u_{i+1}} g \left\{ \frac{1-\mu}{\sqrt{2}} \right. \right.$$
$$\left. \left. \frac{p_a A_m K}{mg} \left(\left(\frac{\rho}{\rho_a}\right) \quad 1 \right) \right\} + \lambda_{\rho i+1} \frac{\rho}{(L-x)} \left\{ u - \beta V_l \frac{\rho_a}{\rho} \right\} \right]$$

where $i = 1,2,\ldots\ldots,(n-1)$.

Thus the Lagrangian multipliers $\lambda_x, \lambda_u, \lambda_\rho$ are continuous across discontinuity and hence continuous throughout the period of the bomb inside the barrel.

7. WEIESTRASS CONDITION

A necessary condition for the minimum value of J is

$$E \geqslant 0$$

where E is Weiestrass function.

On evaluation of Weiestrass function E, we get

$$KM \geqslant K^{\dagger} M^{\dagger}$$

where

$$\left.\begin{aligned} M &= \frac{\lambda_\rho V_l \rho_a}{(L-x)} - \frac{\lambda_u p_a A_m}{m} \left\{\left(\frac{\rho}{\rho_a}\right)^{\gamma} - 1\right\} \\ M^{\dagger} &= \frac{\lambda_\rho V_l \rho_a}{(L-x)} - \frac{\lambda_u p_a A_m}{m^{\dagger}} \left\{\left(\frac{\rho}{\rho_a}\right)^{\gamma} - 1\right\} \end{aligned}\right\}$$

and † denotes functions subjected to finite admissible variations.

Eqn. (23) must be satisfied for all admissible variations, not all zero.

8. SEQUENCE OF ARCS

Differentiating Eqn. (24) with respect to time and simplifying by making use of equations of motion and Euler-Lagrange Equations, we obtain

$$\begin{aligned} \dot{M} = & \frac{\lambda_x p_a A_m}{m} \left\{\left(\frac{\rho}{\rho_a}\right)^{\gamma} - 1\right\} + \frac{\lambda_u \gamma p_a A_m}{m\,(L-x)} \left(\frac{\rho}{\rho_a}\right)^{\gamma} \left[u + V_1 \frac{\rho_a}{\rho}\right] \\ & + \lambda_{\prime} \frac{\rho}{(L-x)} \left[\frac{\gamma R T_a u}{(L-x) V_I} \left(\frac{\rho}{\rho_a}\right)^{\gamma-2} + \frac{p_a A_m}{m} \left\{\left(\frac{\rho}{\rho_a}\right)^{\gamma} - 1\right\}\right] \end{aligned} \tag{25}$$

From Eqns. (18) and (19), we observe that the extremum arc is discontinuous and the sequence of arcs can have the following types :

(a) either (i) $\lambda_k = 0,\ a \neq 0$

or (ii) $\lambda_k = 0,\ a \neq 0$

(b) either (i) $\lambda_m \neq 0, \ \eta = 0$

or (ii) $\lambda_m = 0, \ \eta \neq 0$

The first possibility implies that either $K = K^*$ or $K = K^{**}$ while the second possibility means that $\dot{M} = 0$, i.e., $M = 0$ which is incompatible with Eqn. (25). Accordingly, the first possibility is forced meaning that only arcs of minimum or maximum windage can exist and there is no arc flown with intermediate windage.

Since the possibility of intermediate windage arc is ruled out, we have now to determine the conditions for the existence of minimum and maximum windage arcs. Now whenever $K = K^*$ or $K = K^{**}$, the condition (23) must hold all along the extremal arc, which means that when $M = M^{\dagger}$,then if

$$M < 0, K = K^*$$
$$M > 0, K = K^{**}$$

But when

$$M \neq M^{\dagger} \text{ and } K = K^{\dagger}$$

$$\text{then } M > M^{\dagger}$$

It implies for all admissible variations in m, the value of m should be such that M is always maximum. Thus we conclude that when M changes sign then there is always a change in the nature of windage programming and the arc is of maximum windage if $M < 0$ and of minimum windage if $M > 0$. Hence M is the quantity whose behaviour determines the nature of the extremal arc.

Now the behaviour of M is studied for minimum and maximum windage arcs for all admissible variations in m.

Case I

$$\lambda_m \neq 0, \eta = 0.$$

This condition implies that either $m = m^*$ or $m = m^{**}$, which means that the only arcs of minimum or maximum mass can exist and there is no arc flown with intermediate mass of the bomb.

Case 2

$$\lambda_m = 0, \quad \eta \neq 0.$$

From Eqn. (17), we get

$$\frac{\lambda_u p_a A_m K}{m^2}\left[\left(\frac{\rho}{\rho_a}\right)^{\gamma} - 1\right] = 0$$

Eqn. (28) holds good if $\lambda_u = 0$

Eqn. (24) reduces to

$$M = \frac{\lambda_p V_l \rho_a}{(L-x)} \tag{29}$$

This on differentiation with respect to time t gives

$$\dot{M} = \frac{\lambda_p \gamma \, R T_a \, \rho \, u}{(L-x)^2 V_l} \left(\frac{\rho}{\rho_a}\right)^{\gamma-2} \tag{30}$$

From Eqn. (29) the behaviour of M is depend upon the sign of the multiplier λ_p. Hence $\dot{M}$ also have the same type of behaviour as M as given by Eqn. (30).

Hence the extremal arc can exist even with the intermediate mass of the bomb depending upon the behaviour of the quantity M.

9. EXAMPLES

The above can be explained by taking a few simple examples.

Example 1 Minimum x

In this case $J \equiv x\,(t_f)$

The boundary conditions of the given problem are

$$t = t_o = 0 \quad \begin{matrix} x = 0 \\ u = 0 \\ \rho = \rho_a \end{matrix} \qquad t = t_f \quad \text{(unspecified)} \quad \begin{cases} \rho = \rho_f \\ u = u_f \end{cases}$$

Since the final time is not prescribed and with the aid of prescribed boundary conditions, the transversality condition (21) leads to

$C\,(t_f) = 0, \qquad \lambda_x(t_f) = 1$

Initially, $\lambda_x\,(t_o) = -1, \qquad \lambda u(t_o) = -1, \qquad \lambda_p\,(t_o) = -1.$

Given : $K^* \leq K \leq K^{**}$ and $m^* \leq m \leq m^{**}$.

The possibility of extremal arc can exist only when $\lambda_k \neq 0,\ a = 0$ i.e., either $K = K^*$ or $K = K^{**}$.

Example 2 Minimum u

In this case $J \equiv u(t_f)$

The boundary conditions of the given problem are

$$t = t_o = 0 \quad \begin{array}{l} x = 0 \\ u = 0 \\ \rho = \rho_a \end{array} \qquad t = t_f \text{ (unspecified)} \quad \begin{array}{l} x = x_f \\ \rho = \rho_f \end{array}$$

Since the final time is not prescribed and with the aid of prescribed boundary conditions, the transversality condition leads to

$C(t_f) = 0, \quad \lambda_u(t_f) = 1$

Initially, $\lambda_x(t_o) = -1, \quad \lambda_u(t_o) = -1, \quad \lambda_\rho(t_o) = -1$

In the present study an attempt has been made to obtain theoretically the best value of barrel to bomb clearance based on the scheme given in Example 2.

The succeeding paragraphs explain the scheme for evaluating the minimum mortar windage for a given set of variables involved in the problem.

10. METHODOLOGY

Four different masses 1.50 kg, 1.25 kg 1.00 kg and 0.75 kg of mortar bomb have been choosen for 51 mm bore diameter of mortar. The values of K are choosen to be 0.97 and 0.98. The other various constants involved in the problem are given as

$\mu = 0.25, \quad \gamma = 1.4, \quad p_a = 1.0332$ kg f/cm^2,

$\rho_a = 1.2938573.10^{-6}$ kg/cm^3, $\quad T_a = 303.16$ °K,

$R = 29.27.10^2$ kg f cm per kg per °K, $g = 981$ cm/sec^2.

The differential system composed of the constraints and Euler equations are numerically integrated step by step for different masses and K values with the help of Runge-Kutta[2] method.

The velocities at various distances during the motion of the bomb inside the barrel have been worked out separately in each case for different K values. The velocities worked out at various distances for mass 0.75 kg for both K values are shown in Figs. 2 & 3. The distance x_{opt} dropped by the bomb measured from the muzzle to the minimum windage point of the bomb is corresponding to the value $\lambda_u(t_x)$ equal to 1 for minimum u.

Since the difference in $(L - x_{opt})$ should be small, the process is repeated several times with different values of L untill the difference reached in minimum or met the specified final conditions. The length L is chosen in such a manner that the variation of Lagrange's multiplier λ_2 with respect to time t varied from −1 to +1.

The detailed results giving the variation of x_{opt} corresponding to L in each case are plotted in Fig. 4 and the energy E_{opt} observed at x_{opt} versus x_{opt}/L are plotted in Fig. 5

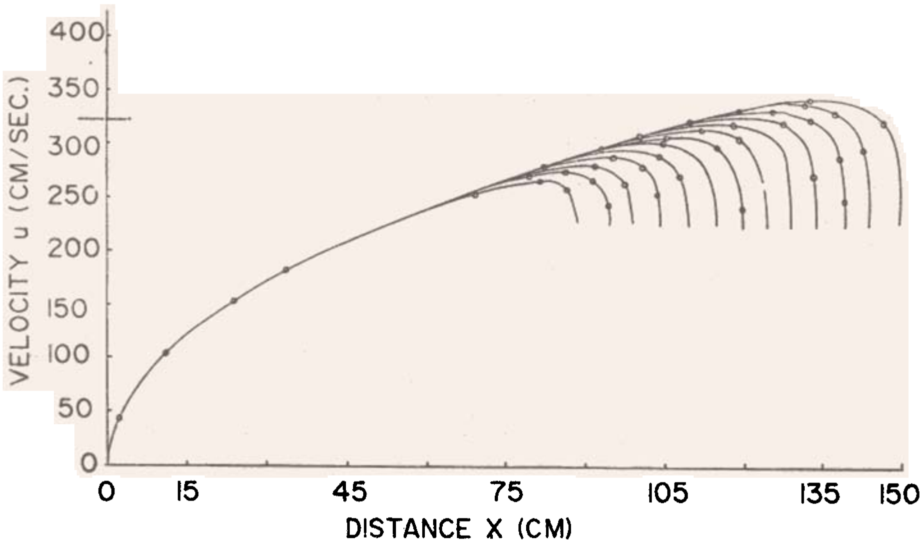

Figure 2. U vs x, M = 0.75 kg, K = 0.98, L = 90, 95, 100, 105, 110, 115, 120, 125, 130, 135, 140, 145 & 150.

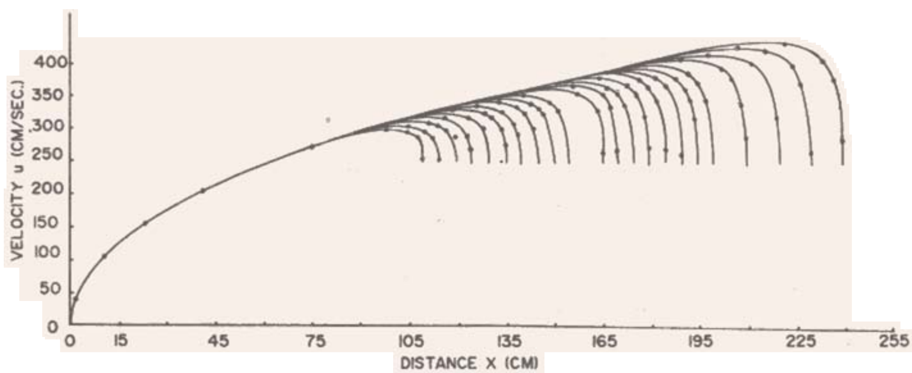

Figure 3. U vs x, M = 0.75 kg, K = 0.97, L = 110, 115, 120, 125, 130, 135, 140, 145, 150, 155, 165, 170, 175, 180, 185, 190, 195, 200, 210, 220, 230, 240.

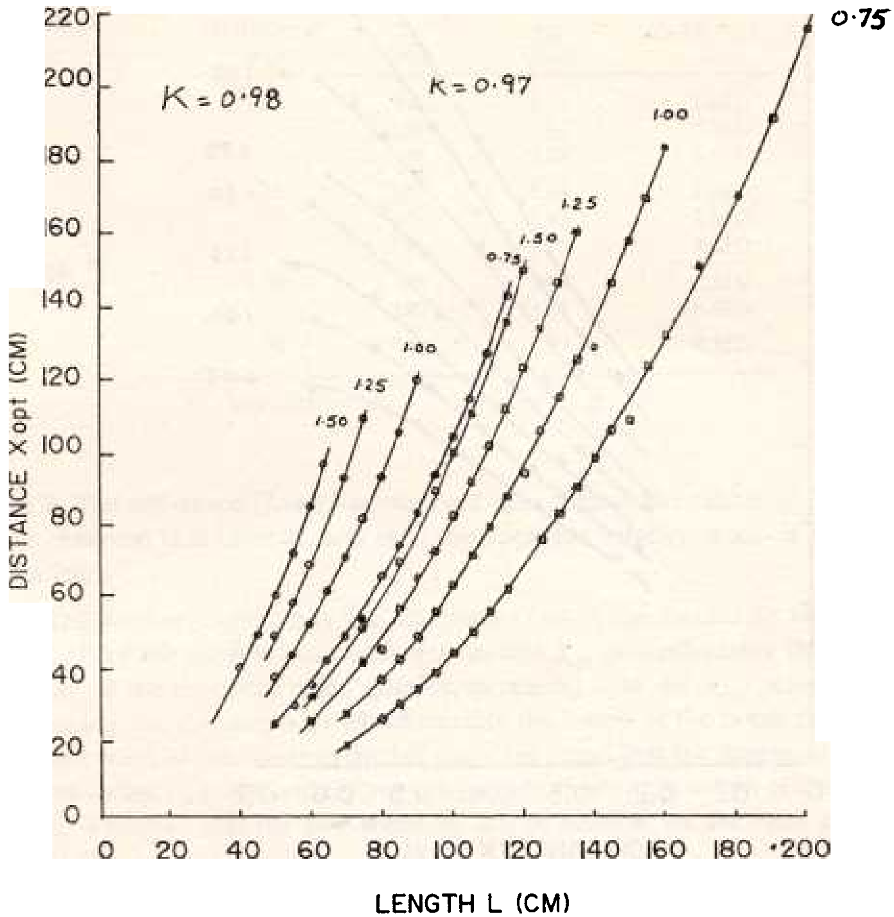

Figure 4. x_{opt} vs L, M = 1.50, 1.25, 1.00, 0.75 kg, K = 0.98, 0.97.

After knowing L and x_{opt} from Fig. 4, the energy corresponding to the ratio x_{opt}/L can be known from the graph in Fig. 5. The process is repeated by varying suitably L and x_{opt} till the results satisfy the desired specified conditions.

Alternately, the process can be carried out rapidly in a backward direction. First, the minimum energy required to hit the firing stud is selected. Corresponding to this energy, x_{opt}/L is found from the graph in Fig. 5.

After knowing the ratio of x_{opt}/L, the appropriate value of L and x_{opt} can be worked out from the graph in Fig. 4 to satisfy the variation of x_{opt} versus L. The process is repeated by giving suitable variation in energy till the best length of mortar for minimum windage obtained which satisfy the final conditions.

The family of extremals obtained for minimum velocity by the various set of parameters corresponding to both values of K are given separately in Table Nos. 1

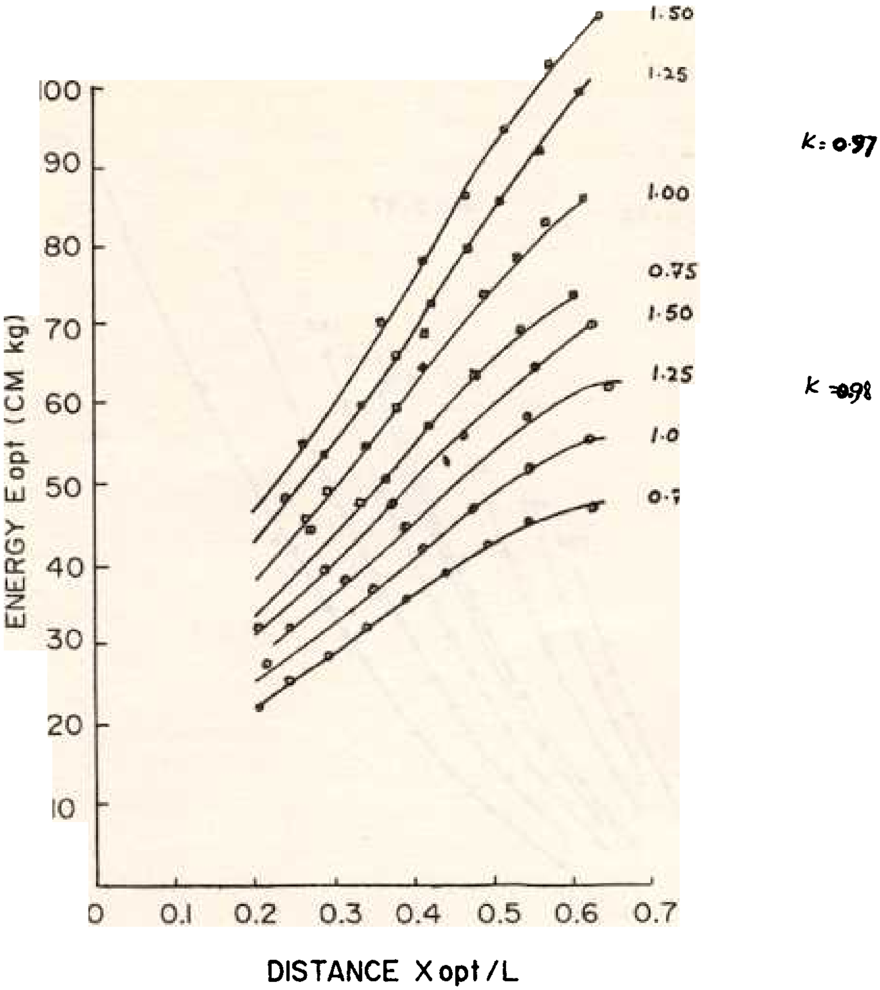

Figure 5. E_{opt} vs x_{opt}/L, M = 1.50, 1.25, 1.00, 0.75 kg, K = 0.98, 0.97.

Table 1. Length and mass of mortar

Bore diameter - 51 mm $K = 0.98$

Mass of bomb (kg)	L (cm)	v_{opt} (m/s)	$(L-x_{opt})$ (cm)
1.50	104	3.02	1.29 D
	100	2.90	2.83 D
	95	2.70	4.39 D
1.25	115	3.12	1.04 D
	110	3.01	3.24 D
1.00	130	3.28	1.82 D
	125	3.18	3.75 D
	120	3.02	5.19 D
0.75	155	3.46	2.08 D
	150	3.43	4.40 D

D – Bore calibre

Table 2. Length and mass of mortar

Bore diameter – 51 mm $K = 0.97$

Mass of bomb (kg)	L (cm)	v_{opt} (m/s)	$(L-x_{opt})$ (cm)
1.50		3.75	
		3.64	
		3.50	
1.25		3.90	
		3.87	
		3.77	
1.00		4.08	
		4.01	
0.75		4.37	

D – Bore calibre

and 2. The difference $(L-x_{opt})$ is observed upto 5 times the calibre of the bore. It is also observed that in each such sets, the optimum velocity at x_{opt} is always higher than 2m/s.

This further suggest that the difference $(L-x_{opt})$ can further be minimised upto $(L-x_{max})$ for the same set of parameters in case x_{opt} occured earlier than x_{max} as the velocity of the dropping bomb goes on increasing upto the x_{max} point. The aim of minimising this distance is to adjust suitably the length of the bomb from minimum windage point of the bomb to the tail end. It is found that the minimum windage for the same mass of bomb and calibre can suitably shorter the length of the barrel. It is further observed that the best solutions can be possible for the ratio x_{opt}/L greater than 0.85.

REFERENCES

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